Fibered Categories
à la Jean Bénabou

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The notion of *fibered category* was introduced by A. Grothendieck for purely geometric reasons. The "logical" aspect of fibered categories and, in particular, their relevance for *category theory over an arbitrary base category with pullbacks* has been investigated and worked out in detail by Jean Bénabou. The aim of these notes is to explain Bénabou's approach to fibered categories which is mostly unpublished but intrinsic to most fields of category theory, in particular to topos theory and categorical logic.

There is no claim for originality by the author of these notes. On the contrary I want to express my gratitude to Jean Bénabou for his lectures and many personal tutorials where he explained to me various aspects of his work on fibered categories. I also want to thank J.-R. Roisin for making me available his handwritten notes [Ben2] of *Des Catégories Fibrées*, a course by Jean Bénabou given at the University of Louvain-la-Neuve back in 1980.

The current notes are based essentially on [Ben2] and quite a few other insights of J. Bénabou that I learnt from him personally. The last four sections are based on results of J.-L. Moens's Thése [Moe] from 1982 which itself was strongly influenced by [Ben2].

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1 Motivation and Examples

If $\mathbf{C}$ is a category then a functor

$$F : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$$

also called a "presheaf over $\mathbf{C}$" is most naturally considered as a "set varying over $\mathbf{C}$". Of course, one may consider also contravariant functors on $\mathbf{C}$ taking their values not in $\mathbf{Set}$ but in some big category of structures like $\mathbf{Grp}$, $\mathbf{Ab}$, $\mathbf{Rng}$, $\mathbf{Sp}$ etc. Typically, a presheaf $G : \mathbf{C}^{\mathrm{op}} \to \mathbf{Grp}$ of groups appears as a group object in $\widehat{\mathbf{C}} = \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ which is a topos if the category $\mathbf{C}$ is small.

More generally, one may consider "presheaves of categories"

$$\mathcal{H} : \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$$

which notion will soon be axiomatized and generalised to our central notion of *fibered category*. But before we consider some examples that (hopefully) will provide some intuition and motivation.

Example 1.1 *Let $\mathbf{C}$ be the category of monoids and monoid homomorphisms. With every monoid M one may associate the category*

$$\mathcal{H}(M) = \mathbf{Set}^{M^{\mathrm{op}}}$$

of right actions of M on some set and with every monoid homomorphism $h : N \to M$ one may associate the functor

$$\mathcal{H}(h) = h^* = \mathbf{Set}^{h^{\mathrm{op}}} : \mathbf{Set}^{M^{\mathrm{op}}} \to \mathbf{Set}^{N^{\mathrm{op}}}$$

where $h^*(X, \alpha) : X \times N \to X : (x, b) \mapsto \alpha(x, h(b))$. $\Diamond$

Example 1.2 *Of course, Example 1.1 can be generalised by taking for* $\mathbf{C}$ *some subcategory of the category of (small) categories and instead of* $\mathbf{Set}$ *some other big category $\mathcal{K}$ (e.g. $\mathcal{K} = \mathbf{Ab}$ and $\mathbf{C} = \mathbf{Cat}$).* $\Diamond$

Example 1.3 *Let $\mathbf{E}$ be an elementary topos (see e.g. [Jo77]). Then*

$$\mathbf{E}(-, \Omega) : \mathbf{E}^{\mathrm{op}} \to \mathbf{Ha}$$

is a contravariant functor from $\mathbf{E}$ to the category $\mathbf{Ha}$ of Heyting algebras and their morphisms. $\Diamond$

Example 1.4 *Let* $\mathbf{C}$ *be the category* $\mathbf{CRng}$ *of commutative rings with* 1. *Then we may consider the functor*

$$\mathcal{H} : \mathbf{CRng}^{\mathrm{op}} \to \mathbf{Cat}$$

where $\mathcal{H}(R)$ *is the category of* R*–modules and for a homomorphism* $h : R' \to R$ *the functor* $\mathcal{H}(h)$ *performs "restriction of scalars", i.e.* $\mathcal{H}(h)(M)$ *is the* R'*–module with the same addition as* M *and scalar multiplication given by* $r \cdot x = h(r) \cdot_M x$. ◊

Example 1.5 *Consider the following instance of Example 1.2. Let* $\mathbf{C} = \mathbf{Set}$ *(where sets are considered as small discrete categories) and* $\mathcal{K} = \mathbf{X}$ *be some (typically not small) category. Then we have*

$$\mathrm{Fam}(\mathbf{X}) : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Cat}$$

where $\mathrm{Fam}(\mathbf{X})(I) = \mathbf{X}^I$ *and*

$$\mathrm{Fam}(\mathbf{X})(u) = \mathbf{X}^u : \mathbf{X}^I \to \mathbf{X}^J$$

for $u : J \to I$ *in* $\mathbf{Set}$.
This example is paradigmatic *for* Fibered Category Theory à la Bénabou *as it allows categories over* $\mathbf{Set}$ *to be considered as fibrations over* $\mathbf{Set}$. *Replacing* $\mathbf{Set}$ *by more general categories* $\mathbf{B}$ *as e.g. toposes or even just categories with pullbacks one may develop a fair amount of* category theory over base $\mathbf{B}$ *!*

Example 1.6 *For a category* $\mathbf{B}$ *with pullbacks we may consider* $\mathcal{H} : \mathbf{B}^{\mathrm{op}} \to \mathbf{Cat}$ *sending* $I \in \mathbf{B}$ *to* $\mathcal{H}(I) = \mathbf{B}/I$ *and* $u : J \to I$ *in* $\mathbf{B}$ *to the pullback functor* $\mathcal{H}(u) = u^* : \mathbf{B}/I \to \mathbf{B}/J$ *which is right adjoint to* $\Sigma_u \equiv u \circ (-)$ *(postcomposition with* u*).*

Notice that this is an example only cum grano salis *as* $u^* : \mathbf{B}/I \to \mathbf{B}/J$ *involves some choice of pullbacks and, accordingly, in general we do not have* $\mathcal{H}(uv) = \mathcal{H}(v) \circ \mathcal{H}(u)$ *but only* $\mathcal{H}(uv) \cong \mathcal{H}(v) \circ \mathcal{H}(u)$ *where the components of the natural isomorphism are given by the respective mediating arrows. Such "functors" preserving composition (and identity) only up to isomorphism are usually called* pseudo–functors. ◊

We definitely do *not* want to exclude the situation of Example 1.6 as it allows one to consider the base category $\mathbf{B}$ as "fibered over itself". Therefore, one might feel forced to accept pseudo–functors and the ensuing bureaucratic handling of "canonical isomorphisms". However, as we will show immediately one may replace pseudo–functors $\mathcal{H} : \mathbf{B}^{\mathrm{op}} \to \mathbf{Cat}$ by fibrations $P : \mathbf{X} \to \mathbf{B}$ where this bureaucracy will turn out as luckily hidden from us.

To motivate the definition of a fibration let us consider a functor $\mathcal{H} : \mathbf{B}^{\mathrm{op}} \to \mathbf{Cat}$ from which we will construct the "fibration" $P = \int \mathcal{H} : \mathbf{X} \to \mathbf{B}$. The objects of $\mathbf{X}$ are pairs (I, X) where $I \in \mathbf{B}$ and $X \in \mathcal{H}(I)$. A morphism in $\mathbf{X}$ from (J, Y) to (I, X) is a pair (u, α) where $u : J \to I$ in $\mathbf{B}$ and $\alpha : Y \to \mathcal{H}(u)(X)$ in $\mathcal{H}(J)$. Composition in $\mathbf{X}$ is defined as follows: for maps $(v, \beta) : (K, Z) \to (J, Y)$ and $(u, \alpha) : (J, Y) \to (I, X)$ in $\int \mathcal{H}$ their composition $(u, \alpha) \circ (v, \beta)$ is given by $(u \circ v, \mathcal{H}(u)(\alpha) \circ \beta)$. It is readily checked that this composition is associative and identities are given by $id_{(I,X)} = (id_I, id_X)$. Let $P = \int \mathcal{H} : \mathbf{X} \to \mathbf{B}$ be the functor sending an object (I, X) in $\mathbf{X}$ to I in $\mathbf{B}$ and a morphism (u, α) in $\mathbf{X}$ to u in $\mathbf{B}$.

Similarly, the pseudo–functor from Example 1.6 may be replaced by the functor $P_{\mathbf{B}} \equiv \partial_1 \equiv \mathsf{cod} : \mathbf{B}^{\mathbf{2}} \to \mathbf{B}$ where $\mathbf{2}$ is the partial order $0 \to 1$, i.e. the ordinal 2. Obviously, $P_{\mathbf{B}}$ sends a commuting square

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "b"'] & A \arrow[d, "a"] \\ J \arrow[r, "u"'] & I \end{tikzcd}$$

to u. Just as we have written ∂_1 for the "codomain" functor cod we will write ∂_0 for the "domain" functor $\mathsf{dom} : \mathbf{B}^{\mathbf{2}} \to \mathbf{B}$. As $P_{\mathbf{B}}$ allows one to consider $\mathbf{B}$ as fibered over itself and this is fundamental for developing category theory over $\mathbf{B}$ we call $P_{\mathbf{B}}$ the *fundamental fibration of* $\mathbf{B}$.

Let $P : \mathbf{X} \to \mathbf{B}$ be a functor as described above. A morphism φ in $\mathbf{X}$ is called *vertical* iff $P(\varphi) = id$. We write $P(I)$ or $\mathbf{X}_I$ for the subcategory of $\mathbf{X}$ which appears as "inverse image of I under P", i.e. which consists of objects X with $P(X) = I$ and morphisms φ with $P(\varphi) = id_I$. If $P = \int \mathcal{H}$ then (u, α) will be called *cartesian* iff α is an isomorphism and if $P = P_{\mathbf{B}}$ then a morphism in $\mathbf{B}^{\mathbf{2}}$ will be called *cartesian* iff the corresponding square is a pullback in $\mathbf{B}$.

2 Basic Definitions

From the examples in the previous section we destill the following definition of fibered category.

Definition 2.1 *Let $P : \mathbf{X} \to \mathbf{B}$ be a functor. A morphism $\varphi : Y \to X$ in $\mathbf{X}$ over $u := P(\varphi)$ is called* cartesian *iff for all $v : K \to J$ in $\mathbf{B}$ and $\theta : Z \to X$ with $P(\theta) = u \circ v$ there exists a unique morphism $\psi : Z \to Y$ with $P(\psi) = v$ and $\theta = \varphi \circ \psi$.*

$$\begin{tikzcd}
Z \arrow[dr, dotted, "\psi"'] \arrow[drr, "\theta"] & & \\
& Y \arrow[r, "\varphi"'] & X
\end{tikzcd}$$

$$\begin{tikzcd}
K \arrow[dr, "v"'] \arrow[drr, "u \circ v"] & & \\
& J \arrow[r, "u"'] & I
\end{tikzcd}$$

A morphism $\alpha : Y \to X$ is called vertical *iff $P(\alpha)$ is an identity morphism in $\mathbf{B}$. For $I \in \mathbf{B}$ we write $\mathbf{X}_I$ or $P(I)$ for the subcategory of $\mathbf{X}$ consisting of those morphism α with $P(\alpha) = id_I$. It is called the* fiber of P over I. $\diamondsuit$

It is straightforward to check that cartesian arrows are closed under composition and that α is an isomorphism in $\mathbf{X}$ iff α is a cartesian morphism over an isomorphism.

Definition 2.2 *A functor $P : \mathbf{X} \to \mathbf{B}$ is called a (Grothendieck)* fibration *or* category fibered over $\mathbf{B}$ *iff for all $u : J \to I$ in $\mathbf{B}$ and $X \in P(I)$ there exists a cartesian arrow $\varphi : Y \to X$ over u called a* cartesian lifting of X along u. $\diamondsuit$

Obviously, the functors $\int \mathcal{H}$ and $P_{\mathbf{B}}$ of the previous section are examples of fibrations and the *ad hoc* notions of "cartesian" as given there coincide with the official ones of Definition 2.2.

Notice that cartesian liftings of $X \in P(I)$ along $u : J \to I$ are unique up to vertical isomorphism: suppose that $\varphi : Y \to X$ and $\psi : Z \to X$ are cartesian over u then there exist vertical arrows $\alpha : Z \to Y$ and $\beta : Y \to Z$ with $\varphi \circ \alpha = \psi$ and $\psi \circ \beta = \varphi$, respectively, from which it follows by cartesianness of φ and ψ that $\beta \circ \alpha = id_Z$ and $\alpha \circ \beta = id_Y$ as $\psi \circ \beta \circ \alpha = \varphi \circ \alpha = \varphi = \varphi \circ id_Y$ and $\varphi \circ \beta \circ \alpha = \psi \circ \alpha = \varphi = \varphi \circ id_Y$.

Definition 2.3 *Let* $P : \mathbf{X} \to \mathbf{B}$ *and* $Q : \mathbf{Y} \to \mathbf{B}$ *be fibrations over* $\mathbf{B}$.
A cartesian *or* fibered *functor from* P *to* Q *is an ordinary functor* $F : \mathbf{X} \to \mathbf{Y}$ *such that*

(1) $Q \circ F = P$ *and*

(2) $F(\varphi)$ *is cartesian w.r.t.* Q *whenever* φ *is cartesian w.r.t.* P.

If F *and* G *are cartesian functors from* P *to* Q *then a* cartesian natural transformation from F to G *is an ordinary natural transformation* $\tau : F \Rightarrow G$ *with* τ_X *vertical for every* $X \in \mathbf{X}$.
The ensuing 2-category will be called $\mathbf{Fib}(\mathbf{B})$. $\Diamond$

Of course, if $\mathbf{B}$ is the terminal category then $\mathbf{Fib}(\mathbf{B})$ is isomorphic to the 2-category $\mathbf{Cat}$.

Remark. What we have called "cartesian" in Definition 2.1 is usually called *hypercartesian* whereas "cartesian" morphisms are defined as follows: a morphism $\varphi : Y \to X$ is called *cartesian* iff for all $\psi : Z \to X$ with $P(\varphi) = P(\psi)$ there is a unique vertical arrow $\alpha : Z \to Y$ with $\varphi \circ \alpha = \psi$. Employing this more liberal notion of "cartesian" one has to strengthen the definition of fibered category by adding the requirement that cartesian arrows are closed under composition. It is a simple exercise to show that this addendum ensures that every cartesian arrow (in the liberal sense) is actually hypercartesian (i.e. cartesian in the more restrictive sense of our definition) and, accordingly, both definitions of fibered category are equivalent.

As the current notes consider only fibrations for which "cartesian" and "hypercartesian" are equivalent anyway we have adopted the somewhat non–canonical Definition 2.1 as in our context it will not lead to any confusion.

Notice, however, that in more recent (unpublished) work by J. Bénabou on *generalised fibrations* the distinction between cartesian arrows (in the liberal sense) and hypercartesian arrows turns out as crucial.

Obviously, a fibration $P : \mathbf{X} \to \mathbf{B}$ is a fibration of groupoids iff all vertical arrows are isos iff all morphism of $\mathbf{X}$ are cartesian and thus P is a *discrete* fibration, i.e. a fibration of discrete categories, iff all vertical arrows are identities.

Lemma 2.1 *Suppose $P : \mathbf{X} \to \mathbf{B}$ and $Q : \mathbf{Y} \to \mathbf{B}$ are fibrations and $F : Q \to P$ is a cartesian functor over $\mathbf{B}$. If P is discrete, i.e. all vertical arrows are identities, then F is a fibration itself.*

Proof. Suppose $Y \in \mathbf{Y}$ and $f : X \to FY$ is a morphism in $\mathbf{X}$. Since Q is a fibration there exists a Q-cartesian arrow $\varphi : Z \to Y$ in $\mathbf{Y}$ above $P(f)$. Since F is cartesian $F(\varphi) : FZ \to FY$ is P-cartesian. We have $P(F(\varphi)) = Q(\varphi) = P(f)$ and thus both $F(\varphi)$ and f are morphism to FY over $P(f)$. Since P is a discrete fibration it follows that $F(\varphi) = f$. It remains to show that φ is F-cartesian. For this purpose suppose $g : U \to X$ and $\psi : V \to Y$ with $F(\psi) = F(\varphi)g$. Then $Q(\psi) = Q(\varphi)P(g)$ and thus, since Q is a fibration, there exists a unique $\theta : V \to Z$ with $\varphi\theta = \psi$ and $Q(\theta) = P(g)$. Thus $F(\varphi)g = F(\psi) = F(\varphi)F(\theta)$ from which it follows that $F(\theta) = g$ since P is a discrete fibration. Suppose $\widetilde{\theta} : V \to Z$ with $\varphi\widetilde{\theta} = \psi$ and $F(\widetilde{\theta}) = g$. Then $Q(\widetilde{\theta}) = P(F(\widetilde{\theta})) = P(g)$. Thus $\theta = \widetilde{\theta}$ as desird. $\square$

In general, i.e. if P is not assumed to be discrete, a cartesian functor $F : Q \to P$ will not be a fibration. For example if $\mathbf{B}$ is nontrivial, $Q = Id_{\mathbf{B}}$ and F is right adjoint to P, i.e. F picks a terminal object in each fiber, then F is not a fibration unless all fibers are equivalent to $\mathbf{1}$. In particular, if $\mathbf{B}$ is the ordinal $\mathbf{2}$ and P is the fundamental fibration $P_{\mathbf{B}}$ of $\mathbf{B}$ then the functor $1 : Id_{\mathbf{B}} \to P_{\mathbf{B}}$ (sending I to id_I) is not a fibration. Thus, it is not sufficient to require that P is faithful, i.e. that P is a fibration of posetal categories, for F being a fibration, too.

3 Split Fibrations and Fibered Yoneda Lemma

If $P : \mathbf{X} \to \mathbf{B}$ is a fibration then using axiom of choice for classes we may select for every $u : J \to I$ in $\mathbf{B}$ and $X \in P(I)$ a cartesian arrow $\mathrm{Cart}(u, X) : u^*X \to X$ over u. Such a choice of cartesian liftings is called a *cleavage* for P and it induces for every map $u : J \to I$ in $\mathbf{B}$ a so-called *reindexing functor* $u^* : P(I) \to P(J)$ in the following way

$$\begin{tikzcd} u^*X \arrow[r, "\mathrm{Cart}(u,X)"] \arrow[d, dashed, "u^*\alpha"'] & X \arrow[d, "\alpha"] \\ u^*Y \arrow[r, "\mathrm{Cart}(u,Y)"'] & Y \end{tikzcd}$$

where $u^*\alpha$ is the unique vertical arrow making the diagram commute. Alas, in general for composable maps $u : J \to I$ and $v : K \to J$ in $\mathbf{B}$ it does not hold that

$$v^* \circ u^* = (u \circ v)^*$$

although the functors are canonically isomorphic via $c_{u,v}$ as shown in the following diagram

$$\begin{tikzcd} v^*u^*X \arrow[dr, "\mathrm{Cart}(v,u^*X)", "\text{cart.}"'] \arrow[dd, dashed, "(c_{u,v})_X"', "\cong"] & & \\ & u^*X \arrow[dr, "\mathrm{Cart}(u,X)", "\text{cart.}"'] & \\ (uv)^*X \arrow[rr, "\text{cart.}", "\mathrm{Cart}(uv,X)"'] & & X \end{tikzcd}$$

where $(c_{u,v})_X$ is the unique vertical arrow making the diagram commute.

Typically, for $P_{\mathbf{B}} = \partial_1 : \mathbf{B}^{\mathbf{2}} \to \mathbf{B}$, the **fundamental fibration** for a category $\mathbf{B}$ with pullbacks, we do not know how to choose pullbacks in a functorial way, i.e. that $\mathrm{Cart}(id, X) = id_X$ and $\mathrm{Cart}(u \circ v, X) = \mathrm{Cart}(u, X) \circ \mathrm{Cart}(v, u^*X)$. Of course, the first condition is easy to achieve but the problem is the second condition since in general one does not know how to choose pullbacks in such a way that they are closed under composition.

But, nevertheless, often such a functorial choice of cartesian liftings is possible in particular situations.

Definition 3.1 *A cleavage* Cart *of a fibration* $P : \mathbf{X} \to \mathbf{B}$ *is called* split *or a* splitting of P *iff the following two conditions are satified*

(1) $\mathrm{Cart}(id, X) = id_X$

(2) $\mathrm{Cart}(uv, X) = \mathrm{Cart}(u, X) \circ \mathrm{Cart}(v, u^*X)$.

A split fibration *is a fibration* endowed *with a split cleavage.*

A split cartesian functor *between split fibrations is a cartesian functor* F *between split fibrations which, moreover, preserves chosen cartesian liftings, i.e. satisfies*

$$F(\mathrm{Cart}(u, X)) = \mathrm{Cart}(u, F(X))$$

for all $u : J \to I$ *in the base and all* X *over* I*. We write* $\mathbf{Sp}(\mathbf{B})$ *for the ensuing category of split fibrations over* $\mathbf{B}$ *and split cartesian functors between them.* $\diamondsuit$

Warning.
(1) There are fibrations which are not splitable. Consider for example the groups $\mathbf{B} = (\mathbb{Z}_2, +_2)$ and $\mathbf{X} = (\mathbb{Z}, +)$ (considered as categories) and the fibration $P : \mathbf{X} \to \mathbf{B} : a \mapsto P(a) := a \bmod 2$. A splitting of P would give rise to a functor $F : \mathbf{B} \to \mathbf{X}$ with $P \circ F = \mathrm{Id}_{\mathbf{B}}$ but that cannot exist as there is no group homomorphism $h : (\mathbf{Z}_2, +_2) \to (\mathbf{Z}, +)$ with $h(1)$ an odd number of $\mathbf{Z}$.
(2) Notice that different splittings of the same fibration may give rise to the same presheaf of categories. Consider for example $\mathcal{H} : \mathbf{2}^{\mathrm{op}} \to \mathbf{Ab}$ with $\mathcal{H}(1) = \mathcal{O}$, the zero group, and $\mathcal{H}(0)$ some non–trivial abelian group A. Then every $g \in A$ induces a splitting Cart_g of $P \equiv \int \mathcal{H}$ by putting

$$\mathrm{Cart}_g(u, \star) = (u, g) \qquad \text{for } u : 0 \to 1 \text{ in } \mathbf{2}$$

but all these Cart_g induce the same functor $\mathbf{2}^{\mathrm{op}} \to \mathbf{Cat}$, namely $\mathcal{H}$!

In the light of (2) it might appear as more appropriate to define split fibrations over $\mathbf{B}$ as functors from $\mathbf{B}^{\mathrm{op}}$ to $\mathbf{Cat}$. The latter may be considered as categories internal to $\widehat{\mathbf{B}} = \mathbf{Set}^{\mathbf{B}^{\mathrm{op}}}$ and organise into the (2-)category $\mathbf{cat}(\mathbf{B})$ of categories and functors internal to $\widehat{\mathbf{B}}$. However, as $\mathbf{Sp}(\mathbf{B})$ and $\mathbf{cat}(\mathbf{B})$ are strongly equivalent as 2-categories we will not distiguish them any further in the rest of these notes.

Next we will presented the *Fibered Yoneda Lemma* making precise the relation between fibered categories and split fibrations (over the same base).

Fibered Yoneda Lemma

Though, as we have seen, not every fibration $P \in \mathbf{Fib}(\mathbf{B})$ is isomorphic to a splitable fibration there is always a distinguished *equivalent* split fibration as ensured by the so-called *Fibered Yoneda Lemma*. Before giving the full formulation of the Fibered Yoneda Lemma we motivate the construction of a canonical split fibration $Sp(P)$ equivalent to a given fibration $P \in \mathbf{Fib}(\mathbf{B})$.

