

TRIPOSES AS A GENERALIZATION OF LOCALIC GEOMETRIC MORPHISMS

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ABSTRACT. In [Hyland et.al. 1980] Hyland, Johnstone and Pitts introduced the notion of *tripos* for the purpose of organizing the construction of realizability toposes in a way that generalizes the construction of localic toposes from complete Heyting algebras. In [Pitts 1999] one finds a generalization of this notion eliminating an unnecessary assumption of [Hyland et.al. 1980].

The aim of this paper is to characterize triposes over a base topos \mathcal{S} in terms of so-called *constant objects* functors from \mathcal{S} to some elementary topos. Our characterization is slightly different from the one in Pitts’s PhD Thesis [Pitts 1981] and motivated by the fibered view of geometric morphisms as described in [Streicher 2020]. In particular, we discuss the question whether triposes over **Set** giving rise to equivalent toposes are already equivalent as triposes.

1. Introduction

As described in [Johnstone 1977] localic geometric morphisms to a topos \mathcal{S} are given by functors F from \mathcal{S} to some topos \mathcal{E} such that

- (1) F preserves finite limits,
- (2) every objects $A \in \mathcal{E}$ appears as subquotient of some FI and
- (3) F has a right adjoint.

In the appendix of [Streicher 2020] one finds a proof of M. Jibladze’s Theorem [Jibladze 1988] saying that fibered toposes over \mathcal{S} having internal sums correspond to finite limit preserving functors from \mathcal{S} to some topos. In particular, a finite limit preserving functor $F : \mathcal{S} \rightarrow \mathcal{E}$ gives rise to the fibration $P_F = F^*P_{\mathcal{E}}$ over \mathcal{S} obtained by change of base along F from the fundamental (“codomain”) fibration $P_{\mathcal{E}} = \text{cod} : \mathcal{E}^{\mathbf{2}} \rightarrow \mathcal{E}$ (where $\mathbf{2}$ is the small category $0 \rightarrow 1$ corresponding to the ordinal 2). But every fibered topos $P : \mathcal{X} \rightarrow \mathcal{S}$ with internal sums is equivalent to P_{Δ} where \mathcal{E} is the fiber of P over 1 and $\Delta : \mathcal{S} \rightarrow \mathcal{E}$ sends $I \in \mathcal{S}$ to $\Delta(I) = \coprod_I 1_I$ in \mathcal{E} .

Moreover, as also shown in [Streicher 2020] for terminal object preserving $F : \mathcal{S} \rightarrow \mathcal{E}$ the fibration P_F is locally small iff F has a right adjoint. Thus, as observed by J. Bénabou already in his 1974 Montreal lectures inverse image parts of geometric morphisms correspond to terminal object preserving functors F between toposes such that the fibration P_F has internal sums and is locally small.

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Moreover, as also observed in [Streicher 2020] for a finite limit preserving functor $F : \mathcal{S} \rightarrow \mathcal{E}$ between toposes condition (2) is equivalent to the requirement that every map $a : A \rightarrow FI$ in \mathcal{E} fits into a commuting diagram

$$\begin{array}{ccc} C & \xrightarrow{e} & A \\ m \downarrow & & \downarrow a \\ FJ & \xrightarrow{Fu} & FI \end{array}$$

where e is epic and m is monic. Obviously, this condition entails (2) instantiating I by a terminal object. For the reverse direction choose $m : C \rightarrow FJ$ and $e : C \rightarrow A$ (which exist by condition (2)) and observe that

$$\begin{array}{ccccc} & & C & \xrightarrow{e} & A \\ & \nearrow m & \downarrow \langle m, ae \rangle & & \downarrow a \\ FJ & \xleftarrow{F\pi_1} & F(J \times I) & \xrightarrow{F\pi_2} & FI \end{array}$$

using the assumption that F preserves finite limits and thus finite products. Thus, condition (2) amounts to the requirement that every object of \mathcal{E} can be covered by a(n internal) sum of subterminals (in the appropriate fibrational sense!). As observed in [Streicher 2020] under assumption (3) this is equivalent to the requirement that g in

$$\begin{array}{ccc} G & \xrightarrow{\quad} & 1_{\mathcal{E}} \\ g \downarrow & \lrcorner & \downarrow \top_{\mathcal{E}} \\ FU\Omega_{\mathcal{E}} & \xrightarrow{\varepsilon_{\Omega_{\mathcal{E}}}} & \Omega_{\mathcal{E}} \end{array}$$

is a generating family for the fibration P_F (where U is right adjoint to F).