For an object $I \in \mathbf{B}$ let $\underline{I} = P_I = \partial_0 : \mathbf{B}/I \to \mathbf{B}$ be the discrete fibration corresponding to the representable presheaf $Y_{\mathbf{B}}(I) = \mathbf{B}(-, I)$ and for $u : J \to I$ in $\mathbf{B}$ let $\underline{u} = P_u = \Sigma_u$ be the cartesian functor from $\underline{J}$ to $\underline{I}$ as given by postcomposition with u and corresponding to the presheaf morphism $Y_{\mathbf{B}}(u) = \mathbf{B}(-, u) : Y_{\mathbf{B}}(J) \to Y_{\mathbf{B}}(I)$. Then cartesian functors from $\underline{I}$ to $P : \mathbf{X} \to \mathbf{B}$ in $\mathbf{Fib}(\mathbf{B})$ correspond to choices of cartesian liftings for an object $X \in P(I)$. There is an obvious functor $E_{P,I} : \mathbf{Fib}(\mathbf{B})(\underline{I}, P) \to P(I)$ sending F to $F(id_I)$ and $\tau : F \to G$ to $\tau_{id_I} : F(id_I) \to G(id_I)$. It is a straightforward exercise to show that $E_{P,I}$ is full and faithful and using the axiom of choice for classes we also get that $E_{P,I}$ is surjective on objects, i.e. that $E_{P,I} : \mathbf{Fib}(\mathbf{B})(\underline{I}, P) \to P(I)$ is an equivalence of categories. Now we can define $Sp(P) : \mathbf{B}^{\mathrm{op}} \to \mathbf{Cat}$ as

$$Sp(P)(I) = \mathbf{Fib}(\mathbf{B})(\underline{I}, P)$$

for objects I in $\mathbf{B}$ and

$$Sp(P)(u) = \mathbf{Fib}(\mathbf{B})(\underline{u}, P) : Sp(P)(I) \to Sp(P)(J)$$

for morphisms $u : J \to I$ in $\mathbf{B}$. Let us write $U(Sp(P))$ for $\int Sp(P)$, the fibration obtained from $Sp(P)$ via the Grothendieck construction. Then the $E_{P,I}$ as described above arise as the components of a cartesian functor $E_P : U(Sp(P)) \to P$ sending objects (I, X) in $U(Sp(P)) = \int Sp(P)$ to $E_{P,I}(X)$ and morphism $(u, \alpha) : G \to F$ in $U(Sp(P)) = \int Sp(P)$ over $u : J \to I$ to the morphism $F(u{:}u{\to}id_I) \circ \alpha_{id_J} : G(id_J) \to F(id_I)$ in $\mathbf{X}$. As all fibers of E_P are equivalences it follows[1] that E_P is an equivalence in the 2-category $\mathbf{Fib}(\mathbf{B})$.

Actually, the construction of $Sp(P)$ from P is just the object part of a 2-functor $Sp : \mathbf{Fib}(\mathbf{B}) \to \mathbf{Sp}(\mathbf{B})$ right adjoint to the forgetful 2-functor from $\mathbf{Sp}(\mathbf{B})$ to $\mathbf{Fib}(\mathbf{B})$ as described in the following theorem (which, however, will not be used any further in the rest of these notes).

[1] We leave it as an exercise to show that under assumption of axiom of choice for classes a cartesian functor is an equivalence in $\mathbf{Fib}(\mathbf{B})$ iff all its fibers are equivalences of categories.

Theorem 3.1 (Fibered Yoneda Lemma)
For every category $\mathbf{B}$ *the forgetful* 2*-functor* $U : \mathbf{Sp}(\mathbf{B}) \to \mathbf{Fib}(\mathbf{B})$ *has a right* 2*-adjoint* $Sp : \mathbf{Fib}(\mathbf{B}) \to \mathbf{Sp}(\mathbf{B})$*, i.e. there is an equivalence of categories*

$$\mathbf{Fib}(\mathbf{B})(U(S), P) \simeq \mathbf{Sp}(\mathbf{B})(S, Sp(P))$$

naturally in $S \in \mathbf{Sp}(\mathbf{B})$ *and* $P \in \mathbf{Fib}(\mathbf{B})$*, whose counit* $E_P : U(Sp(P)) \to P$ *at* P *is an equivalence in* $\mathbf{Fib}(\mathbf{B})$ *for all* $P \in \mathbf{Fib}(\mathbf{B})$*.*

However, in general the unit $H_S : S \to Sp(U(S))$ *at* $S \in \mathbf{Sp}(\mathbf{B})$ *is not an equivalence in* $\mathbf{Sp}(\mathbf{B})$ *although* $U(H_S)$ *is always an equivalence in* $\mathbf{Fib}(\mathbf{B})$*.*

Proof. The functor $U : \mathbf{Sp}(\mathbf{B}) \to \mathbf{Fib}(\mathbf{B})$ just forgets cleavages. The object part of its right adjoint Sp is as described above, namely

$$Sp(P)(I) = \mathbf{Fib}(\mathbf{B})(\underline{I}, P) \qquad Sp(P)(u) = \mathbf{Fib}(\mathbf{B})(\underline{u}, P)$$

for $P \in \mathbf{Fib}(\mathbf{B})$. For cartesian functors $F : P \to Q$ in $\mathbf{Fib}(\mathbf{B})$ we define $Sp(F) : Sp(P) \to Sp(Q)$ as

$$Sp(F)_I = \mathbf{Fib}(\mathbf{B})(\underline{I}, F)$$

for objects I in $\mathbf{B}$. Under assumption of axiom of choice for classes the counit for $U \dashv Sp$ at P is given by the equivalence $E_P : U(Sp(P)) \to P$ as described above. The unit $H_S : S \to Sp(U(S))$ for $U \dashv Sp$ at $S \in \mathbf{Sp}(\mathbf{B})$ sends $X \in P(I)$ to the cartesian functor from $\underline{I}$ to P which chooses cartesian liftings as prescribed by the underlying cleavage of S and arrows $\alpha : X \to Y$ in $P(I)$ to the cartesian natural transformation $H_S(\alpha) : H_S(X) \to H_S(Y)$ with $H_S(\alpha)_{id_I} = \alpha$. We leave it as a tedious, but straightforward exercise to show that these data give rise to an equivalence

$$\mathbf{Fib}(\mathbf{B})(U(S), P) \simeq \mathbf{Sp}(\mathbf{B})(S, Sp(P))$$

naturally in S and P.

As all components of H_S are equivalences of categories it follows that $U(H_S)$ is an equivalence in $\mathbf{Fib}(\mathbf{B})$. However, it cannot be the case that all H_S are equivalences as otherwise a split cartesian functor F were an equivalence in $\mathbf{Sp}(\mathbf{B})$ already if $U(F)$ is an equivalence in $\mathbf{Fib}(\mathbf{B})$ and this is impossible as not every epi in $\widehat{\mathbf{B}}$ is a split epi. $\square$

As $E_P : U(Sp(P)) \to P$ is always an equivalence it follows that for fibrations P and Q

$$Sp_{P,Q} : \mathbf{Fib}(\mathbf{B})(P, Q) \to \mathbf{Sp}(\mathbf{B})(Sp(P), Sp(Q))$$

is an equivalence of categories.

However, in general $Sp_{P,Q}$ is not an isomorphism of categories. An arbitrary split cartesian functor $G : Sp(P) \to Sp(Q)$ corresponds via the 2-adjunction $U \dashv Sp$ to a cartesian functor $E_Q \circ U(G) : U(Sp(P)) \to Q$ which, however, need not factor as $E_Q \circ U(G) = F \circ E_P$ for some cartesian $F : P \to Q$.[2] One may characterise the split cartesian functors of the form $Sp(F)$ for some cartesian $F : P \to Q$ as those split cartesian functors $G : Sp(P) \to Sp(Q)$ satisfying $Sp(E_Q) \circ Sp(U(G)) = G \circ Sp(E_P)$. One easily sees that this condition is necessary and if it holds then an F with $G = Sp(F)$ can be obtained as $E_Q \circ U(G) \circ E'_P$ for some E'_P with $E_P \circ E'_P = Id_P$ because we have $Sp(F) = Sp(E_Q \circ U(G) \circ E'_P) = Sp(E_Q) \circ Sp(U(G)) \circ Sp(E'_P) = G \circ Sp(E_P) \circ Sp(E'_P) = G \circ Sp(E_P \circ E'_P) = G$.

Although Sp is not full and faithful the adjunction $U \dashv Sp$ nevertheless is of the type "full reflective subcategory" albeit in the appropriate 2-categorical sense. This suggests that $\mathbf{Fib}(\mathbf{B})$ is obtained from $\mathbf{Sp}(\mathbf{B})$ by "freely quasi-inverting weak equivalences in $\mathbf{Fib}(\mathbf{B})$" which can be made precise as follows.

A split cartesian functor F is called a *weak equivalence* iff all its fibers are equivalences of categories, i.e. iff $U(F)$ is an equivalence in $\mathbf{Fib}(\mathbf{B})$. Let us write Σ for the class of weak equivalences in $\mathbf{Sp}(\mathbf{B})$. For a 2-category $\mathfrak{X}$ and a 2-functor $\Phi : \mathbf{Sp}(\mathbf{B}) \to \mathfrak{X}$ we say that Φ *quasi-inverts* a morphism F in $\mathbf{Sp}(\mathbf{B})$ iff $\Phi(F)$ is an equivalence in $\mathfrak{X}$. Obviously, the 2-functor $U : \mathbf{Sp}(\mathbf{B}) \to \mathbf{Fib}(\mathbf{B})$ quasi-inverts all weak equivalences. That U freely inverts the maps in Σ can be seen as follows. Suppose that a 2-functor $\Phi : \mathbf{Sp}(\mathbf{B}) \to \mathfrak{X}$ quasi-inverts all weak equivalences. Then there exists a 2-functor $\Psi : \mathbf{Fib}(\mathbf{B}) \to \mathfrak{X}$ unique up to equivalence with the property that $\Psi \circ U \simeq \Phi$. As by assumption Φ quasi-inverts weak equivalences we have $\Phi \circ Sp \circ U \simeq \Phi$ because all H_S are weak equivalences. On the other hand if $\Psi \circ U \simeq \Phi$ then we have $\Psi \simeq \Psi \circ U \circ Sp \simeq \Phi \circ Sp$ (because all E_P are equivalences) showing that Ψ is unique up to equivalence.

A Left Adjoint Splitting

The forgetful functor $U : \mathbf{Sp}(\mathbf{B}) \to \mathbf{Fib}(\mathbf{B})$ admits also a left adjoint $L : \mathbf{Fib}(\mathbf{B}) \to \mathbf{Sp}(\mathbf{B})$ which like the right adjoint splitting discussed previously was devised by J. Giraud in the late 1960s.

This left adjoint splitting $L(P)$ of a fibration $P : \mathbf{X} \to \mathbf{B}$ is constructed as follows. First choose a cleavage Cart_P of P which is *normalized* in the

[2] For example, if $Q = U(Sp(P))$ and $E_Q \circ U(G) = Id_{U(Sp(P))}$ and E_P is not one-to-one on objects which happens to be the case whenever cartesian liftings are not unique in P.

sense that $\mathrm{Cart}_P(id_I, X) = id_X$ for all X over I. From this cleavage one may construct a presheaf $S(P) : \mathbf{B}^{\mathrm{op}} \to \mathbf{Cat}$ of categories giving rise to the desired split fibration $L(P)$ over $\mathbf{B}$. For $I \in \mathbf{B}$ the objects of $S(P)(I)$ are pairs (a, X) where X is an object of $\mathbf{X}$ and $a : I \to P(X)$. Morphisms from (b, Y) to (a, X) are vertical morphism $\alpha : b^*Y \to a^*X$ and composition in $S(P)(I)$ is inherited from $\mathbf{X}$, i.e. $P(I)$. For $u : J \to I$ in $\mathbf{B}$ the functor $S(P)(u) : S(P)(I) \to S(P)(J)$ is constructed as follows. For (a, X) in $S(P)(I)$ let $\mathrm{Cart}_{L(P)}(u, (a, X)) : (au)^*X \to u^*X$ be the unique cartesian arrow φ over u with $\mathrm{Cart}_P(a, X) \circ \varphi = \mathrm{Cart}_P(au, X)$. Let $\alpha : b^*Y \to a^*X$ be a morphism from (b, Y) to (a, X) in $S(P)(I)$. Then we define $S(P)(u)(\alpha)$ as the unique vertical morphism making the diagram

$$\begin{tikzcd}
 & Y \\
(bu)^*Y \arrow[ur, "\mathrm{Cart}_P(bu, Y)"] \arrow[r, "\mathrm{Cart}_{L(P)}(u, (b, Y))"'] \arrow[d, "S(P)(u)(\alpha)"'] & b^*Y \arrow[u, "\mathrm{Cart}_P(b, Y)"'] \arrow[d, "\alpha"] \\
(au)^*X \arrow[r, "\mathrm{Cart}_{L(P)}(u, (a, X))"] \arrow[dr, "\mathrm{Cart}_P(au, X)"'] & a^*X \arrow[d, "\mathrm{Cart}_P(a, X)"] \\
 & X
\end{tikzcd}$$

commute. One readily checks that $S(P)$ is indeed a functor from $\mathbf{B}^{\mathrm{op}}$ to $\mathbf{Cat}$ since $\mathrm{Cart}_{L(P)}(uv, (a, X) = \mathrm{Cart}_{L(P)}(u, (a, X)) \circ \mathrm{Cart}_{L(P)}(v, (au, X))$ and $\mathrm{Cart}_{L(P)}(id_I, (a, X)) = id_{a^*X}$ as one can see easily. Objects of the total category of $L(P)$ are objects of $S(P)(I)$ for some $I \in \mathbf{B}$ and morphisms from (b, Y) to (a, X) are just morphisms $b^*Y \to a^*X$ whose composition is inherited from $\mathbf{X}$. The functor $L(P)$ sends (a, X) to the domain of a and $f : b^*Y \to a^*X$ to $P(f)$. The splitting of $L(P)$ is given by $\mathrm{Cart}_{L(P)}$ as defined above for specifying the morphism part of $S(P)$. The unit $H_P : P \to U(L(P))$ of the (2-categorical) adjunction $L \dashv U$ sends X to $(id_{P(X)}, X)$ and $f : Y \to X$ to $f : H_P(Y) \to H_P(X)$.

Notice that the above construction of $L(P)$ is based on a choice of a cleavage for P. But this may be avoided by defining morphisms from (b, Y) to (a, X) over $u : J \to I$ as equivalence classes of spans (ψ, f) in $\mathbf{X}$ where ψ is a cartesian morphism to Y over b and f is a morphism to A over au where (ψ, f) and (ψ', f') get identified iff there is a vertical isomorphism ι with $\psi \circ \iota = \psi'$ and $f \circ \iota = f'$. For a given cleavage Cart_P of P the equivalence class of (ψ, f) contains a unique pair whose first component is $\mathrm{Cart}_P(b, Y)$.

4 Closure Properties of Fibrations

In this section we will give some examples of fibrations and constructions of new fibrations from already given ones. Keeping in mind that we think of fibrations over **B** as generalisations of fibrations of the form Fam(**C**) over **Set** it will appear that most of these constructions are generalisations of well-known constructions in **Cat**.

Fundamental Fibrations

For a category **B** the codomain functor

$$P_{\mathbf{B}} \equiv \partial_1 : \mathbf{B}^{\mathbf{2}} \to \mathbf{B}$$

is a fibration if and only if **B** has pullbacks. In this case $P_{\mathbf{B}}$ is called the **fundamental fibration** of **B**.

Externalisations of Internal Categories

Let C be a category internal to **B** as given by domain and codomain maps $d_0, d_1 : C_1 \to C_0$, the identity map $i : C_0 \to C_1$ and a composition map $m : C_1 \times_{C_0} C_1 \to C_1$. Then one may construct the fibration $P_C : \underline{C} \to \mathbf{B}$ called *externalisation of* C. The objects of $\underline{C}$ over I are pairs $(I, a : I \to C_0)$ and a morphism in $\underline{C}$ from (J, b) to (I, a) over $u : J \to I$ is given by a morphism $f : J \to C_1$ with $d_0 \circ f = b$ and $d_1 \circ f = a \circ u$. Composition in C is defined using m analogous to Fam(**C**). The fibration P_C itself is defined as

$$P_C(I, a) = I \qquad\qquad P_C(u, f) = u$$

and the cartesian lifting of (I, a) along $u : J \to I$ is given by $i \circ a \circ u$.

In particular, every object $I \in \mathbf{B}$ can be considered as a *discrete* internal category of **B**. Its externalisation is given by $P_I = \partial_0 : \mathbf{B}/I \to \mathbf{B}$ for which (by a convenient abuse of notation) we often also write $\underline{I}$.

Change of Base and "Glueing"

If $P \in \mathbf{Fib}(\mathbf{B})$ and $F : \mathbf{C} \to \mathbf{B}$ is an ordinary functor then $F^*P \in \mathbf{Fib}(\mathbf{C})$ where

$$\begin{tikzcd} \mathbf{Y} \arrow[r, "K"] \arrow[d, "F^*P"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{X} \arrow[d, "P"] \\ \mathbf{C} \arrow[r, "F"'] & \mathbf{B} \end{tikzcd}$$

is a pullback in $\mathbf{Cat}$. One says that fibration F^*P is obtained from P *by change of base along* F. Notice that (u, φ) in $\mathbf{Y}$ is cartesian w.r.t. F^*P iff φ is cartesian w.r.t. P. Accordingly, K preserves cartesianness of arrows as $K(u, \varphi) = \varphi$.

When instantiating P by the fundamental fibration $P_{\mathbf{B}}$ we get the following important particular case of change of base

$$\begin{tikzcd} \mathbf{B}{\downarrow}F \arrow[r, "\partial_1^* F"] \arrow[d, "P_F"'] \arrow[dr, phantom, "\lrcorner"] & \mathbf{B}^{\mathbf{2}} \arrow[d, "P_{\mathbf{B}}"] \\ \mathbf{C} \arrow[r, "F"'] & \mathbf{B} \end{tikzcd}$$

where we write P_F for $F^*P_{\mathbf{B}}$. This is often referred to as *(Artin) glueing* in which case one often writes $\mathsf{gl}(F)$ for P_F and $\mathsf{Gl}(F)$ for $\mathbf{B}{\downarrow}F$. Typically, in applications the functor F will be the inverse image part of a geometric morphism $F \dashv U : \mathbf{E} \to \mathbf{S}$ between toposes. But already if F is a pullback preserving functor between toposes $\mathsf{Gl}(F) = \mathbf{E}{\downarrow}F$ is again a topos and the functor $P_F = \mathsf{gl}(F) : \mathbf{E}{\downarrow}F \to \mathbf{S}$ is *logical*, i.e. preserves all topos structure. The glueing construction will get very important later on when we discuss the *Fibrational Theory of Geometric Morphisms* à la J.-L. Moens.

We write $\mathbf{Fib}$ for the (non–full) subcategory of $\mathbf{Cat}^{\mathbf{2}}$ whose objects are fibrations and whose morphisms are commuting squares

$$\begin{tikzcd} \mathbf{Y} \arrow[r, "K"] \arrow[d, "Q"'] & \mathbf{X} \arrow[d, "P"] \\ \mathbf{C} \arrow[r, "F"'] & \mathbf{B} \end{tikzcd}$$

with K cartesian over F, i.e. $K(\varphi)$ is cartesian over $F(u)$ whenever φ is cartesian over u. Obviously, $\mathbf{Fib}$ is fibered over $\mathbf{Cat}$ via the restriction of $\partial_1 : \mathbf{Cat}^{\mathbf{2}} \to \mathbf{Cat}$ to $\mathbf{Fib}$ for which we write $\mathbf{Fib/Cat} : \mathbf{Fib} \to \mathbf{Cat}$. A morphism of $\mathbf{Fib}$ is cartesian iff it is a pullback square in $\mathbf{Cat}$.

We write $\mathbf{Fib(B)/B}$ for the fibration obtained from $\mathbf{Fib/Cat}$ by change of base along the functor $\Sigma : \mathbf{B} \to \mathbf{Cat}$ sending I to $\mathbf{B}/I$ and $u : J \to I$ to $\Sigma_u : \mathbf{B}/J \to \mathbf{B}/I : v \mapsto u \circ v$

$$\begin{tikzcd} \mathbf{Fib}{\downarrow}\Sigma \arrow[r] \arrow[d, "\mathbf{Fib(B)/B}"'] \arrow[dr, phantom, "\lrcorner"] & \mathbf{Fib} \arrow[d, "\mathbf{Fib/Cat}"] \\ \mathbf{B} \arrow[r, "\Sigma"'] & \mathbf{Cat} \end{tikzcd}$$

We leave it as an exercise to show that $P : \mathbf{X} \to \mathbf{B}/I$ is a fibration iff $P_I \circ P$ is a fibration over $\mathbf{B}$ and $P \in \mathbf{Fib}(\mathbf{B})(P_I \circ P, P_I)$. Accordingly, fibrations over $\mathbf{B}/I$ may be considered as I-indexed families of fibrations over $\mathbf{B}$ in analogy with ordinary functors to a discrete category I which may be considered as I-indexed families of categories.

Composition and Product of Fibrations

First notice that fibrations are closed under composition. Even more we have the following

Theorem 4.1 *Let $P : \mathbf{X} \to \mathbf{B}$ be a fibration and $F : \mathbf{Y} \to \mathbf{X}$ be an arbitrary functor. Then F itself is a fibration over $\mathbf{X}$ iff*

(1) *$Q \equiv P \circ F$ is a fibration and F is a cartesian functor from Q to P over $\mathbf{B}$ and*

(2) *all $F_I : \mathbf{Y}_I \to \mathbf{X}_I$ are fibrations and cartesian arrows w.r.t. these fibrations are stable under reindexing, i.e. for every commuting diagram*

$$\begin{tikzcd} Y_1 \arrow[r, "\varphi_1"] \arrow[d, "\theta"'] & X_1 \arrow[d, "\psi"] \\ Y_2 \arrow[r, "\varphi_2"'] & X_2 \end{tikzcd}$$

in $\mathbf{Y}$ with φ_1 and φ_2 cartesian w.r.t. Q over the same arrow $u : J \to I$ in $\mathbf{B}$ and $Q(\psi) = id_I$ and $Q(\theta) = id_J$ it holds that θ is cartesian w.r.t. F_J whenever ψ is cartesian w.r.t. F_I.

Proof. Exercise left to the reader. $\square$

The second condition means that the commuting diagram

$$\begin{tikzcd} \mathbf{Y}_I \arrow[r, "u^*"] \arrow[d, "F_I"'] & \mathbf{Y}_J \arrow[d, "F_J"] \\ \mathbf{X}_I \arrow[r, "u^*"'] & \mathbf{X}_J \end{tikzcd}$$

is a morphism in **Fib**. (Notice that due to condition (1) of Theorem 4.1 one can choose the reindexing functor $u^* : \mathbf{Y}_I \to \mathbf{Y}_J$ in such a way that the diagram actually commutes. For arbitrary cartesian functors this need not be possible although for all choices of the u^* the diagram always commutes up to isomorphism.)

The relevance of Theorem 4.1 is that it characterises "fibered fibrations" as those fibered functors which are themselves ordinary fibrations. This handy characterisation cannot even be formulated in the framework of indexed categories and, therefore, is considered as a typical example of the superiority of the fibrational point of view.

For fibrations P and Q over $\mathbf{B}$ their product $P{\times_{\mathbf{B}}}Q$ in $\mathbf{Fib}(\mathbf{B})$ is given by $P \circ P^*Q = Q \circ Q^*P$ as in

$$
\begin{tikzcd}
\mathbf{P} \arrow[r, "Q^*P"] \arrow[d, "P^*Q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Y} \arrow[d, "Q"] \\
\mathbf{X} \arrow[r, "P"'] & \mathbf{B}
\end{tikzcd}
$$

and it follows from Theorem 4.1 that $P{\times_{\mathbf{B}}}Q$ is a fibration and that the projections P^*Q and Q^*P are cartesian functors.

Fibrations of Diagrams

Let $\mathbf{D}$ be a category and $P : \mathbf{X} \to \mathbf{B}$ a fibration. Then the *fibration* $P^{(\mathbf{D})}$ *of diagrams of shape* $\mathbf{D}$ is given by

$$
\begin{tikzcd}
\mathbf{X}^{(\mathbf{D})} \arrow[r] \arrow[d, "P^{(\mathbf{D})}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{X}^{\mathbf{D}} \arrow[d, "P^{\mathbf{D}}"] \\
\mathbf{B} \arrow[r, "\Delta_{\mathbf{D}}"'] & \mathbf{B}^{\mathbf{D}}
\end{tikzcd}
$$

where the "diagonal functor" $\Delta_{\mathbf{D}}$ sends $I \in \mathbf{B}$ to the constant functor with value I and a morphism u in $\mathbf{B}$ to the natural transformation all whose components are u.

Somewhat surprisingly, as shown by A. Kurz in spring 2019, the functor $P^{\mathbf{D}}$ is also a fibration, however, over $\mathbf{B}^{\mathbf{D}}$.

Exponentiation of Fibrations

For fibrations P and Q over $\mathbf{B}$ we want to construct a fibration $[P{\to}Q]$ such that there is an equivalence

$$\mathbf{Fib}(\mathbf{B})(R, [P{\to}Q]) \simeq \mathbf{Fib}(\mathbf{B})(R{\times_{\mathbf{B}}}P, Q)$$

naturally in $R \in \mathbf{Fib}(\mathbf{B})$.

Analogous to the construction of exponentials in $\widehat{\mathbf{B}} = \mathbf{Set}^{\mathbf{B}^{\mathrm{op}}}$ the fibered Yoneda lemma (Theorem 3.1) suggest us to put

$$[P{\to}Q](I) = \mathbf{Fib}(\mathbf{B})(\underline{I}{\times_{\mathbf{B}}}P, Q) \qquad [P{\to}Q](u) = \mathbf{Fib}(\mathbf{B})(\underline{u}{\times_{\mathbf{B}}}P, Q)$$

where $\underline{u}$ is given by

$$\begin{tikzcd}
\mathbf{B}/J \arrow[rr, "\Sigma_u"] \arrow[dr, "P_J"'] & & \mathbf{B}/I \arrow[dl, "P_I"] \\
& \mathbf{B} &
\end{tikzcd}$$

for $u : J \to I$ in $\mathbf{B}$. We leave it as a tedious, but straightforward exercise to verify that

$$\mathbf{Fib}(\mathbf{B})(R, [P{\to}Q]) \simeq \mathbf{Fib}(\mathbf{B})(R{\times_{\mathbf{B}}}P, Q)$$

holds naturally in $R \in \mathbf{Fib}(\mathbf{B})$.

Notice that we have

$$\mathbf{Fib}(\mathbf{B})(P_I{\times_{\mathbf{B}}}P, Q) \simeq \mathbf{Fib}(\mathbf{B}/I)(P_{/I}, Q_{/I})$$

naturally in $I \in \mathbf{B}$ where $P_{/I} = {P_I}^*P$ and $Q_{/I} = {P_I}^*Q$ are obtained by change of base along P_I. Usually $P_{/I}$ is referred to as "localisation of P to I". The desired equivalence follows from the fact that change of base along P_I is right adjoint to postcomposition with P_I and the precise correspondence between $F \in \mathbf{Fib}(\mathbf{B})(P_I{\times_{\mathbf{B}}}P, Q)$ and $G \in \mathbf{Fib}(\mathbf{B}/I)(P_{/I}, Q_{/I})$ is indicated by the following diagram

$$\begin{tikzcd}
\cdot \arrow[dr, "G"] \arrow[drr, "F"] \arrow[ddr, "P_{/I}"'] & & \\
& \cdot \arrow[r] \arrow[d, "Q_{/I}"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Y} \arrow[d, "Q"] \\
& \mathbf{B}/I \arrow[r, "P_I"'] & \mathbf{B}
\end{tikzcd}$$

5 The Opposite of a Fibration

If $P : \mathbf{X} \to \mathbf{B}$ is a fibration thought of "as of the form $\mathrm{Fam}(\mathbf{C})$" then one may want to construct the fibration P^{op} thought of "of the form $\mathrm{Fam}(\mathbf{C}^{\mathrm{op}})$". It might be tempting at first sight to apply $(-)^{\mathrm{op}}$ to the functor P giving rise to the functor $\mathbf{X}^{\mathrm{op}} \to \mathbf{B}^{\mathrm{op}}$ which, however, has the wrong base even if it were a fibration (which in general will not be the case). If $P = \int \mathcal{H}$ for some $\mathcal{H} : \mathbf{B}^{\mathrm{op}} \to \mathbf{Cat}$ then one may consider $\mathcal{H}^{\mathrm{op}} = (-)^{\mathrm{op}} \circ \mathcal{H} : \mathbf{B}^{\mathrm{op}} \to \mathbf{Cat}$, i.e. the assignment

$$I \mapsto \mathcal{H}(I)^{\mathrm{op}} \qquad\qquad u : J \to I \;\mapsto\; \mathcal{H}(u)^{\mathrm{op}} : \mathcal{H}(I)^{\mathrm{op}} \to \mathcal{H}(I)^{\mathrm{op}}$$

where $(-)^{\mathrm{op}}$ is applied to the fibers of $\mathcal{H}$ and to the reindexing functors. Now we express $P^{\mathrm{op}} = \int \mathcal{H}^{\mathrm{op}}$ in terms of $P = \int \mathcal{H}$ directly.

The fibration $P^{\mathrm{op}} : \mathbf{Y} \to \mathbf{B}$ is constructed from the fibration $P : \mathbf{X} \to \mathbf{B}$ in the following way. The objects of $\mathbf{Y}$ and $\mathbf{X}$ are the same but for $X \in P(I)$, $Y \in P(J)$ and $u : J \to I$ the collection of morphisms in $\mathbf{Y}$ from Y to X over u is constructed as follows. It consists of all spans (α, φ) with $\alpha : Z \to Y$ vertical and $\varphi : Z \to X$ is cartesian over u *modulo* the equivalence relation $\sim_{Y,u,X}$ (also denoted simply as $\sim$) where $(\alpha, \varphi) \sim_{Y,u,X} (\alpha', \varphi')$ iff

$$\begin{tikzcd}
& Z \arrow[dl, "\alpha"'] \arrow[dr, "\varphi", "\text{cart}"'] & \\
Y & & X \\
& Z' \arrow[ul, "\alpha'"] \arrow[ur, "\varphi'"', "\text{cart}"] \arrow[uu, "\iota", "\cong"'] &
\end{tikzcd}$$

for some (necessarily unique) vertical isomorphism $\iota : Z' \to Z$. Composition of arrows in $\mathbf{Y}$ is defined as follows: if $[(\alpha, \varphi)]_\sim : Y \to X$ over $u : J \to I$ and $[(\beta, \psi)]_\sim : Z \to Y$ over $v : K \to J$ then $[(\alpha, \varphi)]_\sim \circ [(\beta, \psi)]_\sim := [(\beta \circ \widetilde{\alpha}, \varphi \circ \widetilde{\psi})]_\sim$ where

$$\begin{tikzcd}
& & (uv)^*X \arrow[dl, "\widetilde{\alpha}"'] \arrow[dr, "\widetilde{\psi}", "\text{cart}"'] & & \\
& v^*Y \arrow[dl, "\beta"'] \arrow[dr, "\psi"', "\text{cart}"] & \text{p.b.} & u^*X \arrow[dl, "\alpha"] \arrow[dr, "\varphi", "\text{cart}"'] & \\
Z & & Y & & X
\end{tikzcd}$$

with $\widetilde{\alpha}$ vertical.

Actually, this definition does not depend on the choice of $\widetilde{\psi}$ as morphisms in $\mathbf{Y}$ are equivalence classes modulo $\sim$ which forgets about all distinctions made by choice of cleavages. On objects P^{op} behaves like P and $P^{\mathrm{op}}([(\alpha,\varphi)]_\sim)$ is defined as $P(\varphi)$. The P^{op}-cartesian arrows are the equivalence classes $[(\alpha,\varphi)]_\sim$ where α is a vertical isomorphism.

Though most constructions appear more elegant from the fibrational point of view the construction of P^{op} from P may appear as somewhat less immediate though (hopefully!) not too unelegant. Notice, however, that for small fibrations, i.e. externalisations of internal categories, the construction can be performed as in the case of presheaves of categories as we have $P_{C^{\mathrm{op}}} \simeq P_C^{\mathrm{op}}$ for internal categories C.

Anyway, we have generalised now enough constructions from ordinary category theory to the fibrational level so that we can perform (analogues of) the various constructions of (covariant and contravariant) functor categories on the level of fibrations. In particular, for a category C internal to a category $\mathbf{B}$ with pullbacks we may construct the fibration $[P_C^{\mathrm{op}} \to P_{\mathbf{B}}]$ which may be considered as the fibration of (families of) $\mathbf{B}$-valued presheaves over the internal category C. Moreover, for categories C and D internal to $\mathbf{B}$ the fibration of (families of) distributors from C to D is given by $[P_D^{\mathrm{op}} \times P_C \to P_{\mathbf{B}}]$.[3]

[3]For an equivalent, but non-fibrational treatment of internal presheaves and distributors see [Jo77].

6 Internal Sums

Suppose that $\mathbf{C}$ is a category. We will identify a purely fibrational property of the fibration $\mathrm{Fam}(\mathbf{C}) \to \mathbf{Set}$ equivalent to the requirement that the category $\mathbf{C}$ has small sums. This will provide a basis for generalising the property of "having small sums" to fibrations over arbitrary base categories with pullbacks.

Suppose that category $\mathbf{C}$ has small sums. Consider a family of objects $A = (A_i)_{i \in I}$ and a map $u : I \to J$ in $\mathbf{Set}$. Then one may construct the family $B := (\coprod_{i \in u^{-1}(j)} A_i)_{j \in J}$ together with the morphism $(u, \varphi) : (I, A) \to (J, B)$ in $\mathrm{Fam}(\mathbf{C})$ where $\varphi_i = \mathrm{in}_i : A_i \to B_{u(i)} = \coprod_{k \in u^{-1}(u(i))} A_k$, i.e. the restriction of φ to $u^{-1}(j)$ is the cocone for the sum of the family $(A_i)_{i \in u^{-1}(j)}$.

One readily observes that $(u, \varphi) : A \to B$ satisfies the following universal property: whenever $v : J \to K$ and $(v \circ u, \psi) : A \to C$ then there exists a unique $(v, \theta) : B \to C$ such that $(v, \theta) \circ (u, \varphi) = (v \circ u, \psi)$, i.e. $\theta_{u(i)} \circ \mathrm{in}_i = \psi_i$ for all $i \in I$. Arrows (u, φ) satisfying this universal property are called *cocartesian* and are determined uniquely up to vertical isomorphism.

Moreover, the cocartesian arrows of $\mathrm{Fam}(\mathbf{C})$ satisfy the following so-called[4] *Beck–Chevalley Condition* (BCC) which says that for every pullback

$$\begin{tikzcd} K \arrow[r, "\widetilde{u}"] \arrow[d, "\widetilde{h}"'] \arrow[dr, phantom, "\lrcorner\ (1)"] & L \arrow[d, "h"] \\ I \arrow[r, "u"'] & J \end{tikzcd}$$

in $\mathbf{Set}$ and cocartesian arrow $\varphi : A \to B$ over u it holds that for every commuting square

$$\begin{tikzcd} C \arrow[r, "\widetilde{\varphi}"] \arrow[d, "\widetilde{\psi}"'] & D \arrow[d, "\psi"] \\ A \arrow[r, "\varphi"'] & B \end{tikzcd}$$

over the pullback square (1) in $\mathbf{B}$ with ψ and $\widetilde{\psi}$ cartesian the arrow $\widetilde{\varphi}$ is cocartesian, too.

Now it is a simple exercise to formulate the obvious generalisation to fibrations over an arbitrary base category with pullbacks.

[4] Chevalley had this condition long before Beck who later independently found it again.

Definition 6.1 *Let* $\mathbf{B}$ *be a category with pullbacks and* $P : \mathbf{X} \to \mathbf{B}$ *a fibration over* $\mathbf{B}$*. An arrow* $\varphi : X \to Y$ *over* $u : I \to J$ *is called* cocartesian *iff for every* $v : J \to K$ *in* $\mathbf{B}$ *and* $\psi : X \to Z$ *over* $v \circ u$ *there is a unique arrow* $\theta : Y \to Z$ *over* v *with* $\theta \circ \varphi = \psi$*.*
The fibration P has internal sums *iff the following two conditions are satisfied.*

(1) *For every* $X \in P(I)$ *and* $u : I \to J$ *in* $\mathbf{B}$ *there exists a cocartesian arrow* $\varphi : X \to Y$ *over* u*.*

(2) *The* Beck–Chevalley Condition (BCC) *holds, i.e. for every commuting square in* $\mathbf{X}$

$$\begin{tikzcd} C \arrow[r, "\widetilde{\varphi}"] \arrow[d, "\widetilde{\psi}"'] & D \arrow[d, "\psi"] \\ A \arrow[r, "\varphi"'] & B \end{tikzcd}$$

over a pullback in the base it holds that $\widetilde{\varphi}$ *is cocartesian whenever* φ *is cocartesian and* ψ *and* $\widetilde{\psi}$ *are cartesian.* $\Diamond$

Remark.
(1) One easily sees that for a fibration $P : \mathbf{X} \to \mathbf{B}$ an arrow $\varphi : X \to Y$ is cocartesian iff for all $\psi : X \to Z$ over $P(\varphi)$ there exists a unique vertical arrow $\alpha : Y \to Z$ with $\alpha \circ \varphi = \psi$.
(2) It is easy to see that BCC of Definition 6.1 is equivalent to the requirement that for every commuting square in $\mathbf{X}$

$$\begin{tikzcd} C \arrow[r, "\widetilde{\varphi}"] \arrow[d, "\widetilde{\psi}"'] & D \arrow[d, "\psi"] \\ A \arrow[r, "\varphi"'] & B \end{tikzcd}$$

over a pullback in the base it holds that ψ is cartesian whenever $\widetilde{\psi}$ is cartesian and φ and $\widetilde{\varphi}$ are cocartesian.

Next we give a less phenomenological explanation of the concept of internal sums where, in particular, the Beck–Chevalley Condition arises in a less *ad hoc* way. For this purpose we first generalise the Fam construction from ordinary categories to fibrations.

Definition 6.2 *Let* $\mathbf{B}$ *be a category with pullbacks and* $P : \mathbf{X} \to \mathbf{B}$ *be a fibration. Then the* family fibration $\mathrm{Fam}(P)$ for P *is defined as* $P_{\mathbf{B}} \circ \mathit{Fam}(P)$ *where*

$$
\begin{tikzcd}
P{\downarrow}\mathbf{B} \arrow[r] \arrow[d, "\mathit{Fam}(P)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{X} \arrow[d, "P"] \\
\mathbf{B^2} \arrow[r, "\partial_0"'] & \mathbf{B}
\end{tikzcd}
$$

The cartesian functor $\mathit{Fam}(P) : \mathrm{Fam}(P) \to P_{\mathbf{B}}$ *is called the* fibered family fibration of P.

The cartesian functor $\eta_P : P \to \mathrm{Fam}(P)$ *is defined as in the diagram*

$$
\begin{tikzcd}
\mathbf{X} \arrow[dd, "P"'] \arrow[dr, dashed, "\eta_P"] \arrow[drr, equal] & & \\
& P{\downarrow}\mathbf{B} \arrow[r] \arrow[d, "\mathit{Fam}(P)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{X} \arrow[d, "P"] \\
\mathbf{B} \arrow[r, "\Delta_{\mathbf{B}}"'] \arrow[dr, equal] & \mathbf{B^2} \arrow[r, "\partial_0"'] \arrow[d, "\partial_1 = P_{\mathbf{B}}"] & \mathbf{B} \\
& \mathbf{B} &
\end{tikzcd}
$$

where $\Delta_{\mathbf{B}}$ *sends* $u : I \to J$ *to*

$$
\begin{tikzcd}
I \arrow[r, "u"] \arrow[d, equal] & J \arrow[d, equal] \\
I \arrow[r, "u"'] & J
\end{tikzcd}
$$

in $\mathbf{B^2}$. *More explicitly,* η_P *sends* $\varphi : X \to Y$ *over* $u : I \to J$ *to*

$$
\begin{tikzcd}
X \arrow[r, "\varphi"] & Y \\
I \arrow[r, "u"] \arrow[d, equal] & J \arrow[d, equal] \\
I \arrow[r, "u"'] & J
\end{tikzcd}
$$

in $P{\downarrow}\mathbf{B}$. *Obviously, the functor* η_P *preserves cartesianness of arrows, i.e.* η_P *is cartesian.* $\diamondsuit$

Remark.
(1) If $\mathit{Fam}(P)(\varphi)$ is cocartesian w.r.t. $P_{\mathbf{B}}$ then φ is cartesian w.r.t. $\mathit{Fam}(P)$ iff φ is cocartesian w.r.t. $\mathit{Fam}(P)$. Moreover, for every morphism

$$\begin{tikzcd} A \arrow[r, "v"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ I \arrow[r, "u"'] & J \end{tikzcd}$$

in $\mathbf{B^2}$ we have

$$\begin{tikzcd} A \arrow[r, "v"] \arrow[d, equal] \arrow[dr, phantom, "1_v"] & B \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\varphi_b"] & B \arrow[d, "b"] \\ A \arrow[r, "v"'] & B \arrow[r, "b"'] & I \end{tikzcd} = \begin{tikzcd} A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\varphi_a"] & A \arrow[r, "v"] \arrow[d, "a"] & B \arrow[d, "b"] \\ A \arrow[r, "a"'] & I \arrow[r, "u"'] & J \end{tikzcd}$$

where φ_a and φ_b are cocartesian w.r.t. $P_{\mathbf{B}}$. Using these two observations one can show that for fibrations P and Q over $\mathbf{B}$ a cartesian functor $F : \mathit{Fam}(P) \to \mathit{Fam}(Q)$ is determined uniquely up to isomorphism by its restriction along the inclusion $\Delta_{\mathbf{B}} : \mathbf{B} \to \mathbf{B^2}$ from which it follows that F is isomorphic to $\mathit{Fam}(\Delta_{\mathbf{B}}^* F)$. Thus, up to isomorphism all cartesian functors from $\mathit{Fam}(P)$ to $\mathit{Fam}(Q)$ are of the form $\mathit{Fam}(F)$ for some cartesian functor $F : P \to Q$.
(2) Notice, however, that not every cartesian functor $\mathrm{Fam}(P) \to \mathrm{Fam}(Q)$ over $\mathbf{B}$ is isomorphic to one of the form $\mathrm{Fam}(F)$ for some cartesian functor $F : P \to Q$. An example for this failure is the cartesian functor $\mu_P : \mathrm{Fam}^2(P) \to \mathrm{Fam}(P)$ sending $((X, v), u)$ to (X, uv) for nontrivial $\mathbf{B}$.[5]
(3) If $\mathbf{X}$ is a category we write $\mathrm{Fam}(\mathbf{X})$ for the category of families in $\mathbf{X}$ and $\mathit{Fam}(\mathbf{X}) : \mathrm{Fam}(\mathbf{X}) \to \mathbf{Set}$ for the family fibration.

The analogon of (1) in ordinary category theory is that for categories $\mathbf{X}$ and $\mathbf{Y}$ a cartesian functor $F : \mathit{Fam}(\mathbf{X}) \to \mathit{Fam}(\mathbf{Y})$ is isomorphic to $\mathit{Fam}(F_1)$ (the fiber of F at $1 \in \mathbf{Set}$).

The analogon of (2) in ordinary category theory is that not every ordinary functor $F : \mathrm{Fam}(\mathbf{X}) \to \mathrm{Fam}(\mathbf{Y})$ is isomorphic to one of the form $\mathrm{Fam}(G)$ for some $G : \mathbf{X} \to \mathbf{Y}$.

Next we characterise the property of having internal sums in terms of the family monad Fam.

[5] One can show that η and μ are natural transformations giving rise to a monad $(\mathrm{Fam}, \eta, \mu)$ on $\mathbf{Fib}(\mathbf{B})$.

Theorem 6.1 *Let* $\mathbf{B}$ *be a category with pullbacks and* $P : \mathbf{X} \to \mathbf{B}$ *be a fibration. Then* P *has internal sums iff* $\eta_P : P \to \mathrm{Fam}(P)$ *has a fibered left adjoint* $\coprod_P : \mathrm{Fam}(P) \to P$ *, i.e.* $\coprod_P \dashv \eta_P$ *where* $\coprod_P$ *is cartesian and unit and counit of the adjunction are cartesian natural transformations.*

Proof. The universal property of the unit of the adjunction $\coprod_P \dashv \eta_P$ at (u, X) is explicitated in the following diagram

$$
\begin{tikzcd}
 & & Z \arrow[dd, dotted, no head] \\
X \arrow[r, "\eta_{(u,X)}"'] \arrow[urr, "\psi"] \arrow[d, dotted, no head] & Y \arrow[ur, dashed, "\theta"'] \arrow[d, dotted, no head] & \\
I \arrow[r, "u"] \arrow[d, "u"'] & J \arrow[r, "v"] \arrow[d, equal] & K \arrow[d, equal] \\
J \arrow[r, equal] & J \arrow[r, "v"'] & K
\end{tikzcd}
$$

whose left column is the unit at (u, X). From this it follows that $\eta_{(u,X)} : X \to Y$ is cocartesian over u.

Cartesianness of $\coprod_P$ says that the cartesian arrow f as given by

$$
\begin{tikzcd}
X \arrow[r, "\psi", "\text{cart}"'] & Y \\
K \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & L \arrow[d, "v"] \\
I \arrow[r, "u"'] & J
\end{tikzcd}
$$

in $P{\downarrow}\mathbf{B}$ is sent by $\coprod_P$ to the cartesian arrow $\coprod_P f$ over u satisfying

$$
\begin{tikzcd}
X \arrow[r, "\psi", "\text{cart}"'] \arrow[d, "\eta_{(p,X)}"'] & Y \arrow[d, "\eta_{(v,Y)}"] \\
A \arrow[r, "\text{cart}", "\coprod_P f"'] & B
\end{tikzcd}
$$

where $\eta_{(p,X)}$ and $\eta_{(v,Y)}$ are the cocartesian units of the adjunction above p and v, respectively. Thus, according to the second remark after Definition 6.1 cartesianness of $\coprod_P$ is just the Beck–Chevalley Condition for internal sums.

On the other hand if P has internal sums then the functor $\coprod_P$ left adjoint to P is given by sending a morphism f in $P{\downarrow}\mathbf{B}$ as given by

$$X \xrightarrow{\psi} Y$$

$$\begin{array}{ccc} K & \xrightarrow{q} & L \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle v} \\ I & \xrightarrow[u]{} & J \end{array}$$

to the morphism $\coprod_P f$ over u satisfying

$$\begin{array}{ccc} X & \xrightarrow{\psi} & Y \\ {\scriptstyle \varphi_1}\downarrow & & \downarrow{\scriptstyle \varphi_2} \\ A & \xrightarrow[\coprod_P f]{} & B \end{array}$$

where φ_1 and φ_2 are cocartesian over p and v, respectively. It is easy to check that $\coprod_P$ is actually left adjoint to η_P using for the units of the adjunction the cocartesian liftings guaranteed for P. Cartesianness of $\coprod_P$ is easily seen to be equivalent to the Beck–Chevalley condition. $\square$

7 Internal Products

Of course, by duality a fibration $P : \mathbf{X} \to \mathbf{B}$ has internal products iff the dual fibration P^{op} has internal sums. After some explicitation (left to the reader) one can see that the property of having internal products can be characterised more elementarily as follows.