ve notion of [Hyland et.al. 1980] as “traditional tripos” as in the following

1.1. DEFINITION. *A tripos over a topos \mathcal{S} is a finite limit preserving functor F from \mathcal{S} to a topos \mathcal{E} such that every $A \in \mathcal{E}$ appears as subquotient of FI for some $I \in \mathcal{S}$.*

A tripos is called traditional iff $F^\text{Sub}_{\mathcal{E}}$ admits a generic family, i.e. a mono $t : T \rightarrow$*

$F\Sigma$ such that every mono $m : P \rightarrow FI$ fits into a pullback diagram

$$\begin{array}{ccc} P & \longrightarrow & T \\ \downarrow m & \lrcorner & \downarrow t \\ FI & \xrightarrow{Fp} & F\Sigma \end{array}$$

for some (typically not unique) $p : I \rightarrow \Sigma$.

With a tripos $F : \mathcal{S} \rightarrow \mathcal{E}$ in the sense of Def. 1.1 one may associate the fibered poset $\mathcal{P}_F = F^*\mathbf{Sub}_{\mathcal{E}}$ which is a first order hyperdoctrine, i.e. a fibration of Heyting algebras with internal sums and products, since $\mathbf{Sub}_{\mathcal{E}}$ has and change of base along the finite limit preserving functor F preserves the required properties (see [Streicher 2020]). Next we show that \mathcal{P}_F is actually a fibered poset which allows one to interpret higher order (intuitionistic) logic.

1.2. LEMMA. *Let $F : \mathcal{S} \rightarrow \mathcal{E}$ be a tripos. Then for every object I in \mathcal{S} there is an object $P(I)$ in \mathcal{S} and $\varepsilon_I \in \mathcal{P}_F(I \times P(I))$ such that for all J in \mathcal{S} and $\rho \in \mathcal{P}_F(I \times J)$ the **comprehension principle***

$$\forall j \in J. \exists p \in P(I). \forall i \in I. \rho(i, j) \leftrightarrow i \varepsilon_I p$$

holds in the internal logic of \mathcal{P}_F .

PROOF. Let I be an object of \mathcal{S} . Then since \mathcal{P} is a tripos there is an object $P(I)$ in \mathcal{S} such that $\mathcal{P}(FI)$ appears as subquotient of $F(P(I))$, i.e. there is a subobject $m_I : C_I \rightarrow F(P(I))$ such that there exists an epi $e_I : C_I \rightarrow \mathcal{P}(FI)$. Consider

$$\begin{array}{ccccc} \varepsilon_I & \xlongequal{\quad} & \varepsilon_I & \longrightarrow & \varepsilon_{FI} \\ \downarrow & & \downarrow & \lrcorner & \downarrow \\ FI \times F(P(I)) & \xleftarrow{FI \times m_I} & FI \times C_I & \xrightarrow{FI \times e_I} & FI \times \mathcal{P}(FI) \end{array}$$

giving rise to a subobject ε_I of $F(I \times P(I)) \cong FI \times F(P(I))$. Since the left square in the above diagram is a pullback we have $(FI \times m_I)^*\varepsilon_I = (FI \times e_I)^*\varepsilon_{FI}$.

Suppose $\rho : R \rightarrow F(I \times J) \cong FI \times FJ$. Then

$$\begin{array}{ccc} R & \longrightarrow & \varepsilon_{FI} \\ \downarrow \rho & \lrcorner & \downarrow \\ FI \times FJ & \xrightarrow{FI \times r} & FI \times \mathcal{P}(FI) \end{array}$$

for a unique $r : FJ \rightarrow \mathcal{P}(FI)$. Consider the pullback

$$\begin{array}{ccc}
 C & \xrightarrow{\tilde{r}} & C_I \\
 \downarrow e & \lrcorner & \downarrow e_I \\
 FJ & \xrightarrow{r} & \mathcal{P}(FI)
 \end{array}$$

where e is epic since in a topos epis are stable under arbitrary pullbacks. Thus, we have

$$\begin{aligned}
 (FI \times e)^* \rho &= (FI \times e)^*(FI \times r)^* \in_{FI} \cong (FI \times er)^* \in_{FI} = \\
 &= (FI \times e_I \tilde{r})^* \in_{FI} \cong (FI \times \tilde{r})^*(FI \times e_I)^* \in_{FI} = \\
 &= (FI \times \tilde{r})^*(FI \times m_I)^* \varepsilon_I \cong \\
 &\cong (FI \times m_I \tilde{r})^* \varepsilon_I
 \end{aligned}$$