Theorem 7.1 *Let* $\mathbf{B}$ *be a category with pullbacks. Then a fibration* $P : \mathbf{X} \to \mathbf{B}$ *has internal products iff the following two conditions are satisfied.*

(i) *For every* $u : I \to J$ *in* $\mathbf{B}$ *and* $X \in P(I)$ *there is a span* $\varphi : u^*E \to E$, $\varepsilon : u^*E \to X$ *with* φ *cartesian over* u *and* ε *vertical such that for every span* $\theta : u^*Z \to Z$, $\alpha : u^*Z \to X$ *with* θ *cartesian over* u *and* α *vertical there is a unique vertical arrow* $\beta : Z \to E$ *such that* $\alpha = \varepsilon \circ u^*\beta$ *where* $u^*\beta$ *is the vertical arrow with* $\varphi \circ u^*\beta = \beta \circ \theta$ *as illustrated in the diagram*

$$\begin{tikzcd} & u^*Z \arrow[r, "\theta", "\mathrm{cart}"'] \arrow[dl, "\alpha"'] \arrow[d, "u^*\beta"] & Z \arrow[d, "\beta"] \\ X & u^*E \arrow[l, "\varepsilon"] \arrow[r, "\mathrm{cart}", "\varphi"'] & E \end{tikzcd}$$

Notice that the span (φ, ε) *is determined uniquely up to vertical isomorphism by this universal property and is called an* evaluation span for X along u.

(ii) *Whenever*

$$\begin{tikzcd} L \arrow[r, "\widetilde{v}"] \arrow[d, "\widetilde{u}"'] \arrow[dr, phantom, "\lrcorner\ (1)"] & I \arrow[d, "u"] \\ K \arrow[r, "v"'] & J \end{tikzcd}$$

is a pullback in $\mathbf{B}$ *and* $\varphi : u^*E \to E$, $\varepsilon : u^*E \to X$ *is an evaluation* *span for X along u then for every diagram*

$$\begin{tikzcd}
\widetilde{v}^*X \arrow[r, "\psi", "\mathrm{cart}"'] & X \\
\widetilde{u}^*\widetilde{E} \arrow[u, "\widetilde{\varepsilon}"] \arrow[r, "\widetilde{\theta}", "\mathrm{cart}"'] \arrow[d, "\widetilde{\varphi}"'] & u^*E \arrow[u, "\varepsilon"'] \arrow[d, "\varphi"] \\
\widetilde{E} \arrow[r, "\mathrm{cart}", "\theta"'] & E
\end{tikzcd}$$

where the lower square is above pullback (1) in $\mathbf{B}$ *and* $\widetilde{\varepsilon}$ *is vertical it holds that* $(\widetilde{\varphi}, \widetilde{\varepsilon})$ *is an evaluation span for* $\widetilde{v}^*X$ *along* $\widetilde{u}$.

Proof. Tedious, but straightforward explicitation of the requirement that P^{op} has internal sums. $\square$

Condition (ii) is called Beck–Chevally Condition (BCC) for internal products and essentially says that evaluation spans are stable under reindexing.

Examples.
(1) $\mathrm{Mon}(\mathcal{E})$ fibered over topos $\mathcal{E}$ has both internal sums and internal products.
(2) For every category $\mathbf{B}$ with pullbacks the fundamental fibration $P_{\mathbf{B}} = \partial_1 : \mathbf{B}^{\mathbf{2}} \to \mathbf{B}$ has internal sums which look as follows

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "a"'] & A \arrow[d, "\coprod_u a"] \\
I \arrow[r, "u"'] & J
\end{tikzcd}$$

The fundamental fibration $P_{\mathbf{B}}$ has internal products iff for every $u : I \to J$ in $\mathbf{B}$ the pullback functor $u^* : \mathbf{B}/J \to \mathbf{B}/I$ has a right adjoint $\prod_u$. For $\mathbf{B} = \mathbf{Set}$ this right adjoint gives *dependent products* (as known from Martin-Löf Type Theory).

Models of Martin–Löf Type Theory

A category $\mathbf{B}$ with finite limits such that its fundamental fibration $P_{\mathbf{B}}$ has internal products—usually called a *locally cartesian closed category*—allows one to interpret Σ, Π and Identity Types of Martin–Löf Type Theory. Dependent sum Σ and dependent product Π are interpreted as internal sums and internal products.The fiberewise diagonal δ_a

$$
\begin{tikzcd}
A \arrow[drr, equal] \arrow[ddr, equal] \arrow[dr, "\delta_a"] & & \\
& A \times_I A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "a"] \\
& A \arrow[r, "a"'] & I
\end{tikzcd}
$$

is used for interpreting identity types: the sequent $i{:}I, x, y{:}A \vdash \mathrm{Id}_A(x,y)$ is interpreted as δ_a when $i{:}I \vdash A$ is interpreted as a.

One may interpret W-types in $\mathbf{B}$ iff for $b : B \to A$ and $a : A \to I$ there is a "least" $w : W \to I$ such that $W \cong \coprod_a \prod_b b^* w$ mimicking on a categorical level the requirement that W is the "least" solution of the recursive type equation $W \cong \Sigma x{:}A.W^{B(x)}$.

8 Fibrations of Finite Limit Categories and Complete Fibrations

Let $\mathbf{B}$ be a category with pullbacks remaining fixed for this section.

Lemma 8.1 *For a fibration $P : \mathbf{X} \to \mathbf{B}$ we have that*

(1) *a commuting square of cartesian arrows in $\mathbf{X}$ over a pullback in $\mathbf{B}$ is always a pullback in $\mathbf{X}$*

(2) *a commuting square*

$$\begin{tikzcd} Y_1 \arrow[r, "\varphi_1", "\mathrm{cart}"'] \arrow[d, "\beta"'] & X_1 \arrow[d, "\alpha"] \\ Y_2 \arrow[r, "\mathrm{cart}", "\varphi_2"'] & X_2 \end{tikzcd}$$

in $\mathbf{X}$ is a pullback in $\mathbf{X}$ whenever the φ_i are cartesian and α and β vertical.

Proof. Straightforward exercise. □

Definition 8.1 *$P : \mathbf{X} \to \mathbf{B}$ is a* fibration of categories with pullbacks *iff every fiber $P(I)$ has pullbacks and these are stable under reindexing along arbitrary morphisms in the base.* ◊

Lemma 8.2 *If $P : \mathbf{X} \to \mathbf{B}$ is a fibration of categories with pullbacks then every pullback in some fiber $P(I)$ is also a pullback in $\mathbf{X}$.*

Proof. Suppose

$$\begin{tikzcd} Z \arrow[r, "\beta_2"] \arrow[d, "\beta_1"'] \arrow[dr, phantom, "\lrcorner\ (\dagger)"] & X_2 \arrow[d, "\alpha_2"] \\ X_1 \arrow[r, "\alpha_1"'] & Y \end{tikzcd}$$

is a pullback in $P(I)$ and θ_1, θ_2 is a cone over α_1, α_2 in $\mathbf{X}$, i.e. $\alpha_1 \circ \theta_1 = \alpha_2 \circ \theta_2$. Obviously, θ_1 and θ_2 are above the same arrow u in $\mathbf{B}$. For $i = 1, 2$ let $\varphi_i : u^* X_i \to X_i$ be a cartesian arrow over u and $\gamma_i : V \to u^* X_i$ be a vertical arrow with $\varphi_i \circ \gamma_i = \theta_i$. As the image of (†) under u^* is a pullback in its fiber there is a vertical arrow γ with $\gamma_i = u^*\beta_i \circ \gamma$ for $i = 1, 2$. The situation is illustrated in the following diagram

$$
\begin{tikzcd}
V \arrow[dr, "\gamma"] \arrow[drr, "\theta"] \arrow[ddr, "\gamma_i"'] & & \\
& u^*Z \arrow[r, "\psi", "\mathrm{cart}"'] \arrow[d, "u^*\beta_i"] & Z \arrow[d, "\beta_i"] \\
& u^*X_i \arrow[r, "\mathrm{cart}", "\varphi_i"'] \arrow[d, "u^*\alpha_i"] & X_i \arrow[d, "\alpha_i"] \\
& u^*Y \arrow[r, "\mathrm{cart}", "\varphi"'] & Y
\end{tikzcd}
$$

where φ and ψ are cartesian over u. From this diagram it is obvious that $\theta := \psi \circ \gamma$ is a mediating arrow as desired. If θ' were another such mediating arrow then for $\theta' = \psi \circ \gamma'$ with γ' vertical it holds that $\gamma' = \gamma$ as both are mediating arrows to $u^*(\dagger)$ for the cone given by γ_1 and γ_2 and, therefore, it follows that $\theta = \theta'$. Thus θ is the unique mediating arrow. $\square$

Now we can give a simple characterisation of fibrations of categories with pullbacks in terms of a preservation property.

Theorem 8.3 *$P : \mathbf{X} \to \mathbf{B}$ is a fibration of categories with pullbacks iff $\mathbf{X}$ has and P preserves pullbacks.*

Proof. Suppose that $P : \mathbf{X} \to \mathbf{B}$ is a fibration of categories with pullbacks. For $i = 1, 2$ let $f_i : Y_i \to X$ be arrows in $\mathbf{X}$ and $f_i = \varphi_i \circ \alpha_i$ be some vertical/cartesian factorisations. Consider the diagram

$$
\begin{tikzcd}
U \arrow[r, "\beta_2"] \arrow[d, "\beta_1"'] \arrow[dr, phantom, "(4)"] & \cdot \arrow[r, "\varphi_1''"] \arrow[d, "\alpha_2'"] \arrow[dr, phantom, "(3)"] & Y_2 \arrow[d, "\alpha_2"] \\
\cdot \arrow[r, "\alpha_1'"] \arrow[d, "\varphi_2''"'] \arrow[dr, phantom, "(2)"] & \cdot \arrow[r, "\varphi_1'"] \arrow[d, "\varphi_2'"] \arrow[dr, phantom, "(1)"] & Z_2 \arrow[d, "\varphi_2"] \\
Y_1 \arrow[r, "\alpha_1"'] & Z_1 \arrow[r, "\varphi_1"'] & X
\end{tikzcd}
$$

where the φ's are cartesian and the α's and β's are vertical. Square (1) is a pullback in $\mathbf{X}$ over a pullback in $\mathbf{B}$ by Lemma 8.1(1). Squares (2) and (3) are pullbacks in $\mathbf{X}$ by Lemma 8.1(2). Square (4) is a pullback in $\mathbf{X}$ by Lemma 8.2. Accordingly, the big square is a pullback in $\mathbf{X}$ over a pullback in $\mathbf{B}$. Thus, $\mathbf{X}$ has and P preserves pullbacks.

For the reverse direction assume that $\mathbf{X}$ has and P preserves pullbacks. Then every fiber of P has pullbacks and they are preserved under reindexing for the following reason. For every pullback

$$\begin{tikzcd}
Z \arrow[r, "\beta_2"] \arrow[d, "\beta_1"'] \arrow[dr, phantom, "(\dagger)"] & X_2 \arrow[d, "\alpha_2"] \\
X_1 \arrow[r, "\alpha_1"'] & Y
\end{tikzcd}$$

in $P(I)$ and $u : J \to I$ in $\mathbf{B}$ by Lemma 8.1 we have

$$\begin{tikzcd}
u^*Z \arrow[r, "\theta", "\text{cart}"'] \arrow[d, "u^*\beta_i"'] & Z \arrow[d, "\beta_i"] \\
u^*X_i \arrow[r, "\text{cart}", "\varphi_i"'] \arrow[d, "u^*\alpha_i"'] & X_i \arrow[d, "\alpha_i"] \\
u^*Y \arrow[r, "\text{cart}", "\varphi"'] & Y
\end{tikzcd}$$

and, therefore, the image of pullback ($\dagger$) under u^* is isomorphic to the pullback of ($\dagger$) along φ in $\mathbf{X}$. As pullback functors preserve pullbacks it follows that the reindexing of ($\dagger$) along u is a pullback, too. $\square$

Definition 8.2 *A fibration $P : \mathbf{X} \to \mathbf{B}$ is a* fibration of categories with terminal objects *iff every fiber $P(I)$ has a terminal object and these are stable under reindexing.* $\Diamond$

One easily sees that P is a fibration of categories with terminal objects iff for every $I \in \mathbf{B}$ there is an object $1_I \in P(I)$ such that for every $u : J \to I$ in $\mathbf{B}$ and $X \in P(J)$ there is a unique arrow $f : X \to 1_I$ in $\mathbf{X}$ over u. Such a 1_I is called an "I–indexed family of terminal objects". It is easy to see that this property is stable under reindexing.

Lemma 8.4 *Let $\mathbf{B}$ have a terminal object (besides having pullbacks). Then $P : \mathbf{X} \to \mathbf{B}$ is a fibration of categories with terminal objects iff $\mathbf{X}$ has a terminal object $1_{\mathbf{X}}$ with $P(1_{\mathbf{X}})$ terminal in $\mathbf{B}$.*

Proof. Simple exercise. $\square$

Theorem 8.5 *For a category* $\mathbf{B}$ *with finite limits a fibration* $P : \mathbf{X} \to \mathbf{B}$ *is a fibration of categories with finite limits, i.e. all fibers of* P *have finite limits preserved by reindexing along arbitrary arrows in the base, iff* $\mathbf{X}$ *has finite limits and* P *preserves them.*

Proof. Immediate from Theorem 8.3 and Lemma 8.4. $\square$

From ordinary category theory one knows that $\mathbf{C}$ has small limits iff $\mathbf{C}$ has finite limits and small products. Accordingly, one may define "completeness" of a fibration over a base category $\mathbf{B}$ with finite limits by the requirements that

(1) P is a fibration of categories with finite limits and

(2) P has internal products (satisfying BCC).

In [Bor] vol.2, Ch.8 it has been shown that for a fibration P complete in the sense above it holds for all $C \in \mathbf{cat}(\mathbf{B})$ that the fibered "diagonal" functor $\Delta_C : P \to [P_C \to P]$ has a fibered right adjoint $\prod_C$ sending diagrams of shape C to their limiting cone (in the appropriate fibered sense). Thus, requirement (2) above is necessary and sufficient for internal completeness under the assumption of requirement (1).

9 Elementary Fibrations and Representability

A fibration $P : \mathbf{X} \to \mathbf{B}$ is called *discrete* iff all its fibers are discrete categories, i.e. iff P reflects identity morphisms. However, already in ordinary category theory discreteness of categories is not stable under equivalence (though, of course, it is stable under isomorphism of categories). Notice that a category $\mathbf{C}$ is equivalent to a discrete one iff it is a *posetal groupoid*, i.e. Hom–sets contain at most one element and all morphisms are isomorphisms. Such categories will be called *elementary*.

This looks even nicer from a fibrational point of view.

Theorem 9.1 *Let $P : \mathbf{X} \to \mathbf{B}$ be a fibration. Then we have*

(1) *P is a fibration of groupoids iff P is* conservative, *i.e. P reflects isomorphism.*

(2) *P is a fibration of posetal categories iff P is faithful.*

(3) *P is a fibration of elementary categories iff P is faithful and reflects isomorphisms.*

Fibrations $P : \mathbf{X} \to \mathbf{B}$ are called elementary *iff P is faithful and reflects isomorphisms.*

Proof. Straightforward exercise. □

It is well known that a presheaf $A : \mathbf{B}^{op} \to \mathbf{Set}$ is representable iff $\int A : \mathbf{Elts}(A) \to \mathbf{B}$ has a terminal object. This motivates the following definition.

Definition 9.1 *An elementary fibration $P : \mathbf{X} \to \mathbf{B}$ is* representable *iff $\mathbf{X}$ has a terminal object, i.e. there is an object $R \in P(I)$ such that for every $X \in \mathbf{X}$ there is a unique classifying morphism $u : P(X) \to I$ in $\mathbf{B}$ with $X \cong u^*R$, i.e. fibration P is equivalent to $P_I = \partial_0 : \mathbf{B}/I \to \mathbf{B}$ for some $I \in \mathbf{B}$, i.e. P is equivalent to some small discrete fibration over $\mathbf{B}$.* ◊

10 Local Smallness

Definition 10.1 *Let $P : \mathbf{X} \to \mathbf{B}$ be a fibration. For objects $X, Y \in P(I)$ let $\mathbf{Hom}_I(X, Y)$ be the category defined as follows. Its objects are spans*

$$\begin{tikzcd} & U \arrow[dl, "\varphi"', "\text{cart}"] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

with $P(\varphi) = P(f)$ and φ cartesian. A morphism from (ψ, g) to (φ, f) is a morphism θ in $\mathbf{X}$ such that $\varphi \circ \theta = \psi$ and $f \circ \theta = g$

$$\begin{tikzcd} & X & \\ V \arrow[ur, "\psi", "\text{cart}"'] \arrow[rr, "\theta"'] \arrow[dr, "g"'] & & U \arrow[ul, "\varphi"', "\text{cart}"] \\ & Y \arrow[ur, "f"'] & \end{tikzcd}$$

Notice that θ is necessarily cartesian and fully determined by $P(\theta)$. The category $\mathbf{Hom}_I(X, Y)$ is fibered over $\mathbf{B}/I$ by sending (φ, f) to $P(\varphi)$ and θ to $P(\theta)$. The fibration P is called locally small *iff for all $X, Y \in P(I)$ the elementary fibration $\mathbf{Hom}_I(X, Y)$ over $\mathbf{B}/I$ is representable, i.e. $\mathbf{Hom}_I(X, Y)$ has a terminal object.* $\diamondsuit$

The intuition behind this definition can be seen as follows. Let (φ_0, f_0) be terminal in $\mathbf{Hom}_I(X, Y)$. Let $d := P(\varphi_0) : \mathrm{hom}_I(X, Y) \to I$. Let $f_0 = \psi_0 \circ \mu_{X,Y}$ with ψ_0 cartesian and $\mu_{X,Y}$ vertical. Then for every $u : J \to I$ and $\alpha : u^*X \to u^*Y$ there exists a unique $v : J \to \mathrm{hom}_I(X, Y)$ with $d \circ v = u$ such that $\alpha = v^*\mu_{X,Y}$ as illustrated in the following diagram.

$$\begin{tikzcd} & X \\ u^*X \arrow[ur, "\varphi"] \arrow[r, "\theta"'] \arrow[d, "\alpha = v^*\mu_{X,Y}"'] & d^*X \arrow[u, "\varphi_0"'] \arrow[d, "\mu_{X,Y}"] \\ u^*Y \arrow[r, "\text{cart}"] \arrow[dr, "\text{cart}"] & d^*Y \arrow[d, "\psi_0"] \\ & Y \end{tikzcd}$$

Theorem 10.1 *Let $P : \mathbf{X} \to \mathbf{B}$ be a locally small fibration and $\mathbf{B}$ have binary products. Then for all objects X, Y in $\mathbf{X}$ there exist morphisms $\varphi_0 : Z_0 \to X$ and $f_0 : Z_0 \to Y$ with φ_0 cartesian such that for morphisms $\varphi : Z \to X$ and $f : Z \to Y$ with φ cartesian there exists a unique $\theta : Z \to Z_0$ making the diagram*

$$\begin{tikzcd}
 & X \\
Z \arrow[ur, "\varphi"] \arrow[r, dashed, "\theta"'] \arrow[dr, "f"'] & Z_0 \arrow[u, "\varphi_0"'] \arrow[d, "f_0"] \\
 & Y
\end{tikzcd}$$

commute.

Proof. Let $p : K \to I$, $q : K \to J$ be a product cone in $\mathbf{B}$. Then the desired span (φ_0, f_0) is obtained by putting

$$\varphi_0 := \varphi_X \circ \widetilde{\varphi} \qquad f_0 := \varphi_Y \circ \widetilde{f}$$

where $(\widetilde{\varphi}, \widetilde{f})$ is terminal in $\mathbf{Hom}_K(p^*X, q^*Y)$ and $\varphi_X : p^*X \to X$ and $\varphi_Y : p^*Y \to Y$ are cartesian over p and q, respectively. We leave it as a straightforward exercise to verify that (φ_0, f_0) satisfies the desired universal property. $\square$

Locally small categories are closed under a lot of constructions as e.g. finite products and functor categories. All these arguments go through for locally small fibrations (see e.g. [Bor] vol. 2, Ch. 8.6). There arises the question what it means that $\mathbf{B}$ fibered over itself is locally small. The answer given by the following Theorem is quite satisfactory.

Theorem 10.2 *Let $\mathbf{B}$ be a category with pullbacks. Then the fundamental fibration $P_{\mathbf{B}} = \partial_0 : \mathbf{B}^{\mathbf{2}} \to \mathbf{B}$ is locally small if and only if for every $u : J \to I$ in $\mathbf{B}$ the pullback functor $u^* : \mathbf{B}/I \to \mathbf{B}/J$ has a right adjoint Π_u or, equivalently, iff every slice of $\mathbf{B}$ is cartesian closed. Such categories are usually called* locally cartesian closed.

Proof. Lengthy but straightforward exercise. $\square$

Some further uses of local smallness are the following.

Observation 10.3 *Let* $\mathbf{B}$ *be a category with an initial object* 0 *and* $P : \mathbf{X} \to \mathbf{B}$ *be a locally small fibration. Then for* $X, Y \in P(0)$ *there is precisely one vertical morphism from* X *to* Y*.*

Proof. Let (φ_0, f_0) be terminal in $\mathbf{Hom}_0(X,Y)$. Then there is a 1–1–correspondence between vertical arrows $\alpha : X \to Y$ and sections θ of φ_0

$$\begin{tikzcd}
 & X \\
X \arrow[ur, equal] \arrow[r, dashed, "\theta"'] \arrow[dr, "\alpha"'] & Z_0 \arrow[u, "\varphi_0"'] \arrow[d, "f_0"] \\
 & Y
\end{tikzcd}$$

As there is precisely one map from 0 to $P(Z_0)$ there is precisely one section θ of φ_0. Accordingly, there is precisely one vertical arrow $\alpha : X \to Y$. $\square$

Observation 10.4 *Let* $P : \mathbf{X} \to \mathbf{B}$ *be a locally small fibration. Then every cartesian arrow over an epimorphism in* $\mathbf{B}$ *is itself an epimorphism in* $\mathbf{X}$*.*

Proof. Let $\varphi : Y \to X$ be cartesian with $P(\varphi)$ epic in $\mathbf{B}$. For φ being epic in $\mathbf{X}$ it suffices to check that φ is epic w.r.t. vertical arrows. Suppose that $\alpha_1 \circ \varphi = \alpha_2 \circ \varphi$ for vertical $\alpha_1, \alpha_2 : X \to Z$. Due to local smallness of P there is a terminal object (φ_0, f_0) in $\mathbf{Hom}_{P(X)}(X,Z)$. Thus, for i=1,2 there are unique cartesian arrows ψ_i with $\varphi_0 \circ \psi_i = id_X$ and $f_0 \circ \psi_i = \alpha_i$. We have

$$\varphi_0 \circ \psi_1 \circ \varphi = \varphi = \varphi_0 \circ \psi_2 \circ \varphi \qquad \text{and}$$

$$f_0 \circ \psi_1 \circ \varphi = \alpha_1 \circ \varphi = \alpha_2 \circ \varphi = f_0 \circ \psi_2 \circ \varphi$$

from which it follows that $\psi_1 \circ \varphi = \psi_2 \circ \varphi$. Thus, $P(\psi_1) \circ P(\varphi) = P(\psi_2) \circ P(\varphi)$ and, therefore, as $P(\varphi)$ is epic by assumption it follows that $P(\psi_1) = P(\psi_2)$. As $\varphi_0 \circ \psi_1 = \varphi_0 \circ \psi_2$ and $P(\psi_1) = P(\psi_2)$ it follows that $\psi_1 = \psi_2$ as φ_0 is cartesian. Thus, we finally get

$$\alpha_1 = f_0 \circ \psi_1 = f_0 \circ \psi_2 = \alpha_2$$

as desired. $\square$

Next we introduce the notion of generating family.

Definition 10.2 *Let $P : \mathbf{X} \to \mathbf{B}$ be a fibration. A* generating family *for P is an object $G \in P(I)$ such that for every parallel pair of distinct vertical arrows $\alpha_1, \alpha_2 : X \to Y$ there exist morphisms $\varphi : Z \to G$ and $\psi : Z \to X$ with φ cartesian and $\alpha_1 \circ \psi \neq \alpha_2 \circ \psi$.*

For locally small fibrations we have the following useful characterisation of generating families provided the base has binary products.

Theorem 10.5 *Let $\mathbf{B}$ have binary products and $P : \mathbf{X} \to \mathbf{B}$ be a locally small fibration. Then $G \in P(I)$ is a generating family for P iff for every $X \in \mathbf{X}$ there are morphisms $\varphi_X : Z_X \to G$ and $\psi_X : Z_X \to X$ such that φ_X is cartesian and ψ_X is* collectively epic *in the sense that vertical arrows $\alpha_1, \alpha_2 : X \to Y$ are equal iff $\alpha_1 \circ \psi_X = \alpha_2 \circ \psi_X$.*

Proof. The implication from right to left is trivial.

For the reverse direction suppose that $G \in P(I)$ is a generating family. Let $X \in P(J)$. As $\mathbf{B}$ is assumed to have binary products by Theorem 10.1 there exist $\varphi_0 : Z_0 \to G$ and $\psi_0 : Z_0 \to X$ with φ_0 cartesian such that for morphisms $\varphi : Z \to G$ and $\psi : Z \to X$ with φ cartesian there exists a unique $\theta : Z \to Z_0$ with

$$\begin{tikzcd}
 & G \\
Z \arrow[ur, "\varphi"] \arrow[r, dashed, "\theta"'] \arrow[dr, "\psi"'] & Z_0 \arrow[u, "\varphi_0"'] \arrow[d, "\psi_0"] \\
 & X
\end{tikzcd}$$

Now assume that $\alpha_1, \alpha_2 : X \to Y$ are distinct vertical arrows. As G is a generating family for P there exist $\varphi : Z \to G$ and $\psi : Z \to X$ with φ cartesian and $\alpha_1 \circ \psi \neq \alpha_2 \circ \psi$. But there is a $\theta : Z \to Z_0$ with $\psi = \psi_0 \circ \theta$. Then we have $\alpha_1 \circ \psi_0 \neq \alpha_2 \circ \psi_0$ (as otherwise $\alpha_1 \circ \psi = \alpha_1 \circ \psi_0 \circ \theta = \alpha_2 \circ \psi_0 \circ \theta = \alpha_2 \circ \psi$). Thus, we have shown that ψ_0 is collectively epic and we may take φ_0 and ψ_0 as φ_X and ψ_X, respectively. $\square$

Intuitively, this means that "every object can be covered by a sum of G_i's" in case the fibration has internal sums.

11 Well-Poweredness

For ordinary categories well-poweredness means that for every object the collection of its subobjects can be indexed by a set. Employing again the notion of representability (of elementary fibrations) we can define a notion of well–poweredness for (a wide class of) fibrations.

Definition 11.1 *Let $P : \mathbf{X} \to \mathbf{B}$ be a fibration where vertical monos are stable under reindexing. For $X \in P(I)$ let $\mathrm{Sub}_I(X)$ be the following category. Its objects are pairs (φ, m) where $\varphi : Y \to X$ is cartesian and $m : S \to Y$ is a vertical mono. A morphism from (ψ, n) to (φ, m) is a morphism θ such that $\varphi \circ \theta = \psi$ and $\theta \circ n = m \circ \widetilde{\theta}$*

$$\begin{tikzcd}
T \arrow[r, "\widetilde{\theta}", "\mathrm{cart}"'] \arrow[d, "n"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "m"] \\
Z \arrow[r, "\mathrm{cart}", "\theta"'] \arrow[dr, "\psi"'] & Y \arrow[d, "\varphi"] \\
 & X
\end{tikzcd}$$

for a (necessarily unique) cartesian arrow $\widetilde{\theta}$. The category $\mathrm{Sub}_I(X)$ is fibered over $\mathbf{B}/I$ by sending objects (φ, m) to $P(\varphi)$ and morphisms θ to $P(\theta)$.

The fibration P is called well–powered *iff for every $I \in \mathbf{B}$ and $X \in P(I)$ the elementary fibration $\mathrm{Sub}_I(X)$ over $\mathbf{B}/I$ is representable, i.e. $\mathrm{Sub}_I(X)$ has a terminal object.* $\diamondsuit$

If (φ_X, m_X) is terminal in $\mathrm{Sub}_I(X)$ then, roughly speaking, for every $u : J \to I$ and $m \in \mathrm{Sub}_{P(J)}(u^*X)$ there is a unique map $v : u \to P(\varphi_X)$ in $\mathbf{B}/I$ with $v^*(m_X) \cong m$. We write $\sigma_X : \mathrm{S}_I(X) \to I$ for $P(\varphi_X)$.

Categories with finite limits whose fundamental fibration is well-powered have the following pleasant characterisation.

Theorem 11.1 *A category $\mathbf{B}$ with finite limits is a topos if and only if its fundamental fibration $P_{\mathbf{B}} = \partial_1 : \mathbf{B^2} \to \mathbf{B}$ is well–powered.*
Thus, in this particular case well–poweredness entails local smallness as every topos is locally cartesian closed.

Proof. Lengthy, but straightforward exercise. □

One may find it reassuring that for categories $\mathbf{B}$ with finite limits we have

$P_{\mathbf{B}}$ is locally small iff $\mathbf{B}$ is locally cartesian closed

$P_{\mathbf{B}}$ is wellpowered iff $\mathbf{B}$ is a topos

i.e. that important properties of $\mathbf{B}$ can be expressed by simple conceptual properties of the corresponding fundamental fibration.

12 Definability

If $\mathbf{C}$ is a category and $(A_i)_{i \in I}$ is a family of objects in $\mathbf{C}$ then for every subcategory $\mathbf{P}$ of $\mathbf{C}$ one may want to form the subset

$$\{i \in I \mid A_i \in \mathbf{P}\}$$

of I consisting of all those indices $i \in I$ such that object A_i belongs to the subcategory $\mathbf{P}$. Though intuitively "clear" it is somewhat disputable from the point of view of *formal axiomatic set theory* (e.g. ZFC or GBN) whether the set $\{i \in I \mid A_i \in \mathbf{P}\}$ actually exists. The reason is that the usual *separation axiom* guarantees the existence of (sub)sets of the form $\{ i \in I \mid P(i) \}$ only for predicates $P(i)$ that can be expressed[6] in the formal language of set theory. Now this may appear as a purely "foundationalist" argument to the working mathematician. However, we don't take any definite position w.r.t. this delicate foundational question but, instead, investigate the mathematically clean concept of *definability* for fibrations.

Definition 12.1 *Let $P : \mathbf{X} \to \mathbf{B}$ be a fibration. A class $\mathcal{C} \subseteq \mathrm{Ob}(\mathbf{X})$ is called* P–stable *or simply* stable *iff for P–cartesian arrows $\varphi : Y \to X$ it holds that $Y \in \mathcal{C}$ whenever $X \in \mathcal{C}$, i.e. iff the class $\mathcal{C}$ is stable under reindexing (w.r.t. P).* $\diamondsuit$

Definition 12.2 *Let $P : \mathbf{X} \to \mathbf{B}$ be a fibration. A stable class $\mathcal{C} \subseteq \mathrm{Ob}(\mathbf{X})$ is called* definable *iff for every $X \in P(I)$ there is a subobject $m_0 : I_0 \rightarrowtail I$ such that*

(1) *$m_0^* X \in \mathcal{C}$ and*

(2) *$u : J \to I$ factors through m_0 whenever $u^* X \in \mathcal{C}$.* $\diamondsuit$

Notice that $u^* X \in \mathcal{C}$ makes sense as stable classes $\mathcal{C} \subseteq \mathrm{Ob}(\mathbf{X})$ are necessarily closed under (vertical) isomorphisms.

Remark. If $\mathcal{C} \subseteq \mathrm{Ob}(\mathbf{X})$ is stable then $\mathcal{C}$ is definable iff for every $X \in P(I)$ the elementary fibration $P_{\mathcal{C},X} : \mathcal{C}_X \to \mathbf{B}/I$ is representable where $\mathcal{C}_X$ is the full subcategory of $\mathbf{X}/X$ on cartesian arrows and $P_{\mathcal{C},X} = P_{/X}$ sends

$$\begin{tikzcd} Z \arrow[rr, "\theta"] \arrow[dr, "\psi"', "\text{cart}"] & & Y \arrow[dl, "\varphi", "\text{cart}"'] \\ & X & \end{tikzcd}$$

[6]i.e. by a first order formula using just the binary relation symbols $=$ and $\in$

in $\mathcal{C}_X$ to

$$\begin{tikzcd} P(Z) \arrow[rr, "P(\theta)"] \arrow[dr, "P(\psi)"'] & & P(Y) \arrow[dl, "P(\varphi)"] \\ & P(X) & \end{tikzcd}$$

in $\mathbf{B}/I$. Representability of the elementary fibration $P_{\mathcal{C},X}$ then means that there is a cartesian arrow $\varphi_0 : X_0 \to X$ with $X_0 \in \mathcal{C}$ such that for every cartesian arrow $\psi : Z \to X$ with $Z \in \mathcal{C}$ there exists a unique arrow $\theta : Z \to X_0$ with $\varphi_0 \circ \theta = \psi$. This θ is necessarily cartesian and, therefore, already determined by $P(\theta)$. From uniqueness of θ it follows immediately that $P(\varphi_0)$ is monic.

One also could describe the situation as follows. Every $X \in P(I)$ gives rise to a subpresheaf C_X of $\mathrm{Y}_{\mathbf{B}}(P(X))$ consisting of the arrows $u : J \to I$ with $u^*X \in \mathcal{C}$. Then $\mathcal{C}$ is definable iff for every $X \in \mathbf{X}$ the presheaf C_X is representable, i.e.

$$\begin{tikzcd} \mathcal{C}_X \arrow[r, tail] \arrow[d, "\cong"] & \mathrm{Y}_{\mathbf{B}}(P(X)) \\ \mathrm{Y}_{\mathbf{B}}(I_X) \arrow[ur, "\mathrm{Y}_{\mathbf{B}}(m_X)"'] & \end{tikzcd}$$

where m_X is monic as $\mathrm{Y}_{\mathbf{B}}$ reflects monos. $\Diamond$

Next we give an example demonstrating that *definability is not vacuously true*. Let $\mathbf{C} = \mathbf{FinSet}$ and $\mathbf{X} = \mathrm{Fam}(\mathbf{C})$ fibered over $\mathbf{Set}$. Let $\mathcal{C} \subseteq \mathrm{Fam}(\mathbf{C})$ consist of those families $(A_i)_{i \in I}$ such that $\exists n \in \mathbb{N}. \forall i \in I. |A_i| \leq n$. Obviously, the class $\mathcal{C}$ is stable but it is not definable as for the family

$$K_n = \{i \in \mathbb{N} \mid i < n\} \qquad (n \in \mathbb{N})$$

there is no greatest subset $P \subseteq \mathbb{N}$ with $\exists n \in \mathbb{N}. \forall i \in P. i < n$. Thus, the requirement of definability is non–trivial already when the base is $\mathbf{Set}$.

For a fibration $P : \mathbf{X} \to \mathbf{B}$ one may consider the fibration $P^{(\mathbf{2})} : \mathbf{X}^{(\mathbf{2})} \to \mathbf{B}$ of (vertical) arrows in $\mathbf{X}$. Thus, it is clear what it means that a class $\mathcal{M} \subseteq \mathrm{Ob}(\mathbf{X}^{(\mathbf{2})})$ is ($P^{(\mathbf{2})}$-)stable. Recall that $\mathrm{Ob}(\mathbf{X}^{(\mathbf{2})})$ is the class of P–vertical arrows of $\mathbf{X}$. Then $\mathcal{M}$ is stable iff for all $\alpha : X \to Y$ in $\mathcal{M}$ and cartesian arrows $\varphi : X' \to X$ and $\psi : Y' \to Y$ over the same arrow u in $\mathbf{B}$ the unique vertical arrow $\alpha' : X' \to Y'$ with $\psi \circ \alpha' = \alpha \circ \varphi$ is in $\mathcal{M}$, too. In other words $u^*\alpha \in \mathcal{M}$ whenever $\alpha \in \mathcal{M}$.

Definition 12.3 *Let $P : \mathbf{X} \to \mathbf{B}$ be a fibration and $\mathcal{M}$ a stable class of vertical arrows in $\mathbf{X}$. Then $\mathcal{M}$ is called* definable *iff for every $\alpha \in P(I)$ there is a subobject $m_0 : I_0 \rightarrowtail I$ such that $m_0^*\alpha \in \mathcal{M}$ and $u : J \to I$ factors through m_0 whenever $u^*\alpha \in \mathcal{M}$.* ♢

Next we discuss what is an appropriate notion of subfibration for a fibration $P : \mathbf{X} \to \mathbf{B}$. Keeping in mind the analogy with Fam($\mathbf{C}$) over **Set** we have to generalise the situation Fam($\mathbf{S}$) $\subseteq$ Fam($\mathbf{C}$) where $\mathbf{S}$ is a subcategory of $\mathbf{C}$ which is *replete* in the sense that Mor($\mathbf{S}$) is stable under composition with isomorphisms in $\mathbf{C}$. In this case the objects of Fam($\mathbf{S}$) are stable under reindexing and so are the vertical arrows of Fam($\mathbf{S}$). This motivates the following

Definition 12.4 *Let $P : \mathbf{X} \to \mathbf{B}$. A* subfibration *of P is given by a subcategory $\mathbf{Z}$ of $\mathbf{X}$ such that*

(1) *cartesian arrows of $\mathbf{X}$ are in $\mathbf{Z}$ whenever their codomain is in $\mathbf{Z}$ (i.e. a cartesian arrow $\varphi : Y \to X$ is in $\mathbf{Z}$ whenever $X \in \mathbf{Z}$) and*

(2) *for every commuting square in $\mathbf{X}$*

$$\begin{tikzcd} X' \arrow[r, "\varphi", "\text{cart}"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "\text{cart}", "\psi"'] & Y \end{tikzcd}$$

the morphism $f' \in \mathbf{Z}$ whenever $f \in \mathbf{Z}$ and φ and ψ are cartesian. ♢

Notice that a subfibration $\mathbf{Z}$ of $P : \mathbf{X} \to \mathbf{B}$ is determined uniquely by $\mathcal{V} \cap \mathbf{Z}$ where $\mathcal{V}$ is the class of vertical arrows of $\mathbf{X}$ w.r.t. P. Thus, $\mathbf{Z}$ gives rise to *replete* subcategories

$$S(I) = \mathbf{Z} \cap P(I) \qquad\qquad (I \in \mathbf{B})$$

which are stable under reindexing in the sense that for $u : J \to I$ in $\mathbf{B}$

(S_{obj}) $\quad u^*X \in S(J)$ whenever $X \in S(I)$ and

(S_{mor}) $\quad u^*\alpha \in S(J)$ whenever $\alpha \in S(I)$.

On the other hand for every such such system $S = (S(I) \mid I \in \mathbf{B})$ of replete subcategories of the fibers of P which is stable under reindexing in the sense that the above conditions (S_{obj}) and (S_{mor}) are satisfied we can define a subfibration $\mathbf{Z}$ of $P : \mathbf{X} \to \mathbf{B}$ as follows: $f : Y \to X$ in $\mathbf{Z}$ iff $X \in S(P(X))$ and $\alpha \in S(P(Y))$ where the diagram

$$\begin{tikzcd} Y \arrow[d, "\alpha"'] \arrow[dr, "f"] & \\ Z \arrow[r, "\mathrm{cart}", "\varphi"'] & X \end{tikzcd}$$

commutes and α is vertical and φ is cartesian. Obviously, this subcategory $\mathbf{Z}$ satisfies condition (1) of Definition 12.4. For condition (2) consider the diagram

$$\begin{tikzcd} X' \arrow[r, "\varphi", "\mathrm{cart}"'] \arrow[d, "\alpha'"'] & X \arrow[d, "\alpha"] \\ Z' \arrow[r, "\mathrm{cart}"'] \arrow[d, "\theta'"'] & Z \arrow[d, "\theta"] \\ Y' \arrow[r, "\mathrm{cart}", "\psi"'] & Y \end{tikzcd}$$

where α' and α are vertical, θ' and θ are cartesian and $f' = \theta' \circ \alpha'$ and $f = \theta \circ \alpha$ from which it is clear that $\alpha' \in S(P(X'))$ whenever $\alpha \in S(P(X))$ and, therefore, $f' \in \mathbf{Z}$ whenever $f \in \mathbf{Z}$.

Now we are ready to define the notion of definability for subfibrations.

Definition 12.5 *A subfibration* $\mathbf{Z}$ *of a fibration* $P : \mathbf{X} \to \mathbf{B}$ *is* definable *iff* $\mathcal{C} = \mathrm{Ob}(\mathbf{Z})$ *and* $\mathcal{M} = \mathcal{V} \cap \mathbf{Z}$ *are definable classes of objects and morphisms, respectively.* $\Diamond$

Without proof we mention a couple of results illustrating the strength of definability. Proofs can be found in [Bor] vol.2, Ch.8.

(1) Locally small fibrations are closed under definable subfibrations.

(2) Let $P : \mathbf{X} \to \mathbf{B}$ be a locally small fibration over $\mathbf{B}$ with finite limits. Then the class of vertical isomorphisms of $\mathbf{X}$ is a definable subclass of objects of $\mathbf{X}^{(\mathbf{2})}$ w.r.t. $P^{(\mathbf{2})}$.

(3) If, moreover, $\mathbf{X}$ has finite limits and P preserves them then vertical monos (w.r.t. their fibers) form a definable subclass of objects of $\mathbf{X}^{(\mathbf{2})}$ w.r.t. $P^{(\mathbf{2})}$ and fiberwise terminal objects form a definable subclass of objects of $\mathbf{X}$ w.r.t. P.

(4) Under the assumptions of (3) for every finite category $\mathbf{D}$ the fiberwise limiting cones of fiberwise $\mathbf{D}$–diagrams from a definable class.

A pleasant consequence of (3) is that under the assumptions of (3) the class of pairs of the form (α, α) for some vertical arrow α form a definable subclass of the objects of $\mathbf{X}^{(\mathbb{G})}$ w.r.t. $P^{(\mathbb{G})}$ where $\mathbb{G}$ is the category with two objects 0 and 1 and two nontrivial arrows from 0 to 1. In other words under the assumptions of (3) *equality of morphisms is definable.*

On the negative side we have to remark that for most fibrations the class $\{(X, X) \mid X \in \mathrm{Ob}(X)\}$ is not definable as a subclass of $\mathbf{X}^{(2)}$ (where $2 = 1+1$ is the discrete category with 2 objects) simply because this class is not even stable (under reindexing). Actually, stability fails already if some of the fibers contains different isomorphic objects! This observation may be interpreted as confirming the old suspicion that equality of objects is somewhat "fishy" at least for non–split fibrations. Notice, however, that even for discrete split fibrations equality need not be definable which can be seen as follows. Consider a presheaf $A \in \widehat{\mathbb{G}}$ (where $\mathbb{G}$ is defined as in the previous paragraph) which may most naturally be considered as a directed graph. Then for A considered as a discrete split fibration equality of objects is definable if and only if A is subterminal, i.e. both $A(1)$ and $A(0)$ contain at most one element. Thus, for interesting graphs equality of objects is not definable!

We conclude this section with the following positive result.

Theorem 12.1 *Let* $\mathbf{B}$ *be a topos and* $P : \mathbf{X} \to \mathbf{B}$ *a fibration. If* $\mathcal{C}$ *is a definable class of objects of* $\mathbf{X}$ *(w.r.t.* P*) then for every cartesian arrow* $\varphi : Y \to X$ *over an epimorphism in* $\mathbf{B}$ *we have that* $X \in \mathcal{C}$ *iff* $Y \in \mathcal{C}$ *(often refered to as "descent property").*

Proof. The implication from left to right follows from stability of $\mathcal{C}$.

For the reverse direction suppose that $\varphi : Y \to X$ is cartesian over an epi e in $\mathbf{B}$. Then by definability of $\mathcal{C}$ we have $e = m \circ f$ where m is a mono in $\mathbf{B}$ with $m^*X \in \mathcal{C}$. But as e is epic and m is monic and we are in a topos it follows that m is an isomorphism and, therefore, $X \cong m^*X \in \mathcal{C}$. $\square$

Notice that this Theorem can be generalised to regular categories $\mathbf{B}$ where, however, one has to require that $P(\varphi)$ is a regular epi (as a monomorphism m in a regular category is an isomorphism if $m \circ f$ is a regular epimorphism for some morphism f in $\mathbf{B}$).

13 Preservation Properties of Change of Base

We know already that for an arbitrary functor $F : \mathbf{A} \to \mathbf{B}$ we have that $F^*P \in \mathbf{Fib}(\mathbf{A})$ whenever $P \in \mathbf{Fib}(\mathbf{B})$. The (2-)functor $F^* : \mathbf{Fib}(\mathbf{B}) \to \mathbf{Fib}(\mathbf{A})$ is known as *change of base along F*. In this section we will characterize those functors F which by change of base along F preserve "all good properties of fibrations".

Lemma 13.1 *Let $F : \mathbf{A} \to \mathbf{B}$ be a functor. Then $F^* : \mathbf{Fib}(\mathbf{B}) \to \mathbf{Fib}(\mathbf{A})$ preserves smallness of fibrations if and only if F has a right adjoint U.*

Proof. Suppose that F has a right adjoint U. If $C \in \mathbf{cat}(\mathbf{B})$ then F^*P_C is isomorphic to $P_{U(C)}$ where $U(C)$ is the image of C under U which preserves all existing limits as it is a right adjoint.

Suppose that F^* preserves smallness of fibrations. Consider for $I \in \mathbf{B}$ the small fibration $P_I = \underline{I} = \partial_0 : \mathbf{B}/I \to \mathbf{B}$. Then F^*P_I is isomorphic to $\partial_0 : F{\downarrow}I \to \mathbf{B}$ which is small iff there exists $U(I) \in \mathbf{A}$ such that $F^*P_I \cong P_{U(I)}$, i.e. $\mathbf{B}(F(-), I) \cong \mathbf{A}(-, U(I))$. Thus, if F^* preserves smallness of fibrations then for all $I \in \mathbf{B}$ we have $\mathbf{B}(F(-), I) \cong \mathbf{A}(-, U(I))$ for some $U(I) \in \mathbf{A}$, i.e. F has a right adjoint U. $\square$

As a consequence of Lemma 13.1 we get that for $u : I \to J$ in $\mathbf{B}$ change of base along $\Sigma_u : \mathbf{B}/I \to \mathbf{B}/J$ preserves smallness of fibrations iff Σ_u has a right adjoint $u^* : \mathbf{B}/J \to \mathbf{B}/I$, i.e. pullbacks in $\mathbf{B}$ along u do exist. Analogously, change of base along $\Sigma_I : \mathbf{B}/I \to \mathbf{B}$ preserves smallness of fibrations iff Σ_I has a right adjoint I^*, i.e. for all $K \in \mathbf{B}$ the cartesian product of I and K exists. One can show that change of base along u^* and I^* is right adjoint to change of base along Σ_u and Σ_I, respectively. Thus, again by Lemma 13.1 change of base along u^* and I^* preserves smallness of fibrations iff u^* and I^* have right adjoints Π_u and Π_I, respectively.