from which it readily follows that

$$\forall j \in J. \exists p \in P(I). \forall i \in I. \rho(i, j) \leftrightarrow i \varepsilon_I p$$

holds in the internal logic of \mathcal{P}_F . ■

From Lemma 1.2 and the results of [Pitts 1999] it follows that fibered posets of the form $F^* \mathbf{Sub}_{\mathcal{E}}$ for some tripos $F : \mathcal{S} \rightarrow \mathcal{E}$ may up to equivalence be characterized as Heyting algebras \mathcal{P} fibered over \mathcal{S} with internal sums \exists and internal products \forall such that for every object I in \mathcal{S} there is an ε_I in $\mathcal{P}(I \times P(I))$ for some $P(I)$ in \mathcal{S} such that the *comprehension* principle

$$\forall j \in J. \exists p \in P(I). \forall i \in I. \rho(i, j) \leftrightarrow i \varepsilon_I p$$

holds in the internal logic of \mathcal{P} for all $\rho \in \mathcal{P}(I \times J)$.

Notice that a tripos $F : \mathcal{S} \rightarrow \mathcal{E}$ is traditional iff the corresponding fibered poset \mathcal{P} is a tripos in the sense of [Hyland et.al. 1980], i.e. there exists a $T \in \mathcal{P}(\Sigma)$ from which all $P \in \mathcal{P}(I)$ can be obtained by reindexing along some map $p : I \rightarrow \Sigma$. Notice that these maps are all unique if F has a right adjoint U and $\Sigma = U\Omega_{\mathcal{E}}$. Finally, we notice that F is the inverse image part of a localic geometric morphism to \mathcal{S} , i.e. F has a right adjoint, iff $F^* \mathbf{Sub}_{\mathcal{E}}$ is locally small iff P_F is locally small as discussed in [Streicher 2020].

2. Constant objects functors are not unique

For arbitrary base toposes \mathcal{S} triposes $F, G : \mathcal{S} \rightarrow \mathcal{E}$ need not be equivalent since if \mathcal{S} is $\mathbf{Sh}(X)$ and \mathcal{E} is $\mathbf{Sh}(Y)$ for some sober spaces X and Y then there are at least as many triposes $\mathcal{S} \rightarrow \mathcal{E}$ (up to equivalence) as there are continuous maps from Y to X . But even if \mathcal{S} is \mathbf{Set} there are in general many non-equivalent triposes over \mathbf{Set} giving rise to the same topos as shown by the following simple counterexample

2.1. THEOREM. *For every natural number $n > 0$ the functor $F_n : \mathbf{Set} \rightarrow \mathbf{Set} : I \mapsto I^n$ is a tripos. The triposes $F_n^* \mathbf{Sub}_{\mathbf{Set}}$ and $F_m^* \mathbf{Sub}_{\mathbf{Set}}$ are equivalent if and only if $n = m$.*

PROOF. Obviously, the F_n preserve finite limits since they have right adjoints and every $I \in \mathbf{Set}$ appears as split subobject of $F_n(I)$. Thus, all F_n are triposes but $F_n^* \mathbf{Sub}_{\mathbf{Set}}$ and $F_m^* \mathbf{Sub}_{\mathbf{Set}}$ are equivalent as triposes if and only if $n = m$ since the latter is equivalent to $2^n = 2^m$ which in turn is equivalent to $F_n^* \mathbf{Sub}_{\mathbf{Set}}(2) \simeq F_m^* \mathbf{Sub}_{\mathbf{Set}}(2)$. ■

Notice, however, that F_n is a traditional tripos if and only if $n = 1$. Thus, it may still be the case that there exist traditional triposes $F, G : \mathcal{S} \rightarrow \mathcal{E}$ which are not equivalent as triposes. Unfortunately, we have not been able so far to find examples of non-equivalent *traditional* triposes \mathcal{P}_1 and \mathcal{P}_2 over \mathbf{Set} such that the ensuing toposes $\mathbf{Set}[\mathcal{P}_1]$ and $\mathbf{Set}[\mathcal{P}_2]$ are equivalent. However, though a bit annoying, we can't find this as a major problem since our weak notion of tripos is conceptually more adequate than the traditional one for the following two reasons. Firstly, because it characterizes those fibered posets which give rise to elementary toposes when “adding subquotients”. Secondly, because from a logical point of view adding the above mentioned comprehension principles to first order posetal hyperdoctrines appears as much more natural than requiring that they are witnessed by Skolem functions in the base \mathcal{S} , i.e. requiring for all $\rho \in \mathcal{P}(I \times J)$ the existence of a function $r : J \rightarrow P(I)$ such that

$$\forall j \in J. \forall i \in I. \rho(i, j) \leftrightarrow i \varepsilon_I r(j)$$

holds in the logic of \mathcal{P} . At the end of [Pitts 1999] the author expresses a similar view in a slightly more cautious way.