From now on we make the reasonable assumption that all base categories have pullbacks as otherwise their fundamental fibrations would not exist.

Lemma 13.2 *Let $\mathbf{A}$ and $\mathbf{B}$ be categories with pullbacks and $F : \mathbf{A} \to \mathbf{B}$ an arbitrary functor. Then the following conditions are equivalent*

(1) *F preserves pullbacks*

(2) *$F^* : \mathbf{Fib}(\mathbf{B}) \to \mathbf{Fib}(\mathbf{A})$ preserves the property of having internal sums*

(3) *$\partial_1 : \mathbf{B}{\downarrow}F \to \mathbf{A}$ has internal sums.*

Proof. The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are easy. The implication (3) $\Rightarrow$ (1) can be seen as follows. Suppose that the bifibration $\partial_1 : \mathbf{B}{\downarrow}F \to \mathbf{A}$ satisfies BCC then for pullbacks

$$\begin{tikzcd}
L \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & K \arrow[d, "v"] \\
J \arrow[r, "u"'] & I
\end{tikzcd}$$

in $\mathbf{A}$ we have

$$\begin{tikzcd}
F(L) \arrow[rr, "F(q)"] \arrow[dd, equal] \arrow[dr, dashed, "\alpha"', "\cong"] & & F(K) \arrow[dd, equal] \arrow[dr, equal] & \\
& {} \arrow[rr] \arrow[dd] & & F(K) \arrow[dd, "F(v)"] \\
F(L) \arrow[rr, "F(q)" near end] \arrow[dr, "F(p)"'] & & F(K) \arrow[dr, "F(v)"] & \\
& F(J) \arrow[rr, "F(u)"'] & & F(I)
\end{tikzcd}$$

As back and front face of the cube are cartesian arrows and the right face is a cocartesian arrow it follows from the postulated BCC for $\partial_1 : \mathbf{B}{\downarrow}F \to \mathbf{A}$ that the left face is a cocartesian arrow, too. Thus, the map α is an isomorphism from which it follows that

$$\begin{tikzcd}
F(L) \arrow[r, "F(q)"] \arrow[d, "F(p)"'] \arrow[dr, phantom, "\lrcorner", very near start] & F(K) \arrow[d, "F(v)"] \\
F(J) \arrow[r, "F(u)"'] & F(I)
\end{tikzcd}$$

is a pullback as required. $\square$

Thus, by the previous two lemmas a functor $F : \mathbf{A} \to \mathbf{B}$ between categories with pullbacks necessarily has to preserve pullbacks and have a right adjoint U whenever F^* preserves "all good properties of fibrations" as being small and having internal sums certainly are such "good properties".

Actually, as pointed out by Bénabou in his 1980 Louvain-la-Neuve lectures [Ben2] these requirements for F are also sufficient for F^* preserving the following good properties of fibrations

- (co)completeness
- smallness
- local smallness
- definability
- well–poweredness.

We will not prove all these claims but instead discuss *en detail* preservation of local smallness. Already in this case the proof is paradigmatic and the other cases can be proved analogously.

Lemma 13.3 *If $F : \mathbf{A} \to \mathbf{B}$ is a functor with right adjoint U and $\mathbf{A}$ and $\mathbf{B}$ have pullbacks then change of base along F preserves local smallness of fibrations.*

Proof. Suppose $P \in \mathbf{Fib}(\mathbf{B})$ is locally small. Let $X, Y \in P(FI)$ and

$$\begin{tikzcd} & X \\ d^*X \arrow[ur, "\varphi_0", "\mathrm{cart}"'] \arrow[dr, "\psi_0"'] & \\ & Y \end{tikzcd}$$

with $d = P(\varphi_0) = P(\psi_0) : \hom_{FI}(X,Y) \to FI$ be the terminal such span. Then consider the pullback (where we write H for $\hom_{FI}(X,Y)$)

$$\begin{tikzcd} \widetilde{H} \arrow[r, "h"] \arrow[d, "\widetilde{d}"'] \arrow[dr, phantom, "\lrcorner", very near start] & UH \arrow[d, "Ud"] \\ I \arrow[r, "\eta_I"'] & UFI \end{tikzcd}$$

in $\mathbf{A}$ where η_I is the unit of $F \dashv U$ at I. Then there is a natural bijection between $v : u \to \widetilde{d}$ in $\mathbf{A}/I$ and $\widehat{v} : F(u) \to d$ in $\mathbf{B}/FI$ by sending v to $\widehat{v} = \varepsilon_H \circ F(h) \circ F(v)$ where ε_H is the counit of $F \dashv U$ at H.

Let θ_1 be a P-cartesian arrow over $\varepsilon_H \circ F(h) : F\widetilde{H} \to H$ to d^*X. Let $\varphi_1 := \varphi_0 \circ \theta_1$ and $\psi_1 := \psi_0 \circ \theta_1$ which are both mapped by P to

$$d \circ \varepsilon_H \circ F(h) = \varepsilon_{FI} \circ F(Ud) \circ F(h) = \varepsilon_{FI} \circ F\eta_I \circ F\widetilde{d} = F\widetilde{d}$$

We show now that the span $\big((\widetilde{d}, \varphi_1), ((\widetilde{d}, \psi_1)\big)$ is a terminal object in the category $\mathbf{Hom}_I(X, Y)$ for F^*P. For that purpose suppose that $u : J \to I$ in $\mathbf{A}$ and $\varphi : Z \to X$, $\psi : Z \to Y$ are arrows over u w.r.t. F^*P with φ cartesian w.r.t. F^*P.

There exists a unique P-cartesian arrow θ_2 with $\varphi = \varphi_0 \circ \theta_2$ and $\psi = \psi_0 \circ \theta_2$. For $\widehat{v} := P(\theta_2)$ we have $d \circ \widehat{v} = F(u)$ as $P(\varphi) = F(u) = P(\psi)$. Then there exists a unique map $v : u \to \widetilde{d}$ with $\varepsilon_H \circ F(h) \circ F(v) = \widehat{v}$. Now let θ be the unique P-cartesian arrow over $F(v)$ with $\theta_2 = \theta_1 \circ \theta$ which exists as $P(\theta_1) = \varepsilon_H \circ F(h)$ and $P(\theta_2) = \widehat{v} = \varepsilon_H \circ F(h) \circ F(v)$. Thus, we have $v : u \to \widetilde{d}$ and a cartesian arrow θ with $P(\theta) = F(v)$, $\varphi_1 \circ \theta = \varphi$ and $\psi_1 \circ \theta = \psi$ as desired.

For uniqueness of (v, θ) with this property suppose that $v' : u \to \widetilde{d}$ and θ' is a cartesian arrow with $P(\theta') = F(v')$, $\varphi_1 \circ \theta' = \varphi$ and $\psi_1 \circ \theta' = \psi$. From the universal property of (φ_0, ψ_0) it follows that $\theta_2 = \theta_1 \circ \theta'$. Thus, we have

$$\widehat{v} = P(\theta_2) = P(\theta_1) \circ P(\theta') = \varepsilon_H \circ F(h) \circ P(\theta') = \varepsilon_H \circ F(h) \circ F(v')$$

from which it follows that $v = v'$ as by assumption we also have $\widetilde{d} \circ v' = u$. From $\theta_2 = \theta_1 \circ \theta'$ and $P(\theta') = F(v') = F(v)$ it follows that $\theta' = \theta$ because θ_1 is cartesian and we have $\theta_2 = \theta_1 \circ \theta$ and $P(\theta') = F(v)$ due to the construction of θ. $\square$

Analogously one shows that under the same premisses as in Lemma 13.3 the functor F^* preserves well–poweredness of fibrations and that, for fibrations $P : \mathbf{X} \to \mathbf{B}$ and definable classes $\mathcal{C} \subseteq \mathbf{X}$, the class

$$F^*(\mathcal{C}) := \{(I, X) \mid X \in P(FI) \wedge X \in \mathcal{C}\}$$

is definable w.r.t. F^*P.

Warning. If $F \dashv U : \mathbf{E} \to \mathbf{S}$ is an unbounded geometric morphism then $P_\mathbf{E} = \partial_1 : \mathbf{E^2} \to \mathbf{E}$ has a generic family though $P_F \cong F^*P_\mathbf{E}$ does not have a generic family as otherwise by Theorem 17.3 (proved later on) the geometric morphism $F \dashv U$ were bounded! Thus, the property of having a generating family is not preserved by change of base along functors that preserve finite limits and have a right adjoint. In this respect the property of having a small generating family is not as "good" as the other properties of fibrations mentioned before which are stable under change of base along functors that preserve pullbacks and have a right adjoint. $\diamondsuit$

The moral of this section is that functors F between categories with pullbacks preserve "all good" (actually "most good") properties of fibrations by change of base along F if and only if F preserves pullbacks and has a right adjoint. In particular, this holds for inverse image parts of geometric morphisms, i.e. finite limit preserving functors having a right adjoint. But there are many more examples which are also important. Let $\mathbf{B}$ be a category with pullbacks and $u : I \to J$ a morphism in $\mathbf{B}$ then $\Sigma_u : \mathbf{B}/I \to \mathbf{B}/J$ preserves pullbacks and has a right adjoint, namely the pullback functor $u^* : \mathbf{B}/J \to \mathbf{B}/I$, but Σ_u preserves terminal objects if and only if u is an isomorphism. Notice that for $I \in \mathbf{B}$ the functor $\Sigma_I = \partial_0 : \mathbf{B}/I \to \mathbf{B}$ always preserves pullbacks but has a right adjoint I^* if and only if for all $K \in \mathbf{B}$ the cartesian product of I and K exists. Thus, for a category $\mathbf{B}$ with pullbacks the functors $\Sigma_I : \mathbf{B}/I \to \mathbf{B}$ preserve "all good properties" of fibrations by change of base if and only if $\mathbf{B}$ has all binary products (but not necessarily a terminal object!).

A typical such example is the full subcategory $\mathbf{F}$ of $\mathbf{Set}^{\mathbb{N}}$ on those $\mathbb{N}$-indexed families of sets which are empty for almost all indices. Notice, however, that every slice of $\mathbf{F}$ actually is a (Grothendieck) topos. This $\mathbf{F}$ is a typical example for Bénabou's notion of *partial topos*, i.e. a category with binary products where every slice is a topos. The above example can be generalised easily. Let $\mathbf{E}$ be some topos and F be a (downward closed) subset of $\mathrm{Sub}_{\mathbf{E}}(1_{\mathbf{E}})$ then $\mathbf{E}_{/F}$, the full subcategory of $\mathbf{E}$ on those objects A whose terminal projection factors through some subterminal in F, is a partial topos whose subterminal objects form a full reflective subcategory of $\mathbf{E}_{/F}$ and have binary infima.

Exercise. Let $\mathbf{B}$ be an arbitrary category. Let $\mathrm{st}(\mathbf{B})$ be the full subcategory of $\mathbf{B}$ on *subterminal* objects, i.e. objects U such that for every $I \in \mathbf{B}$ there is at most one arrow $I \to U$ (possibly none!). We say that $\mathbf{B}$ *has supports* iff $\mathrm{st}(\mathbf{B})$ is a (full) reflective subcategory of $\mathbf{B}$.

Show that for a category $\mathbf{B}$ having pullbacks and supports it holds that $\mathbf{B}$ has binary products iff $\mathrm{st}(\mathbf{B})$ has binary meets!

14 Adjoints to Change of Base

We first show that for a functor $F : \mathbf{A} \to \mathbf{B}$ there is a left (2-)adjoint $\coprod_F$ and a right (2-)adjoint $\prod_F$ to $F^* : \mathbf{Fib}(\mathbf{B}) \to \mathbf{Fib}(\mathbf{A})$, i.e. change of base along F.

The right (2-)adjoint $\prod_F$ is easier to describe as its behaviour is prescribed by the fibered Yoneda Lemma as

$$\prod_F(P)(I) \simeq \mathbf{Fib}(\mathbf{B})(\underline{I}, \prod_F(P)) \simeq \mathbf{Fib}(\mathbf{A})(F^*\underline{I}, P)$$

for $I \in \mathbf{B}$. Accordingly, one verifies easily that the right adjoint $\prod_F$ to F^* is given by

$$\prod_F(P)(I) = \mathbf{Fib}(\mathbf{A})(F^*\underline{I}, P) \qquad\qquad \prod_F(P)(u) = \mathbf{Fib}(\mathbf{A})(F^*\underline{u}, P)$$

for objects I and morphisms u in $\mathbf{B}$. Obviously, as expected if $\mathbf{B}$ is terminal then $\prod P = \prod_F P$ is the category of all cartesian sections of P.

Notice further that in case F has a right adjoint U then $F^*\underline{I} \cong \underline{UI}$ and, accordingly, we have $\prod_F \simeq U^*$.

We now turn to the description of $\coprod_F$. We consider first the simpler case where $\mathbf{B}$ is terminal. Then one easily checks that for a fibration $P : \mathbf{X} \to \mathbf{A}$ the sum $\coprod P = \coprod_F P$ is given by $\mathbf{X}[\mathrm{Cart}(P)^{-1}]$, i.e. the category obtained from $\mathbf{X}$ be freely inverting all cartesian arrows. This we can extend to the case of arbitrary functors F as follows. For $I \in \mathbf{B}$ consider the pullback $P_{(I)}$ of P along $\partial_1 : I/F \to \mathbf{A}$

$$\begin{tikzcd} \mathbf{X}_{(I)} \arrow[r] \arrow[d, "P_{(I)}"'] \arrow[dr, phantom, "\lrcorner"] & \mathbf{X} \arrow[d, "P"] \\ I/F \arrow[r, "\partial_1"'] & \mathbf{A} \end{tikzcd}$$

and for $u : J \to I$ in $\mathbf{B}$ let G_u the mediating cartesian functor from $P_{(I)}$ to $P_{(J)}$ over $\underline{u}/F$ (precomposition by u) in the diagram

$$\begin{tikzcd} \mathbf{X}_{(I)} \arrow[r, "G_u"] \arrow[d, "P_{(I)}"'] \arrow[dr, phantom, "\lrcorner"] & \mathbf{X}_{(J)} \arrow[r] \arrow[d, "P_{(J)}"'] \arrow[dr, phantom, "\lrcorner"] & \mathbf{X} \arrow[d, "P"] \\ I/F \arrow[r, "\underline{u}/F"'] & J/F \arrow[r, "\partial_1"'] & \mathbf{A} \end{tikzcd}$$

bearing in mind that $\partial_1 \circ \underline{u}/F = \partial_1$. Now the reindexing functor $\coprod_F(u) : \coprod_F(I) \to \coprod_F(J)$ is the unique functor H_u with

$$\begin{tikzcd}
\mathbf{X}_{(I)}[\mathrm{Cart}(P_{(I)})^{-1}] \arrow[r, "H_u"] & \mathbf{X}_{(J)}[\mathrm{Cart}(P_{(I)})^{-1}] \\
\mathbf{X}_{(I)} \arrow[u, "Q_I"] \arrow[r, "G_u"'] & \mathbf{X}_{(J)} \arrow[u, "Q_J"']
\end{tikzcd}$$

which exists as $Q_J \circ G_u$ inverts the cartesian arrows of $X_{(I)}$.

Notice, however, that due to the non–local character[7] of the construction of $\coprod_F$ and $\prod_F$ in general the Beck–Chevalley Condition does not hold for $\coprod$ and $\prod$.

As for adjoint functors $F \dashv U$ we have $\prod_F \simeq U^*$ it follows that $F^* \simeq \coprod_U$.

Now we will consider change of base along distributors. Recall that a distributor ϕ from $\mathbf{A}$ to $\mathbf{B}$ (notation $\phi : \mathbf{A} \nrightarrow \mathbf{B}$) is a functor from $\mathbf{B}^{\mathrm{op}} \times \mathbf{A}$ to $\mathbf{Set}$, or equivalently, a functor from $\mathbf{A}$ to $\widehat{\mathbf{B}} = \mathbf{Set}^{\mathbf{B}^{\mathrm{op}}}$. Of course, up to isomorphism distributors from $\mathbf{A}$ to $\mathbf{B}$ are in 1–1–correspondence with cocontinuous functors from $\widehat{\mathbf{A}}$ to $\widehat{\mathbf{B}}$ (by left Kan extension along $\mathrm{Y}_{\mathbf{A}}$). Composition of distributors is defined in terms of composition of the associated cocontinuous functors.[8] For a functor $F : \mathbf{A} \to \mathbf{B}$ one may define a distributor $\phi_F : \mathbf{A} \nrightarrow \mathbf{B}$ as $\phi_F(B, A) = \mathbf{B}(B, FA)$ and a distributor $\phi^F : \mathbf{B} \nrightarrow \mathbf{A}$ in the reverse direction as $\phi^F(A, B) = \mathbf{B}(FA, B)$. Notice that ϕ_F corresponds to $\mathrm{Y}_{\mathbf{B}} \circ F$ and ϕ^F is right adjoint to ϕ_F.

For a distributor $\phi : \mathbf{A} \to \widehat{\mathbf{B}}$ change of base along ϕ is defined as follows (identifying presheaves over $\mathbf{B}$ with their corresponding discrete fibrations)

$$\phi^*(P)(I) = \mathbf{Fib}(\mathbf{B})(\phi(I), P) \qquad\qquad \phi^*(P)(u) = \mathbf{Fib}(\mathbf{B})(\phi(u), P)$$

for objects I and morphisms u in $\mathbf{A}$. From this definition one easily sees that for a functor $F : \mathbf{A} \to \mathbf{B}$ change of base along ϕ^F coincides with $\prod_F$, i.e. we have

$$(\phi^F)^* P \cong \prod_F P$$

[7]Here we mean that $\mathbf{X}_{(I)}[\mathrm{Cart}(P_{(I)})]$ and $\mathrm{Cart}(F^*\underline{I}, P)$ do not depend only on $P(I)$, the fiber of P over I. This phenomenon already turns up when considering reindexing of presheaves which in general for does not preserve exponentials for example.

[8]As the correspondence between distributors and cocontinuous functors is only up to isomorphism composition of distributors is defined also only up to isomorphism. That is the reason why distributors do form only a bicategory and not an ordinary category!

for all fibrations P over $\mathbf{A}$.

This observation allows us to reduce change of base along distributors to change of base along functors and their right adjoints. The reason is that every distributor $\phi : \mathbf{A} \nrightarrow \mathbf{B}$ can be factorised as a composition of the form $\phi^G \phi_F$.[9] Thus, we obtain

$$\phi^* = (\phi^G \phi_F)^* \simeq (\phi_F)^* (\phi^G)_* \simeq F^* \prod_G$$

as $(\phi_F)^* \simeq F^*$ and $(\phi^G)^* \simeq \prod_G$ and change of base along distributors is functorial in a contravariant way (i.e. $(\phi_2\phi_1)^* \simeq \phi_1^*\phi_2^*$). Thus ϕ^* has a left adjoint $\coprod_\phi = G^* \coprod_F$.

One might ask whether for all distributors $\phi : \mathbf{A} \nrightarrow \mathbf{B}$ there also exists a right adjoint $\prod_\phi$ to ϕ^*. Of course, if $\prod_\phi$ exists then by the fibered Yoneda Lemma it must look as follows

$$(\prod_\phi P)(I) \simeq \mathbf{Fib}(\mathbf{B})(\underline{I}, \prod_\phi P) \simeq \mathbf{Fib}(\mathbf{A})(\phi^* \underline{I}, P)$$

from which it is obvious that it restricts to an adjunction between $\widehat{\mathbf{A}}$ and $\widehat{\mathbf{B}}$ as $\mathbf{Fib}(\mathbf{A})(\phi^*\underline{I}, P)$ is discrete whenever P is discrete. Thus, a necessary condition for the existence of $\prod_\phi$ is that the functor $\phi^* : \widehat{\mathbf{B}} \to \widehat{\mathbf{A}}$ is cocontinuous. As $\phi^* : \widehat{\mathbf{B}} \to \widehat{\mathbf{A}}$ is right adjoint to $\widehat{\phi} : \widehat{\mathbf{A}} \to \widehat{\mathbf{B}}$, the left Kan extension of $\phi : \mathbf{A} \to \widehat{\mathbf{B}}$ along $\mathrm{Y}_{\mathbf{B}}$, the distributor ϕ has a right adjoint if and only if ϕ^* is cocontinuous. Thus, a necessary condition for the existence of $\prod_\phi$ is the existence of a right adjoint distributor to ϕ. This, however, is known to be equivalent (see e.g. [Bor] vol.1) to the requirement that $\phi(A)$ is a retract of a representable presheaf for all objects A in $\mathbf{A}$. In case $\mathbf{B}$ is Cauchy complete, i.e. all idempotents in $\mathbf{B}$ split, this means that up to isomorphism ϕ is of the form ϕ_F for some functor $F : \mathbf{A} \to \mathbf{B}$ and then $\prod_F$ provides a right adjoint to ϕ^*. As $\mathbf{Fib}(\mathbf{B})$ is equivalent to $\mathbf{Fib}(\mathrm{IdSp}(\mathbf{B}))$, where $\mathrm{IdSp}(\mathbf{B})$ is obtained from $\mathbf{B}$ by splitting all idempotents, one can show that ϕ^* has a right adjoint $\prod_\phi$ whenever ϕ has a right adjoint distributor. Thus, for a distributor ϕ the change of base functor ϕ^* has a right adjoint $\prod_\phi$ if and only if ϕ has a right adjoint distributor, i.e. if and only if ϕ is essentially a functor. An example of a distributor ϕ where ϕ^* does not have a right adjoint can be obtained as follows. Let $\mathbf{A}$ be a terminal category and $\mathbf{B}$

[9] Let F and G be the inclusions of $\mathbf{A}$ and $\mathbf{B}$, respectively, into the *display category* $\mathbf{D}_\phi$ of ϕ which is obtained by adjoining to the disjoint union of $\mathbf{A}$ and $\mathbf{B}$ the elements of $\phi(B, A)$ as morphisms from B to A and defining $u \circ x \circ v$ as $\phi(v, u)(x)$ for $u : A \to A'$, $x \in \phi(B, A)$ and $v : B' \to B$.

a small category whose splitting of idempotents does not have a terminal object. Let $\phi : \mathbf{A} \to \widehat{\mathbf{B}}$ select a terminal presheaf from $\widehat{\mathbf{B}}$. Then ϕ^* amounts to the global sections functor on $\widehat{\mathbf{B}}$ which, however, does not have a right adjoint as otherwise $\mathrm{IdSp}(\mathbf{B})$ would have a terminal object.

15 Finite Limit Preserving Functors as Fibrations

If $F : \mathbf{B} \to \mathbf{C}$ is a finite limit preserving functor between categories with finite limits then the fibration

$$P_F \equiv F^* P_{\mathbf{C}} = \partial_1 : \mathbf{C}{\downarrow}F \to \mathbf{B}$$

satisfies the following conditions which later will turn out as sufficient for reconstructing the functor $F : \mathbf{B} \to \mathbf{C}$ up to equivalence.

(1) $\mathbf{C}{\downarrow}F$ has finite limits and P_F preserves them.

(2) P_F has internal sums which are *stable* in the sense that cocartesian arrows are stable under pullbacks along arbitrary morphisms.

(3) The internal sums of P_F are *disjoint* in the sense that for every cocartesian arrow $\varphi : X \to Y$ the fiberwise diagonal δ_φ is cocartesian, too.

$$\begin{tikzcd}
X \arrow[dr, "\delta_\varphi"] \arrow[drr, equal, bend left] \arrow[ddr, equal, bend right] & & \\
& \cdot \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\varphi"] \\
& X \arrow[r, "\varphi"'] & Y
\end{tikzcd}$$

We refrain from giving the detailed verifications of properties (1)–(3). Instead we recall some few basic facts needed intrinsically when verifying the claims.