Finally, we observe that triposes over \mathbf{Set} may give rise to non-localic Grothendieck toposes. Let \mathcal{E} be the topos of reflexive graphs, i.e. presheaves over the 3 element monoid $\Delta([1], [1])$ of monotone endomaps of the ordinal 2. As observed by Lawvere the global elements functor $\Gamma : \mathcal{E} \rightarrow \mathbf{Set}$ fits into a sequence of adjoints $\Pi \dashv \Delta \dashv \Gamma \dashv \nabla : \mathbf{Set} \leftrightarrow \mathcal{E}$. The rightmost functor ∇ preserves all limits since it has a left adjoint. Subobjects of objects of the form $\nabla(I)$ are up to isomorphism precisely those reflexive graphs where between two nodes there is at most one edge (i.e. directed graphs as traditionally considered in combinatorics!). But since any reflexive graph can be covered by such a traditional directed graph every object of \mathcal{E} appears as subquotient of some $\nabla(I)$ for which reason ∇ is a tripos over \mathbf{Set} though it is not the inverse image part of a geometric morphism.

3. Regular triposes

It is well known that a morphism $e : Y \rightarrow X$ in an elementary topos \mathcal{E} is epic iff the pullback functor $e^* : \mathbf{Sub}_{\mathcal{E}}(X) \rightarrow \mathbf{Sub}_{\mathcal{E}}(Y)$ reflects maximal subobjects, i.e. a mono $m : P \rightarrow X$ in \mathcal{E} is an iso already if e^*m is an iso. Recall that a poset fibered over a regular category is a *prestack* (w.r.t. the regular cover topology) iff for all regular epis e reindexing along it (preserves and) reflects the order. Thus, for a tripos $F : \mathcal{S} \rightarrow \mathcal{E}$ the fibered poset $F^* \mathbf{Sub}_{\mathcal{E}}$ is a prestack iff F preserves (regular) epis.

This observation strongly suggests to require that triposes $F : \mathcal{S} \rightarrow \mathcal{E}$ also preserve epis since it vacuously holds when \mathcal{S} is **Set** (since in **Set** all epis are split as ensured by the axiom of choice!) and, moreover, by Lemma 6.1 (“Pitts’s Iteration Theorem”) of [Pitts 1981] triposes preserving epis are closed under composition.

3.1. DEFINITION. *A tripos $F : \mathcal{S} \rightarrow \mathcal{E}$ is called regular iff F preserves epis.*

Recall that a functor between regular categories is called *regular* iff it preserves finite limits and regular epis. Thus, regular triposes are regular functors $F : \mathcal{S} \rightarrow \mathcal{E}$ between toposes such that every $A \in \mathcal{E}$ appears as subquotient of FI for some $I \in \mathcal{S}$.

From Prop. 3.14 of [Pitts 1981] it follows that a traditional tripos $F : \mathcal{S} \rightarrow \mathcal{E}$ is regular iff it has “fibrewise quantification”, i.e. there are maps $\bigvee, \bigwedge : \Omega_{\Sigma}^{\Sigma} \rightarrow \Sigma$ such that $\exists_{Fu}(Fp)^*t$ and $\forall_{Fu}(Fp)^*t$ appear as pullbacks of $t : T \rightarrow F\Sigma$ along $F(\lambda i:I. \bigvee\{p(j) \mid u(j) = i\})$ and $F(\lambda i:I. \bigwedge\{p(j) \mid u(j) = i\})$, respectively, for all $u : J \rightarrow I$ and $p : J \rightarrow \Sigma$.

3.2. THEOREM. *Let $F_1 : \mathcal{S} \rightarrow \mathcal{E}_1$ and $F_2 : \mathcal{S} \rightarrow \mathcal{E}_2$ be triposes and $H : F_1 \rightarrow F_2$, i.e. $H : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ with $F_2 = HF_1$. Then H is a tripos iff H preserves finite limits and H is a regular tripos iff H is a regular functor.*

PROOF. The forward directions are trivial. For the backwards directions suppose $A \in \mathcal{E}_2$. Then, since F_2 is a tripos there exists a subobject $m : C \rightarrow F_2I$ and an epi $e : C \twoheadrightarrow A$. Since $F_2 = HF_1$ we have $m : C \rightarrow H(F_1I)$ and $e : C \twoheadrightarrow A$. Thus, we have shown that H validates the second condition required for a tripos. ■

The previous theorem for regular triposes $F_1 : \mathcal{S} \rightarrow \mathcal{E}_1$ and $F_2 : \mathcal{S} \rightarrow \mathcal{E}_2$ suggests that the right notion of morphism from F_1 to F_2 is a functor $H : F_1 \rightarrow F_2$ such that $H : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ is regular since for this definition morphisms to a regular tripos $F : \mathcal{S} \rightarrow \mathcal{E}$ coincide with regular triposes over \mathcal{E} .

In the subsequent Theorem 3.4 we will show that morphisms between traditional regular triposes are precisely the traditional regular triposes. But for this purpose we need the following lemma characterizing traditional regular triposes among regular triposes in terms of a condition which at first sight looks weaker than the one given in Def. 1.1.

3.3. LEMMA. *Let $F : \mathcal{S} \rightarrow \mathcal{E}$ be a regular tripos and $t : T \rightarrow F\Sigma$ be weakly generic for $F^*\text{Sub}_{\mathcal{E}}$, i.e. every mono $m : P \rightarrow FI$ fits into a diagram*

$$\begin{array}{ccccc}
 P & \longleftarrow & \bullet & \longrightarrow & T \\
 \downarrow m & & \downarrow & & \downarrow t \\
 FI & \xleftarrow{Fe} & FJ & \xrightarrow{Fp} & F\Sigma
 \end{array}$$

where both squares are pullbacks and $e : J \rightarrow I$ is epic, then F is a traditional tripos.

PROOF. Suppose $t : T \rightarrow F\Sigma$ is a weakly generic family for $F^*\mathbf{Sub}_{\mathcal{E}}$. Let $E = \{(u, U) \in \Sigma \times P(\Sigma) \mid u \in U\}$ and $p : E \rightarrow \Sigma$ and $q : E \rightarrow P(\Sigma)$ the respective projection maps. We will show that $\exists_q p^* t$ is a generic family for $F^*\mathbf{Sub}_{\mathcal{E}}$.

For this purpose suppose $m \in \mathbf{Sub}_{\mathcal{E}}(FI)$. By assumption there are $e : J \rightarrow I$ and $f : J \rightarrow \Sigma$ such that $e^* m \cong f^* t$. Since $F^*\mathbf{Sub}_{\mathcal{E}}$ is a prestack w.r.t. the regular cover topology we have $m \cong \exists_e e^* m \cong \exists_e f^* t$. Let $g : I \rightarrow P(\Sigma)$ with $g(i) = \{f(j) \mid e(j) = i\}$. Obviously, the map $\langle f, ge \rangle$ factors through $\langle p, q \rangle$ since $f(j) \in g(e(j))$. Consider the following diagram

$$\begin{array}{ccccc}
 & & J & & \\
 & & \downarrow h & \searrow e & \\
 & & R & \xrightarrow{\quad} & I \\
 & \swarrow f & \downarrow r_1 & \lrcorner r_2 & \downarrow g \\
 \Sigma & \xleftarrow{p} & E & \xrightarrow{q} & P(\Sigma)
 \end{array}$$

where $R = \{(u, U, i) \in \Sigma \times P(\Sigma) \times I \mid u \in U = g(i)\}$ with r_1 and r_2 the respective projections and $h(j) = (f(j), g(e(j)), e(j))$. Notice that h is onto since $g(i) = \{f(j) \mid e(j) = i\}$. We have

$$\begin{array}{ll}
 g^* \exists_q p^* t \cong \exists_{r_2} r_1^* p^* t & \text{by Chevalley condition} \\
 \cong \exists_{r_2} \exists_h h^* r_1^* p^* t & \text{since } h \text{ is epic} \\
 \cong \exists_e f^* t & \text{since } e = r_2 h \text{ and } f = p r_1 h \\
 \cong m &
 \end{array}$$

as desired. ■

3.4. THEOREM. *Let $F_1 : \mathcal{S} \rightarrow \mathcal{E}_1$ be a traditional regular tripos and $H : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ a regular functor between toposes. Then H is a traditional regular tripos if and only if $F_2 = HF_1$ is a traditional regular tripos*

PROOF. The forward direction is Pitts's Iteration Theorem.