Notice that a morphism

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\
FI \arrow[r, "Fu"'] & FJ
\end{tikzcd}$$

in $\mathbf{C}{\downarrow}F$ over $u : I \to J$ is cocartesian iff f is an isomorphism.

Notice that pullbacks in $\mathbf{C}{\downarrow}F$ are given by

$$\begin{tikzcd}
D \arrow[rr, "\pi_2"] \arrow[dr, "\pi_1"'] \arrow[dd, "d"'] & & C \arrow[dr, "g"] \arrow[dd, "c"] & \\
& B \arrow[rr, "f"] \arrow[dd, "b"] & & A \arrow[dd, "a"] \\
FL \arrow[rr, "Fq"] \arrow[dr, "Fp"'] & & FK \arrow[dr, "Fv"] & \\
& FJ \arrow[rr, "Fu"'] & & FI
\end{tikzcd}$$

where the top square is a pullback and the bottom square is the image of a pullback under F. From this is is clear that $\partial_0, \partial_1 : \mathbf{C}{\downarrow}F \to \mathbf{B}$ both preserve pullbacks. Condition (3) follows from preservation of pullbacks by ∂_0 and the above characterisation of cocartesian arrows.

Now based on work by J.-L. Moens from his Thése [Moe] we will characterise those fibrations over a category $\mathbf{B}$ with finite limits which up to equivalence are of the form P_F for some finite limit preserving functor F from $\mathbf{B}$ to a category $\mathbf{C}$ with finite limits. It will turn out that the three conditions above are necessary and sufficient. In particular, we will show that the functor F can be recovered from P_F in the following way. First observe that $\partial_0 : \mathbf{C}{\downarrow}F \to \mathbf{C}$ is isomorphic to the functor $\boldsymbol{\Delta} : \mathbf{C}{\downarrow}F \to \mathbf{C}{\downarrow}F1 \cong \mathbf{C}$ given by

$$\begin{tikzcd}
X \arrow[r, "\varphi_X", "\text{cocart}"'] \arrow[d, "f"'] & \boldsymbol{\Delta}(X) \arrow[d, "\boldsymbol{\Delta}(f)"] \\
Y \arrow[r, "\text{cocart}", "\varphi_Y"'] & \boldsymbol{\Delta}(Y)
\end{tikzcd}$$

with $\boldsymbol{\Delta}(f)$ vertical over the terminal object in $\mathbf{B}$. Now the functor F itself can be obtained up to isomorphism as $\Delta = \boldsymbol{\Delta} \circ 1$ (where 1 is the cartesian functor choosing fiberwise terminal objects).

Notice that this construction makes sense for arbitrary fibrations P over $\mathbf{B}$ with internal sums. Our goal now is to show that every fibration P of categories with finite limits over $\mathbf{B}$ with stable disjoint (internal) sums is equivalent to P_Δ where Δ is defined as above and preserves finite limits.

But for this purpose we need a sequence of auxiliary lemmas.

Lemma 15.1 *Let* $\mathbf{B}$ *be category with finite limits and* $P : \mathbf{X} \to \mathbf{B}$ *be a fibration of categories with finite limits and stable disjoint internal sums. Then in*

$$\begin{tikzcd}
U \arrow[dr, "\gamma"] \arrow[drr, "\psi"] \arrow[ddr, equal] & & \\
& W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\varphi"] \\
& U \arrow[r, "\varphi \circ \psi"'] & Y
\end{tikzcd}$$

the arrow γ *is cocartesian whenever* φ *is cocartesian.*

Proof. Consider the diagram

$$\begin{tikzcd}
U \arrow[r, "\psi"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\delta_\varphi"'] \arrow[dr, equal] & \\
W \arrow[r] \arrow[d, "\widetilde{\varphi}"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\varphi"] \\
U \arrow[r, "\psi"'] & X \arrow[r, "\varphi"'] & Y
\end{tikzcd}$$

with $\pi_i \delta_\varphi = id_X$ and $\widetilde{\varphi} \circ \gamma = id_U$. Thus, by stability of sums γ is cocartesian as it appears as pullback of the cocartesian arrow δ_φ. $\square$

Lemma 15.2 *Let* $\mathbf{B}$ *be category with finite limits and* $P : \mathbf{X} \to \mathbf{B}$ *be a fibration of categories with finite limits and stable internal sums, i.e.* P *is also a cofibration whose cocartesian arrows are stable under pullbacks along arbitrary maps in* $\mathbf{X}$.

Then the following conditions are equivalent

(1) *The internal sums of* P *are disjoint.*

(2) *If* φ *and* $\varphi \circ \psi$ *are cocartesian then* ψ *is cocartesian, too.*

(3) *If* α *is vertical and both* φ *and* $\varphi \circ \alpha$ *are cocartesian then* α *is an isomorphism.*

(4) *A commuting diagram*

$$
\begin{tikzcd}
X \arrow[r, "\varphi", "\text{cocart}"'] \arrow[d, "\alpha"'] & U \arrow[d, "\beta"] \\
Y \arrow[r, "\text{cocart}", "\psi"'] & V
\end{tikzcd}
$$

is a pullback in $\mathbf{X}$ *whenever* φ, ψ *are cocartesian and* α, β *are vertical.*

The equivalence of conditions (2)–(4) holds already under the weaker assumption that cocartesian arrows are stable under pullbacks along vertical arrows.

Proof. (1) $\Rightarrow$ (2) : Suppose that both φ and $\varphi \circ \psi$ are cocartesian. Then for the diagram

$$
\begin{tikzcd}
Z \arrow[dr, "\gamma"] \arrow[drr, "\psi"] \arrow[ddr, equal] & & \\
& \cdot \arrow[r, "\theta"'] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\varphi"] \\
& Z \arrow[r, "\varphi \circ \psi"'] & X
\end{tikzcd}
$$

we have that $\psi = \theta \circ \gamma$ is cocartesian as γ is cocartesian by Lemma 15.1 and θ is cocartesian by stability of sums as it appears as pullback of the cocartesian arrow $\varphi \circ \psi$.
(2) $\Rightarrow$ (1) : As π_i and $\pi_i \circ \delta_\varphi = id$ are both cocartesian it follows from assumption (2) that δ_φ is cocartesian, too.
(2) $\Leftrightarrow$ (3) : Obviously, (3) is an instance of (2). For the reverse direction assume (3) and suppose that both φ and $\varphi \circ \psi$ are cocartesian. Let $\psi = \alpha \circ \theta$ with θ cocartesian and α vertical. Then $\varphi \circ \alpha$ is cocartesian from which it follows by (3) that α is a vertical isomorphism and thus $\psi = \alpha \circ \theta$ is cocartesian.
(3) $\Leftrightarrow$ (4) : Obviously, (4) entails (3) instantiating β by identity as isos are stable under pullbacks. For the reverse direction consider the diagram

$$
\begin{tikzcd}
X \arrow[dr, "\iota"] \arrow[drr, "\varphi", "\text{cocart.}"'] \arrow[ddr, "\alpha"'] & & \\
& \cdot \arrow[r, "\theta", "\text{cocart.}"'] \arrow[d, "\pi"] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d, "\beta"] \\
& Y \arrow[r, "\psi"'] & V
\end{tikzcd}
$$

with π vertical. The morphism θ is cocartesian since it arises as pullback of the cocartesian arrow ψ along the vertical arrow β. Moreover, the map ι is vertical since α and π are vertical. Thus, by assumption (3) it follows that ι is an isomorphism. Thus, the outer square is a pullback since it is isomorphic to a pullback square via ι. $\square$

Remark. Alternatively, we could have proved Lemma 15.2 by showing (1) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) where the last three implications have already been established. The implication (1) $\Rightarrow$ (4) was proved in [Moe] as follows. Consider the diagram

$$
\begin{tikzcd}
X \arrow[dr, "\gamma"] \arrow[drr, equal] \arrow[dddr, "\alpha"'] & & \\
& \cdot \arrow[r] \arrow[d, "\theta"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\varphi"] \\
& \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d, "\beta"] \\
& Y \arrow[r, "\psi"'] & V
\end{tikzcd}
$$

where θ is cocartesian by stability of sums since θ appears as pullback of the cocartesian arrow φ. From Lemma 15.1 it follows that γ is cocartesian as by assumption $\beta \circ \varphi = \psi \circ \alpha$ and ψ is cocartesian. Thus, the map $\theta \circ \gamma$ is cocartesian over an isomorphism and, therefore, an isomorphism itself. $\Diamond$

Notice that condition (3) of Lemma 15.2 is equivalent to the requirement that for every map $u : I \to J$ in $\mathbf{B}$ the coproduct functor $\coprod_u : \mathbf{X}_I \to \mathbf{X}_J$ reflects isomorphisms.

As a consequence of Lemma 15.2 we get the following characterisation of disjoint stable sums in terms of extensivity.

Lemma 15.3 *Let $\mathbf{B}$ be category with finite limits and $P : \mathbf{X} \to \mathbf{B}$ be a fibration of categories with finite limits and internal sums. Then the following conditions are equivalent*

(1) *The internal sums of P are stable and disjoint.*

(2) *The internal sums of P are extensive*[10]*, i.e. for all commuting squares*

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & U \arrow[d, "\beta"] \\ Y \arrow[r, "\text{cocart}", "\psi"'] & V \end{tikzcd}$$

where ψ is cocartesian and α and β are vertical it holds that φ is cocartesian iff the square is a pullback.

(3) *The internal sums of P are extensive in the sense of Lawvere*[11]*, i.e. for all commuting squares*

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & U \arrow[d, "\beta"] \\ 1_I \arrow[r, "\text{cocart}", "\varphi_I"'] & \coprod_I 1_I \end{tikzcd}$$

[10] Recall that a category with pullbacks and sums is called *extensive* iff for every family of squares

$$\begin{tikzcd} B_i \arrow[r, "f_i"] \arrow[d, "a_i"'] & B \arrow[d, "b"] \\ A_i \arrow[r, "\mathrm{in}_i"'] & \coprod_{i \in I} A_i \end{tikzcd}$$

all squares are pullbacks iff $f_i : B_i \to B$ is a coproduct cone.

[11] Recall that a category $\mathbf{C}$ is extensive in the sense of Lawvere iff for all sets I the categories $\mathbf{C}^I$ and $\mathbf{C}/\coprod_I 1$ are canonically isomorphic.

where φ_I is cocartesian over $!_I : I \to 1$ in $\mathbf{B}$, 1_I is terminal in its fiber and α and β are vertical it holds that φ is cocartesian iff the square is a pullback.

The equivalence of (2) and (3) holds already under the weaker assumption that cocartesian arrows are stable under pullbacks along vertical arrows.

Proof. (1)$\Leftrightarrow$(2) : The implication from right to left in (2) is just stability of internal sums. The implication from left to right in (2) is just condition (4) of Lemma 15.2 which under assumption of stability of sums by Lemma 15.2 is equivalent to the disjointness of sums.

Obviously, condition (3) is an instance of condition (2). Thus it remains to show that (3) entails (2).

Consider the diagram

$$\begin{tikzcd} V \arrow[r, "\psi_0"] \arrow[d, "\beta"'] \arrow[dr, phantom, "(*)"] & V_0 \arrow[d, "\beta_0"] \\ U \arrow[r, "\text{cocart}", "\varphi_0"'] \arrow[d, "\gamma"'] \arrow[dr, phantom, very near start, "\lrcorner"] & U_0 \arrow[d, "\gamma_0"] \\ 1_I \arrow[r, "\text{cocart}", "\varphi_I"'] & \coprod_I 1_I \end{tikzcd}$$

where β, β_0, γ and γ_0 are vertical, φ_I is cocartesian over $!_I$ and 1_I is terminal in its fiber. The lower square is a pullback due to assumption (3). If the upper square is a pullback then the outer rectangle is a pullback and thus ψ_0 is cocartesian by (3). If ψ_0 is cocartesian then the outer rectangle is a pullback by (3) and thus the upper square is a pullback, too.

Thus we have shown that

($\dagger$) a diagram of the form $(*)$ is a pullback iff ψ_0 is cocartesian.

Now consider a commuting diagram

$$\begin{tikzcd} Y \arrow[r, "\psi"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "(+)"] & V \arrow[d, "\beta"] \\ X \arrow[r, "\text{cocart}", "\varphi"'] & U \end{tikzcd}$$

with α and β vertical.

We have to show that ψ is cocartesian iff (+) is a pullback.

Suppose ψ is cocartesian. Then by (†) the outer rectangle and the right square in

$$\begin{tikzcd} Y \arrow[r, "\psi", "\mathrm{cocart}"'] \arrow[d, "\alpha"'] & V \arrow[r, "\psi_0", "\mathrm{cocart}"'] \arrow[d, "\beta"] & V_0 \arrow[d, "\beta_0"] \\ X \arrow[r, "\mathrm{cocart}", "\varphi"'] & U \arrow[r, "\mathrm{cocart}", "\varphi_0"'] & U_0 \end{tikzcd}$$

are pullbacks from which it follows that the left square, i.e. (+), is a pullback, too, as desired.

Suppose the square (+) is a pullback. Then we have

$$\begin{tikzcd} Y \arrow[r, "\psi"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[r, "\psi_0", "\mathrm{cocart}"'] \arrow[d, "\beta"] & V_0 \arrow[d, "\beta_0"] \\ X \arrow[r, "\mathrm{cocart}", "\varphi"'] & U \arrow[r, "\mathrm{cocart}", "\varphi_0"'] & U_0 \end{tikzcd}$$

As by (†) the right square is a pullback it follows that the outer rectangle is a pullback, too, from which it follows by (†) that $\psi_0\psi$ is cocartesian. Now consider the diagram

$$\begin{tikzcd} Y \arrow[r, "\mathrm{cocart}", "\theta"'] \arrow[dr, "\psi"'] & Z \arrow[r, "\mathrm{cocart}", "\theta_0"'] \arrow[d, "\iota"] & Z_0 \arrow[d, "\iota_0"] \\ & V \arrow[r, "\psi_0"'] & V_0 \end{tikzcd}$$

where ι and ι_0 are vertical. Then ι_0 is an isomorphism because $\theta_0\theta$ and $\psi_0\psi$ start from the same source and are both cocartesian over the same arrow in $\mathbf{B}$. By (†) the right square is a pullback from which it follows that ι is an isomorphism (as isomorphisms are pullback stable) and thus ψ is cocartesian as desired. $\square$

Notice that condition (3) of Lemma 15.3 is equivalent to the requirements that for all $I \in \mathbf{B}$ the coproduct functor $\coprod_I : \mathbf{X}_I \to \mathbf{X}_1$ reflects isomorphisms and $\beta^*\varphi_I$ is cocartesian for all vertical maps $\beta : U \to \coprod_I 1_I$.

An immediate consequence of Lemma 15.3 is the following

Corollary 15.4 *Let* $\mathbf{B}$ *have finite limits and* $P : \mathbf{X} \to \mathbf{B}$ *be a fibration of categories with finite limits and stable disjoint internal sums. Then for every* $u : I \to J$ *in* $\mathbf{B}$ *and* $X \in P(I)$ *the functor* $\coprod_u /X : \mathbf{X}_I/X \to \mathbf{X}_J/\coprod_u X$ *is an equivalence. In particular, we get that* $\mathbf{X}_I \cong \mathbf{X}_I/1_I$ *is equivalent to* $\mathbf{X}_J/\coprod_u 1_I$ *via* $\coprod_u /1_I$ *and that* $\mathbf{X}_I \cong \mathbf{X}_I/1_I$ *is equivalent to* $\mathbf{X}_1/\Delta(I)$ *via* $\coprod_I /1_I$ *where* $\Delta(I) = \coprod_I 1_I$.

Corollary 15.5 *Let* $\mathbf{B}$ *have finite limits and* $P : \mathbf{X} \to \mathbf{B}$ *be fibration of categories with finite limits and stable disjoint internal sums. Then for every* $u : I \to J$ *in* $\mathbf{B}$ *the functor* $\coprod_u : \mathbf{X}_I \to \mathbf{X}_J$ *preserves pullbacks.*

Proof. Notice that $\coprod_u = \Sigma_{\mathbf{X}_J/\coprod_u 1_I} \circ \coprod_u /1_I$ where we identify $\mathbf{X}_I$ and $\mathbf{X}_I/1_I$ via their canonical isomorphism. The functor $\coprod_u /1_I$ preserves pullbacks as it is an equivalence by Corollary 15.4. The functor $\Sigma_{\mathbf{X}_J/\coprod_u 1_I} = \partial_0$ is known to preserve pullbacks anyway. Thus, the functor $\coprod_u$ preserves pullbacks as it arises as the composite of pullback preserving functors. $\square$

Lemma 15.6 *Let* $P : \mathbf{X} \to \mathbf{B}$ *be a fibration of categories with finite limits and stable disjoint internal sums. Then the mediating arrow* θ *is cocartesian for any diagram in* $\mathbf{X}$

$$\begin{tikzcd}
U \arrow[rr, "\phi_2"] \arrow[dr, dashed, "\theta"'] \arrow[dd, "\phi_1"'] & & X_2 \arrow[d, "\varphi_2"] \\
& V \arrow[r, "\beta_2"] \arrow[d, "\beta_1"'] & Y_2 \arrow[d, "\alpha_2"] \\
X_1 \arrow[r, "\varphi_1"'] & Y_1 \arrow[r, "\alpha_1"'] & Y
\end{tikzcd}$$

whenever the φ_i *are cocartesian, the* α_i*,* β_i *are vertical and the outer and the inner square are pullbacks.*

Proof. Consider the diagram

$$
\begin{tikzcd}
U \arrow[r, "\psi_2"] \arrow[d, "\psi_1"'] \arrow[dr, phantom, "\lrcorner"] & U_2 \arrow[r, "\gamma_2"] \arrow[d, "\theta_2"'] \arrow[dr, phantom, "\lrcorner"] & X_2 \arrow[d, "\varphi_2"] \\
U_1 \arrow[r, "\theta_1"] \arrow[d, "\gamma_1"'] \arrow[dr, phantom, "\lrcorner"] & V \arrow[r, "\beta_2"] \arrow[d, "\beta_1"'] \arrow[dr, phantom, "\lrcorner"] & Y_2 \arrow[d, "\alpha_2"] \\
X_1 \arrow[r, "\varphi_1"'] & Y_1 \arrow[r, "\alpha_1"'] & Y
\end{tikzcd}
$$

where by stability of sums the ψ_i and θ_i are cocartesian as they arise as pullbacks of φ_1 or φ_2, respectively. As the big outer square is a pullback we may assume that $\phi_i = \gamma_i \circ \psi_i$ (by appropriate choice of the ψ_i).

Thus, $\theta = \theta_1 \circ \psi_1 = \theta_2 \circ \psi_2$ cocartesian as it arises as composition of cocartesian arrows. $\square$

Lemma 15.7 *Let* $\mathbf{B}$ *have finite limits and* $P : \mathbf{X} \to \mathbf{B}$ *be fibration of categories with finite limits and stable disjoint internal sums.*

Then the functor $\mathbf{\Delta} : \mathbf{X} \to \mathbf{X}_1$ *given by*

$$
\begin{tikzcd}
X \arrow[r, "\varphi_X", "\text{cocart}"'] \arrow[d, "f"'] & \mathbf{\Delta}(X) \arrow[d, dashed, "\mathbf{\Delta}(f)"] \\
Y \arrow[r, "\text{cocart}", "\varphi_Y"'] & \mathbf{\Delta}(Y)
\end{tikzcd}
$$

with $\mathbf{\Delta}(f)$ *over* 1 *preserves finite limits.*

Proof. Clearly, the functor $\mathbf{\Delta}$ preserves the terminal object. It remains to show that it preserves also pullbacks. Let

$$
\begin{tikzcd}
U \arrow[r, "g_2"] \arrow[d, "g_1"'] \arrow[dr, phantom, "\lrcorner"] & X_2 \arrow[d, "f_2"] \\
X_1 \arrow[r, "f_1"'] & Y
\end{tikzcd}
$$

be a pullback in $\mathbf{X}$. Then by Lemma 15.6 the arrow θ is cocartesian in

$$\begin{tikzcd}
U \arrow[rr, "g_2"] \arrow[dd, "g_1"'] \arrow[dr, dashed, "\theta"'] \arrow[dr, phantom, "\lrcorner", very near start] & & X_2 \arrow[d, "\varphi_2"] \\
& V \arrow[r, "\beta_2"] \arrow[d, "\beta_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y_2 \arrow[d, "\alpha_2"] \\
X_1 \arrow[r, "\varphi_1"'] & Y_1 \arrow[r, "\alpha_1"'] & Y
\end{tikzcd}$$

where $f_i = \alpha_i \circ \varphi_i$ with α_i vertical and φ_i cocartesian. From this we get that the square

$$\begin{tikzcd}
\mathbf{\Delta}(U) \arrow[r, "\mathbf{\Delta}(g_2)"] \arrow[d, "\mathbf{\Delta}(g_1)"'] & \mathbf{\Delta}(X_2) \arrow[d, "\mathbf{\Delta}(f_2)"] \\
\mathbf{\Delta}(X_1) \arrow[r, "\mathbf{\Delta}(f_1)"'] & \mathbf{\Delta}(Y)
\end{tikzcd}$$

is a pullback, too, as it is obtained by applying the pullback preserving functor $\coprod_{P(Y)}$ to

$$\begin{tikzcd}
V \arrow[r, "\beta_2"] \arrow[d, "\beta_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y_2 \arrow[d, "\alpha_2"] \\
Y_1 \arrow[r, "\alpha_1"'] & Y
\end{tikzcd}$$

which is a pullback in the fiber over $P(Y)$. $\square$

Now we are ready to prove Moens's Theorem.

Theorem 15.8 *Let* $\mathbf{B}$ *have finite limits and* $P : \mathbf{X} \to \mathbf{B}$ *be fibration of categories with finite limits and stable disjoint internal sums. Then* P *is equivalent to* P_Δ *where* Δ *is the finite limit preserving functor* $\mathbf{\Delta} \circ 1$*.*

More explicitely, the fibered equivalence $E : P \to P_\Delta$ *is given by sending* $f : X \to Y$ *in* $\mathbf{X}$ *over* $u : I \to J$ *to*

$$\begin{tikzcd}
\mathbf{\Delta}(X) \arrow[r, "\mathbf{\Delta}(f)"] \arrow[d, "\mathbf{\Delta}(\alpha)"'] \arrow[dr, phantom, "E(f)"] & \mathbf{\Delta}(Y) \arrow[d, "\mathbf{\Delta}(\beta)"] \\
\mathbf{\Delta}(1_I) \arrow[r, "\Delta(u)"'] & \mathbf{\Delta}(1_J)
\end{tikzcd}$$

where α and β are terminal projections in their fibers.

Proof. As $\Delta(u) = \mathbf{\Delta}(1_u)$ the map $E(f)$ arises as the image under $\mathbf{\Delta}$ of the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"] \\ 1_I \arrow[r, "1_u"'] & 1_J \end{tikzcd}$$

which is a pullback if f is cartesian. As by Lemma 15.7 the functor $\mathbf{\Delta}$ preserves pullbacks it follows that E is cartesian. Thus, the fibered functor E is a fibered equivalence as by Corollary 15.4 all fibers of E are (ordinary) equivalences. $\square$

Thus, for categories $\mathbf{B}$ with finite limits we have established a 1–1–correspondence up to equivalence between fibrations of the form $P_F = \partial_1 : \mathbf{C}{\downarrow}F \to \mathbf{B}$ for some finite limit preserving $F : \mathbf{B} \to \mathbf{C}$ where $\mathbf{C}$ has finite limits and fibrations over $\mathbf{B}$ of categories with finite limits and extensive internal sums.[12]

With little effort we get the following generalization of Moens's Theorem.