For the backward direction suppose that $H : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ is a regular functor such that $F_2 = HF_1$ is a traditional regular tripos. By Theorem 3.2 it is immediate that H is a regular tripos, too. Since F_2 has been assumed to be a traditional tripos there is a $t : T \rightarrow F_2 \Sigma$ generic for $F_2^* \mathbf{Sub}_{\mathcal{E}_2}$. For showing that H is a traditional tripos it suffices by Lemma 3.3 to show that $t : T \rightarrow HF_1 \Sigma$ is weakly generic for $H^* \mathbf{Sub}_{\mathcal{E}_2}$.

Suppose $m : P \rightarrow HA$ for some $A \in \mathcal{E}_1$. Since F_1 is a traditional tripos there exist $n : Q \rightarrow F_1 I$ and $e : Q \rightarrow A$ for some $I \in \mathcal{S}$. Since $F_2 = HF_1$ is a traditional tripos

there exists $p : I \rightarrow \Sigma$ such that $Hn \circ (He)^*m$ arises as pullback of t along HF_1p for some $p : I \rightarrow \Sigma$. Thus we have

$$\begin{array}{ccccc}
 P & \longleftarrow & \bullet & \longrightarrow & T \\
 \downarrow m & & \downarrow (He)^*m & & \downarrow t \\
 HA & \longleftarrow & HQ & & HF_1\Sigma \\
 & \xleftarrow{He} & \downarrow Hn & & \\
 & & HF_1I & \xrightarrow{HF_1p} & HF_1\Sigma
 \end{array}$$

from which it follows that $(He)^*m$ arises as pullback of t along $H(F_1p \circ n) = HF_1p \circ Hn$. Thus, we have

$$\begin{array}{ccccc}
 P & \longleftarrow & \bullet & \longrightarrow & T \\
 \downarrow m & & \downarrow (He)^*m & & \downarrow t \\
 HA & \longleftarrow & HQ & \xrightarrow{H(F_1p \circ n)} & HF_1\Sigma \\
 & \xleftarrow{He} & & &
 \end{array}$$

where both squares are pullbacks as required. \blacksquare

We conclude this section with some observations on the

3.5. PRESERVATION OF ASSEMBLIES BY TRIPOS MORPHISMS. Following [van Oosten 2008] for a tripos $F : \mathcal{S} \rightarrow \mathcal{E}$ one may define *assemblies* as those objects of \mathcal{E} which appear as subobjects of some FI . If $G : \mathcal{S} \rightarrow \mathcal{F}$ is a tripos and $H : F \rightarrow G$ such that $H : \mathcal{E} \rightarrow \mathcal{F}$ preserves finite limits then H preserves assemblies, i.e. sends assemblies w.r.t. F to assemblies w.r.t. G , since $Hm : HP \rightarrow HFI = GI$ whenever $m : P \rightarrow FI$. It follows from the definition of tripos that every object A of \mathcal{E} appears as subquotient of some FI , i.e. we have $A \xleftarrow{e} C \xrightarrow{m} FI$. If $H : F \rightarrow G$ is a regular functor between triposes then $HA \xleftarrow{He} HC \xrightarrow{Hm} HFI = GI$, i.e. H preserves coverings of objects by assemblies in a very strong sense.

4. Relation to Miquel's implicative algebras

In an unpublished note [Miquel 2018] A. Miquel has shown that traditional triposes over **Set** correspond to so called *implicative algebras*

4.1. DEFINITION. An implicative structure is a complete lattice \mathcal{A} together with an operation $\rightarrow : \mathcal{A}^{\text{op}} \times \mathcal{A} \rightarrow \mathcal{A}$ such that $x \rightarrow \bigwedge Y = \bigwedge_{y \in Y} (x \rightarrow y)$ for all $x \in \mathcal{A}$ and $Y \subseteq \mathcal{A}$.

Then $K_{\mathcal{A}} = \bigwedge_{x, y \in \mathcal{A}} x \rightarrow y \rightarrow x$ and $S_{\mathcal{A}} = \bigwedge_{x, y, z \in \mathcal{A}} (x \rightarrow y \rightarrow z) \rightarrow (x \rightarrow y) \rightarrow x \rightarrow z$ are elements of \mathcal{A} .

A separator in an implicative structure $(\mathcal{A}, \rightarrow)$ is an upward closed subset \mathcal{S} of \mathcal{A} such that $K_{\mathcal{A}}, S_{\mathcal{A}} \in \mathcal{S}$ and \mathcal{S} is closed under modus ponens, i.e. $b \in \mathcal{S}$ whenever $a \in \mathcal{S}$ and $a \rightarrow b \in \mathcal{S}$.

An implicative algebra is a triple $(\mathcal{A}, \rightarrow, \mathcal{S})$ such that $(\mathcal{A}, \rightarrow)$ is an implicative structure and \mathcal{S} is a separator in $(\mathcal{A}, \rightarrow)$.

With every implicative algebra \mathcal{A} one associates a **Set**-based tripos $\mathcal{P}^{\mathcal{A}}$ where $\mathcal{P}^{\mathcal{A}}(I)$ is the preorder \vdash_I on \mathcal{A}^I defined as

$$\varphi \vdash_I \psi \quad \text{iff} \quad \bigwedge_{i \in I} (\varphi_i \rightarrow \psi_i) \in \mathcal{S}$$

and reindexing is given by precomposition.

In his unpublished note [Miquel 2018] from spring 2018 A. Miquel has shown that every traditional regular tripos over **Set** is equivalent to $\mathcal{P}^{\mathcal{A}}$ for some implicative algebra \mathcal{A} .

For $i=1,2$ let $F_i : \mathbf{Set} \rightarrow \mathcal{E}_i$ be the constant objects functor for the regular tripos induced by an implicative algebra \mathcal{A}_i in **Set**, i.e. $\mathcal{E}_i = \mathbf{Set}[\mathcal{P}^{\mathcal{A}_i}]$. Due to the remark in subsection 3.5 regular functors $G : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ with $F_2 = GF_1$ correspond to cartesian functors $g : F_1^* \mathbf{Sub}_{\mathcal{E}_1} \rightarrow F_2^* \mathbf{Sub}_{\mathcal{E}_2}$ preserving regular logic, i.e. finite limits and existential quantification. Obviously, such g are uniquely determined by $h = g_{\mathcal{A}_1}(\text{id}_{\mathcal{A}_1}) : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ since $g_I(\varphi : I \rightarrow \mathcal{A}_1) = h \circ \varphi$. This suggests to define a morphism of implicative algebras from \mathcal{A}_1 to \mathcal{A}_2 as a function $h : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ such that the cartesian functor $g : F_1^* \mathbf{Sub}_{\mathcal{E}_1} \rightarrow F_2^* \mathbf{Sub}_{\mathcal{E}_2}$ given by $g_I(\varphi : I \rightarrow \mathcal{A}_1) = h \circ \varphi$ preserves regular logic, i.e. finite limits and existential quantification.

Unfortunately, Miquel's result from [Miquel 2018] does not extend to arbitrary base toposes. The reason is that for a traditional regular tripos $F : \mathcal{S} \rightarrow \mathcal{E}$ there need not exist a subobject S of Σ such that

- (1) its characteristic map $\chi_S : \Sigma \rightarrow \Omega_{\mathcal{S}}$ induces by postcomposition a cartesian functor $\gamma_S : F^* \mathbf{Sub}_{\mathcal{E}} \rightarrow \mathbf{Sub}_{\mathcal{S}}$ preserving finite meets in each fiber and
- (2) $u : 1 \rightarrow \Sigma$ factors through S iff $(Fu)^*t$ is isomorphic to id_{F1} .

Notice that the first condition means that $u^*S \leq v^*S$ whenever $F(u)^*t \leq F(v)^*t$ and that $t \in S$ and $\forall u, v : \Sigma. u \wedge v \in S \leftrightarrow (u \in S \wedge v \in S)$ hold in the internal logic of \mathcal{S} . For base toposes \mathcal{S} which are not well-pointed such S need neither exist nor be unique (for the latter see Example 4.12.12 of [Frey 2013] for a counterexample¹).

¹Take for \mathcal{S} the Sierpiński topos $\mathbf{Set}^{2^{\text{op}}}$ and for F the functor $\text{Id}_{\mathcal{S}}$. Then there are two possible choices for S , namely $\top : 1 \rightarrow \Omega_{\mathcal{S}}$ and the subobject S of $\Omega_{\mathcal{S}}$ with $S_0 = \Omega_0$ and $S_1 = \{\top\}$. In the first case the corresponding γ_S is $\text{id}_{\Omega_{\mathcal{S}}}$ and in the second case it sends a subobject P of A in \mathcal{S} to the subobject $\gamma_S(P)$ of A with $\gamma_S(P)_0 = A_0$ and $\gamma_S(P)_1 = P_1$.

The related stronger condition that $u : I \rightarrow \Sigma$ factors through S iff $F(u)^*t$ is isomorphic to id_{FI} is known as “definability of truth”, i.e. that the full subfibration of $F^*\mathbf{Sub}_{\mathcal{E}}$ on true predicates is definable in the sense of Bénabou (see section 12 of [Streicher 2020]). This stronger condition, however, amounts to the requirement that the fibration $F^*\mathbf{Sub}_{\mathcal{E}}$ is locally small, i.e. equivalent to the externalization of a complete Heyting algebra internal to \mathcal{S} , which in turn is equivalent to the requirement that F is the inverse image part of a localic geometric morphism.

5. Summary and Conclusion

We have shown in which sense (generalized) triposes in the sense of [Pitts 1981] may be understood as a generalizations of localic geometric morphisms. The traditional triposes of [Hyland et.al. 1980] can be characterized as those triposes $F : \mathcal{S} \rightarrow \mathcal{E}$ for which the fibered poset $F^*\mathbf{Sub}_{\mathcal{E}}$ admits a generic family $t : T \rightarrow F\Sigma$.

We have defined regular triposes as triposes $F : \mathcal{S} \rightarrow \mathcal{E}$ where F preserves epis, i.e. $F^*\mathbf{Sub}_{\mathcal{E}}$ is a prestack. As opposed to ordinary triposes regular triposes are known to be closed under composition, i.e. are closed under iteration. A further advantage of regular triposes is that for a regular tripos $F : \mathcal{S} \rightarrow \mathcal{E}$ regular triposes over \mathcal{E} correspond to morphisms of regular triposes from F to some regular tripos $G : \mathcal{S} \rightarrow \mathcal{F}$, i.e. $H : F \rightarrow G$ such that $H : \mathcal{E} \rightarrow \mathcal{F}$ is a regular functor. Somewhat surprisingly, an analogous result holds for traditional regular triposes as well.

Finally, we have recalled a theorem due to A. Miquel characterizing traditional regular triposes in terms of implicative algebras generalizing the notion of complete Heyting algebra and identified a notion of morphism between implicative algebras corresponding to regular morphisms of triposes over **Set**.

We think that the more general notion of tripos as introduced in [Pitts 1999] is more natural since it corresponds to the class of first order posetal hyperdoctrines which give rise to toposes by “adding subquotients”. Moreover, the form of comprehension principle characterising them is more natural than the Skolemized form postulated as an axiom in the definition of traditional triposes.

But restricting to regular triposes seems to be a good idea since the condition is most natural from the point of view of fibered categories and, moreover, allows one to identify regular tripos morphisms to $F : \mathcal{S} \rightarrow \mathcal{E}$ with regular triposes over \mathcal{E} as shown in Theorem 3.2.

We have shown that triposes $F, G : \mathbf{Set} \rightarrow \mathcal{E}$ need not be equivalent. But we do not know whether such F and G are necessarily equivalent under the stronger assumption that both F and G are traditional triposes. There is no conceptual reason why this should hold in general but, alas, we have not been able to find a counterexample so far.

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References

- J. Bénabou *Logique Catégorique* lecture notes of a course at Univ. Montreal (1974)
- J. Bénabou *Des Catégories Fibrées* handwritten lecture notes by J.-R. Roisin of a course at Univ. Louvain-la-Neuve (1980)
- J. Frey *A fibrational study of realizability toposes* Thesis Univ. Paris 7 (2013) [arXiv:1403.3672](#)
- M. Jibladze *Geometric Morphisms and fibered Toposes* In *Categorical topology and its relation to analysis, algebra and combinatorics* (Prague, 1988), pages 10–18. World Scientific Publ. (1989)
- M. Hyland, P. Johnstone, A. Pitts *Tripos Theory* Math. Proc. Cambridge Philos. Soc. 88(2) (1980)
- P. T. Johnstone *Topos Theory* Academic Press (1977)
- A. Miquel *Implicative Algebras II: Completeness w.r.t. Set-based Triposes* unpublished note (2018)
- A. Miquel *Implicative algebras: A new foundation for realizability and forcing*. Mathematical Structures in Computer Science, 30(5), 458-510.
- A. Pitts *The Theory of Triposes* Thesis Univ. Cambridge (1981)
- A. Pitts *Tripos Theory in Retrospect* Math. Struct. in Computer Science 12(3) (2002)
- T. Streicher *Fibered Categories à la Jean Bénabou* (2020) [arXiv:1801.02927](#)
- J. van Oosten *Realizability. An Introduction to its Categorical Side* Elsevier (2008)

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