Theorem 15.9 *Let $\mathbf{B}$ be a category with finite limits. If $\mathbf{C}$ is a category with finite limits and $F : \mathbf{B} \to \mathbf{C}$ preserves terminal objects then P_F is a fibration of finite limit categories and a cofibration where cocartesian arrows are stable under pullbacks along vertical arrows and one of the following equivalent conditions holds*

(1) *if φ and $\varphi \circ \psi$ are cocartesian then ψ is cocartesian*

(2) *if α is vertical and both φ and $\varphi \circ \alpha$ are cocartesian then α is an isomorphism*

[12] This may explain why Lawvere's notion of extensive sums is so important. Notice, however, that Lawvere's original definition only applied to ordinary categories $\mathbf{C}$ with small coproducts in the ordinary sense. That our notion of Lawvere extensivity is slightly more general can be seen from the discussion at the end of section 17 where we give an example (due to Peter Johnstone) of a fibration over **Set** of categories with finite limits and Lawvere extensive small sums which, however, is not of the form Fam($\mathbf{C}$) for some ordinary category $\mathbf{C}$.

(3) *a commuting square*

$$\begin{tikzcd} X \arrow[r, "\varphi", "\text{cocart}"'] \arrow[d, "\alpha"'] & U \arrow[d, "\beta"] \\ Y \arrow[r, "\text{cocart}", "\psi"'] & V \end{tikzcd}$$

is a pullback whenever ψ and φ are cocartesian and α and β are vertical.

Bifibrations P satisfying these properties are equivalent to P_{Δ_P} where Δ_P is the functor $\Delta : \mathbf{B} \to \mathbf{X}_1$ sending I to $\Delta(I) = \coprod_I 1_I$ and $u : J \to I$ to the unique vertical arrow $\Delta(u)$ rendering the diagram

$$\begin{tikzcd} 1_J \arrow[r, "\varphi_J"] \arrow[d, "1_u"'] & \Delta(J) \arrow[d, "\Delta(u)"] \\ 1_I \arrow[r, "\text{cocart}", "\varphi_I"'] & \Delta(I) \end{tikzcd}$$

commutive. This correspondence is an equivalence since Δ_{P_F} is isomorphic to F.

Proof. We have already seen that these conditions are necessary and the equivalence of (1)–(3) follows from Lemma 15.2.

For the reverse direction first observe that the assumptions on P imply that every commuting square

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & U \arrow[d, "\beta"] \\ 1_I \arrow[r, "\text{cocart}", "\varphi_I"'] & \Delta(I) \end{tikzcd}$$

with α and β vertical is a pullback iff φ is cocartesian. Thus pullback along the cocartesian arrow $\varphi_I : 1_I \to \coprod_I 1_I = \Delta(I)$ induces an equivalence between X_I and $\mathbf{X}_1/\coprod_I 1_I$. This extends to an equivalence between P and

$P_\Delta : \mathbf{B} \to \mathbf{X}_1$) since

$$\begin{tikzcd}
\mathbf{X}_J & \mathbf{X}_1/\Delta(J) \arrow[l, "\varphi_J^*"', "\simeq"] \\
\mathbf{X}_I \arrow[u, "u^*"] & \mathbf{X}_1/\Delta(I) \arrow[l, "\simeq"', "\varphi_I^*"] \arrow[u, "\Delta(u)^*"']
\end{tikzcd}$$

commutes up to isomorphism for all $u : J \to I$ in $\mathbf{B}$. $\square$

As apparent from the proof fibrations $P : \mathbf{X} \to \mathbf{B}$ over a finite limit category $\mathbf{B}$ are equivalent to P_F for some terminal object preserving functor F to a finite limit category if and only if P is a bifibration such that $\mathbf{X}$ has and P preserves finite limits and every commuting square of the form

$$\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & U \arrow[d, "\beta"] \\
1_I \arrow[r, "\text{cocart}", "\varphi_I"'] & \Delta(I)
\end{tikzcd}$$

with α and β vertical is a pullback iff φ is cocartesian.

Finally we discuss how the fact that finite limit preserving functors are closed under composition is reflected on the level of their fibrations associated via glueing. Suppose that $F : \mathbf{B} \to \mathbf{C}$ and $G : \mathbf{C} \to \mathbf{D}$ are finite limit preserving functors between categories with finite limits. Then $P_{GF} \cong 1^* F^* \mathit{Fam}(P_G)$ as indicated in

$$\begin{tikzcd}
\mathbf{D}{\downarrow}G{\circ}F \arrow[r, hook] \arrow[d, "P_{GF}"] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[r] \arrow[d, "F^*\mathit{Fam}(P_G)"] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[r] \arrow[d, "\mathit{Fam}(P_G)"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{D}{\downarrow}G \arrow[r] \arrow[d, "P_G"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{D^2} \arrow[d, "\partial_1"] \\
\mathbf{B} \arrow[r, hook, "1"'] \arrow[dr, equal] & \mathbf{C}{\downarrow}F \arrow[r, "\partial_1^* F"'] \arrow[d, "P_F"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{C^2} \arrow[r, "\partial_0"'] \arrow[d, "\partial_1"] & \mathbf{C} \arrow[r, "G"'] & \mathbf{D} \\
& \mathbf{B} \arrow[r, "F"'] & \mathbf{C} & &
\end{tikzcd}$$

because $\partial_0 \circ \partial_1^* F \circ 1 = \partial_0 \circ 1 = F$. The fibration $F^* \mathit{Fam}(P_G)$ is P_G shifted from $\mathbf{C}$ to $\mathbf{C}{\downarrow}F$ via change of base along $\boldsymbol{\Delta} = \partial_0 = \partial_0 \circ \partial_1^* F : \mathbf{C}{\downarrow}F \to \mathbf{C}$.

The fibration P_{GF} appears as a(n in general proper) subfibration of the composite fibration $P_F \circ F^* Fam(P_G)$.

A fibration $Q : \mathbf{Y} \to \mathbf{C}{\downarrow}F$ is isomorphic to one of the form $F^* Fam(P_G)$ iff Q is a fibration of categories with finite limits and stable disjoint internal sums such that $\mathbf{\Delta} : \mathbf{Y} \to \mathbf{Y}_1$ is isomorphic to one of the form $G \circ \partial_0$, i.e. iff $\mathbf{\Delta}$ inverts cocartesian arrows of $C{\downarrow}F$. This latter condition is equivalent to the requirement that 1_φ is cocartesian w.r.t. Q whenever φ is cocartesian w.r.t. P_F.[13] This fails e.g. for $Q \equiv P_{Id_{\mathbf{C}\downarrow F}}$ if not all cocartesian arrows of $\mathbf{C}{\downarrow}F$ are isomorphisms, i.e. $\mathbf{B}$ is not equivalent to the trivial category $\mathbf{1}$.

[13]As $\mathbf{\Delta}(\varphi)$ is an isomorphism iff 1_φ is cocartesian. This can be seen from the diagram

$$\begin{tikzcd} 1_X \arrow[r, "\varphi_X", "\text{cocart}"'] \arrow[d, "1_\varphi"'] & \mathbf{\Delta}(X) \arrow[d, "\mathbf{\Delta}(\varphi)"] \\ 1_Y \arrow[r, "\text{cocart}", "\varphi_Y"'] & \mathbf{\Delta}(Y) \end{tikzcd}$$

where φ_X and φ_Y are cocartesian over the terminal projections of X and Y, respectively, and $\mathbf{\Delta}(\varphi)$ is vertical. If 1_φ is cocartesian then $\mathbf{\Delta}(\varphi)$ is an isomorphism as it is vertical and cocartesian. On the other hand if $\mathbf{\Delta}(\varphi)$ is an isomorphism then $\mathbf{\Delta}(\varphi) \circ \varphi_X$ is cocartesian, too, and thus by Lemma 15.2(2) it follows that 1_φ is cocartesian.

16 Geometric Morphisms as Fibrations

Geometric morphism are adjunctions $F \dashv U : \mathbf{C} \to \mathbf{B}$ where F preserves finite limits. Though introduced originally for toposes the notion of geometric morphism makes sense already if $\mathbf{B}$ and $\mathbf{C}$ have finite limits.

First we will characterise for functors F between categories with finite limits the property that F has a right adjoint in terms of a purely fibrational property of its associated fibration $P_F = F^*P_{\mathbf{C}}$, namely that of having *small global sections*.

First we observe that the requirement $P \dashv 1 \dashv G$ is equivalent to P having small global sections since $1 \dashv G$ says that for every $X \in P(I)$ there is an $\varepsilon_X : 1_{GX} \to X$ such that for every $\sigma : 1_J \to X$ over $u : J \to I$ there is a unique $v : J \to GX$ with

$$
\begin{tikzcd}
 & 1_I \\
1_J \arrow[ur, "1_u"] \arrow[r, dashed, "1_v"] \arrow[dr, "\sigma"'] & 1_{GX} \arrow[u, "1_{P(\varepsilon_X)}"'] \arrow[d, "\varepsilon_X"] \\
 & X
\end{tikzcd}
$$

i.e. that $\mathbf{Hom}_I(1_I, X)$ is representable. If P is a fibration of cartesian closed categories (or even a fibered topos) then P has small global sections iff P is locally small.

Theorem 16.1 *Let $F : \mathbf{B} \to \mathbf{C}$ be a functor between categories with finite limits. Then F has a right adjoint U iff the fibration P_F has small global sections, i.e. $P_F \dashv 1 \dashv G$.*

Proof. Suppose that F has a right adjoint U. We show that $1 \dashv G$ by exhibiting its counit $\widetilde{\varepsilon}_a$ for an arbitrary object $a : A \to FI$ in $\mathbf{C}{\downarrow}F$. For this purpose consider the pullback

$$
\begin{tikzcd}
C \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & UA \arrow[d, "Ua"] \\
I \arrow[r, "\eta_I"'] & UFI
\end{tikzcd}
$$

where η_I is the unit of $F \dashv U$ at $I \in \mathbf{B}$. Then for the transpose $\widehat{q} = \varepsilon_A \circ Fq : FC \to A$ of q we have

$$\begin{tikzcd}
FC \arrow[rr, "\widehat{q}"] \arrow[dr, "Fp"'] & & A \arrow[dl, "a"] \\
& FI &
\end{tikzcd}$$

We show that $(p, \widehat{q}) : 1_C \to a$ is the desired counit $\widetilde{\varepsilon}_a$ of $1 \dashv G$ at a. Suppose that $(u, s) : 1_J \to a$ in $\mathbf{C}{\downarrow}F$, i.e. $u : J \to I$ and $s : FJ \to A$ with $a \circ s = Fu$ as shown in the diagram

$$\begin{tikzcd}
FJ \arrow[r, "s"] \arrow[d, equal] & A \arrow[d, "a"] \\
FJ \arrow[r, "Fu"'] & FI \\
J \arrow[r, "u"'] & I
\end{tikzcd}$$

We have to show that there is a unique $v : J \to C$ with $p \circ v = u$ and $\widehat{q} \circ Fv = s$ as shown in the diagram

$$\begin{tikzcd}
FJ \arrow[dr, "Fv"] \arrow[drr, "s"] \arrow[dd, equal] & & \\
& FC \arrow[r, "\widehat{q}"'] \arrow[d, equal] \arrow[dr, phantom, "\widetilde{\varepsilon}_a"] & A \arrow[d, "a"] \\
FJ \arrow[r, "Fv"'] & FC \arrow[r, "Fp"'] & FI
\end{tikzcd}$$

But $\widehat{q} \circ Fv = s$ iff $q \circ v = Us \circ \eta_J$ due to $F \dashv U$. Thus v satisfies the above requirements iff $p \circ v = u$ and $q \circ v = Us \circ \eta_J$, i.e. iff v is the mediating arrow in the diagram

$$\begin{tikzcd}
J \arrow[dr, "v"] \arrow[drr, "Us \circ \eta_J"] \arrow[ddr, "u"'] & & \\
& \cdot \arrow[r, "q"] \arrow[d, "p"] \arrow[dr, phantom, very near start, "\lrcorner"] & UA \arrow[d, "Ua"] \\
& I \arrow[r, "\eta_I"'] & UFI
\end{tikzcd}$$

from which there follows uniqueness and existence of v with the desired properties. Thus $\widetilde{\varepsilon}_a$ actually is the counit for $1 \dashv G$ at a.

For the reverse direction assume that $P_F \dashv 1 \dashv G$. Thus, for all X over 1 we have $\mathbf{B}(-, GX) \cong \mathbf{C}{\downarrow}F(1_{(-)}, X) \cong \mathbf{C}/F1(F_{/1}(-), X)$, i.e. $F_{/1} : \mathbf{B} \cong \mathbf{B}/1 \to \mathbf{C}/F1$ has a right adjoint (given by the restriction of G to $\mathbf{C}/F1$). Since $\Sigma_{F1} \dashv (F1)^* : \mathbf{C} \to \mathbf{C}/F1$ and $F = \Sigma_{F1} \circ F_{/1} : \mathbf{B} \cong \mathbf{B}/1 \to \mathbf{C}$ the functor F has a right adjoint.

A slightly more abstract proof of the backwards direction goes by observing that the inclusion $I : \mathbf{C}{\downarrow}F1 \hookrightarrow \mathbf{C}{\downarrow}F$ has a left adjoint R sending $a : A \to FI$ to $F!_I \circ a : A \to F1$ and a morphism (u, f) from $b : B \to FJ$ to $a : A \to FI$ to $f : R(b) \to R(a)$ since (u, f) from $a : A \to FI$ to $c : C \to F1$ is in 1-1-correspondence with $f : R(a) \to c$ (because necessarily $u = !_I$). Obviously, we have that $F_{/1} = R \circ 1$ and thus $F_{/1}$ has right adjoint $G \circ I$. Since $F = \Sigma_{F1} \circ F_{/1}$ and $\Sigma_{F1} \dashv (F1)^*$ it follows that F has right adjoint $G \circ I \circ (F1)^*$. $\square$

Notice that the above proof goes through if $\mathbf{C}$ has just pullbacks and Σ_{F1} has a right adjoint $(F1)^*$, i.e. $F1 \times X$ exists for all objects X in $\mathbf{C}$.

Thus, we have the following lemma which has a structure analogous to the one of Lemma 13.2.

Lemma 16.2 *Suppose* $\mathbf{B}$ *has finite limits and* $\mathbf{C}$ *has pullbacks and all products of the form* $F1 \times X$. *Then for a functor* $F : \mathbf{B} \to \mathbf{C}$ *the following conditions are equivalent*

(1) *F has a right adjoint*

(2) $F^* : \mathbf{Fib}(\mathbf{C}) \to \mathbf{Fib}(\mathbf{B})$ *preserves the property of having small global sections*

(3) $P_F = F^*P_{\mathbf{C}} = \partial_1 : \mathbf{C}{\downarrow}F \to \mathbf{B}$ *has small global sections.*

Proof. The proof of (1) $\Rightarrow$ (2) is a special case of the proof of Lemma 13.3. Since $P_{\mathbf{C}}$ has small global sections (3) follows from (2). Finally, claim (1) follows from (3) by Theorem 16.1 and the subsequent remark on its strengthening. $\square$

From Lemma 13.2 and Lemma 16.2 it follows that for a functor $F : \mathbf{B} \to \mathbf{C}$ between categories with finite limits the fibration $P_F = F^*P_{\mathbf{C}}$ has internal sums and small global sections iff F preserves pullbacks and has a right adjoint.[14]

[14] This was already observed by J. Bénabou in [Ben1].

Thus, for categories $\mathbf{B}$ *with finite limits we get a 1–1–correspondence (up to equivalence) between geometric morphisms to* $\mathbf{B}$ *(i.e. adjunctions* $F \dashv U : \mathbf{C} \to \mathbf{B}$ *where* $\mathbf{C}$ *has finite limits and* F *preserves them) and fibrations over* $\mathbf{B}$ *of categories with finite limits, stable disjoint sums and small global sections. Such fibrations are called* **geometric**.

In Appendix A we prove M. Jibladze's theorem [Jib] that in fibered toposes with internal sums these are automatically stable and disjoint. As a consequence *geometric morphisms from toposes to a topos* $\mathbf{S}$ *are (up to equivalence) in 1–1–correspondence with toposes fibered over* $\mathbf{S}$ *that are cocomplete and locally small.*

In the rest of this section we show that in a fibered sense every geometric morphism is of the form $\Delta \dashv \Gamma$.

First we observe that there is a fibered version of the functor $\Delta = \boldsymbol{\Delta} \circ 1$ considered in the previous section

Definition 16.1 *Let* $\mathbf{B}$ *be a category with finite limits and* $P : \mathbf{X} \to \mathbf{B}$ *be a fibration of categories with finite limits and stable disjoint internal sums. Then there is a fibered functor* $\Delta_P : P_{\mathbf{B}} \to P$ *sending the morphism*

$$\begin{tikzcd} I_1 \arrow[r, "v"] \arrow[d, "u_1"'] & I_2 \arrow[d, "u_2"] \\ J_1 \arrow[r, "w"'] & J_2 \end{tikzcd}$$

in $P_{\mathbf{B}}$ *to the arrow* $\Delta_P(w,v)$ *in* $\mathbf{X}$ *over* w *making the following diagram commute*

$$\begin{tikzcd} 1_{I_1} \arrow[r, "1_v"] \arrow[d, "\varphi_{u_1}"'] & 1_{I_2} \arrow[d, "\varphi_{u_2}"] \\ \Delta_P(u_1) \arrow[r, "\Delta_P(w,v)"'] & \Delta_P(u_2) \end{tikzcd}$$

where φ_{u_i} *is cocartesian over* u_i *for* $i = 1, 2$.

Notice that Δ_P actually is cartesian as if the first square is a pullback then $\Delta_P(w,v)$ is cartesian by BCC for internal sums as 1_v is cartesian and the φ_{u_i} are cocartesian.

Now P having small global sections turns out as equivalent to Δ_P having a fibered right adjoint Γ_P.

Theorem 16.3 *Let* $\mathbf{B}$ *be a category with finite limits and* $P : \mathbf{X} \to \mathbf{B}$ *be a fibration of categories with finite limits and stable disjoint internal sums. Then* P *has small global sections iff* Δ_P *has a fibered right adjoint* Γ_P*.*

Proof. For the implication from left to right assume that $P \dashv 1 \dashv G$. For $X \in \mathbf{X}$ let $\widetilde{\varepsilon}_X$ be the unique vertical arrow making the diagram

$$\begin{tikzcd} 1_{GX} \arrow[r, "\varphi", "\text{cocart}"'] \arrow[dr, "\varepsilon_X"'] & \Delta_P(P(\varepsilon_X)) \arrow[d, "\widetilde{\varepsilon}_X"] \\ & X \end{tikzcd}$$

commute where ε_X is the counit of $1 \dashv G$ at X. Then for $u : I \to J$ and $f : \Delta_P(u) \to X$ there is a unique morphism $(w, v) : u \to P(\varepsilon_X)$ with $\widetilde{\varepsilon}_X \circ \Delta_P(w, v) = f$ as can be seen from the following diagram

$$\begin{tikzcd} 1_I \arrow[r, "1_v"] \arrow[d, "\varphi_u"'] & 1_{G(X)} \arrow[d, "\varphi"'] \arrow[ddr, "\varepsilon_X"] & \\ \Delta_P(u) \arrow[r, "\Delta_P(w,v)"] \arrow[drr, "f"'] & \Delta_P(P(\varepsilon_X)) \arrow[dr, "\widetilde{\varepsilon}_X"] & \\ & & X \end{tikzcd}$$

using the universal property of ε_X. It follows that necessarily $w = P(f)$. Thus, for $f : \Delta_P(u) \to X$ its lower transpose $\check{f}$ is given by $(P(f), v) : u \to P(\varepsilon_X)$ where $v : I \to G(X)$ is the unique arrow with $\varepsilon_X \circ 1_v = f \circ \varphi_u$. The induced right adjoint Γ_P sends a morphism $h : Y \to X$ in $\mathbf{X}$ to the morphism

$$\begin{tikzcd} G(Y) \arrow[r, "G(h)"] \arrow[d, "P(\varepsilon_Y)"'] \arrow[dr, phantom, "\Gamma_P(h)"] & G(X) \arrow[d, "P(\varepsilon_X)"] \\ P(Y) \arrow[r, "P(h)"'] & P(X) \end{tikzcd}$$

in $\mathbf{B^2}$ because $G(h)$ is the unique morphism v with $\varepsilon_X \circ 1_v = h \circ \varepsilon_Y = h \circ \widetilde{\varepsilon}_Y \circ \varphi_{P(\varepsilon_Y)}$ and, therefore, $(P(h), G(h))$ is the lower transpose of $h \circ \widetilde{\varepsilon}_Y$ as required. The unit $\widetilde{\eta}_u : u \to \Gamma_P(\Delta_P(u)) = P(\varepsilon_{\Delta_P(u)})$ of $\Delta_P \dashv \Gamma_P$ at $u : I \to J$ is given by $\widetilde{\eta}_u$ making the following diagram commute

$$
\begin{tikzcd}
I \arrow[d, dashed, "\widetilde{\eta}_u"'] & & 1_I \arrow[d, "1_{\widetilde{\eta}_u}"'] \arrow[dr, "\varphi_u", "\text{cocart}"'] & \\
G\Delta_P(u) & & 1_{G\Delta_P(u)} \arrow[r, "\varepsilon_{\Delta_P(u)}"'] & \Delta_P(u)
\end{tikzcd}
$$

because $(id_{P(\Delta_P(u))}, \widetilde{\eta}_u)$ is the lower transpose of $id_{\Delta_P(u)}$. As $P_{\mathbf{B}} \circ \Gamma_P = P$ and the components of $\widetilde{\eta}$ and $\widetilde{\varepsilon}$ are vertical it follows[15] that Γ_P is cartesian and thus $\Delta_P \dashv \Gamma_P$ is a fibered adjunction.

For the implication from right to left suppose that Δ_P has a fibered right adjoint Γ_P. We write $\widetilde{\varepsilon}$ for the counit of this adjunction. For $X \in \mathbf{X}$ we define ε_X as $\widetilde{\varepsilon}_X \circ \varphi$

$$
\begin{tikzcd}
1_{GX} \arrow[r, "\varphi", "\text{cocart}"'] \arrow[dr, "\varepsilon_X"'] & \Delta_P\Gamma_P X \arrow[d, "\widetilde{\varepsilon}_X"] \\
& X
\end{tikzcd}
$$

where φ is cocartesian over $P(\Gamma_P(X)) : G(X) \to P(X)$. To verify the desired universal property of ε_X assume that $\sigma : 1_I \to X$ is a morphism over $u : I \to P(X)$. Let $\sigma = f \circ \varphi_u$ with f vertical and φ_u cocartesian. Then the existence of a unique arrow $v : I \to G(X)$ with $\varepsilon_X \circ 1_v = \sigma$ follows from considering the diagram

$$
\begin{tikzcd}
1_I \arrow[r, "1_v"] \arrow[d, "\varphi_u"'] & 1_{G(X)} \arrow[d, "\varphi"'] \arrow[dr, "\varepsilon_X"] & \\
\Delta_P(u) \arrow[r, "\Delta_P(id_{P(X)}, v)"] \arrow[drr, "f"'] & \Delta_P\Gamma_P X \arrow[dr, "\widetilde{\varepsilon}_X"] & \\
& & X
\end{tikzcd}
$$

[15]This is an instance of a general fact about fibered adjunctions whose formulation and (easy) verification we leave as an exercise to the reader.

using the universal property of $\widetilde{\varepsilon}_X$. Thus, P has small global sections. $\square$

The following explicitation of $\Delta_{P_F} \dashv \Gamma_{P_F}$ for finite limit preserving F will be helpful later on.

Theorem 16.4 *For the geometric fibration $P = P_F$ induced by a geometric morphism $F \dashv U : \mathbf{C} \to \mathbf{B}$ the fibered adjunction $\Delta_P \dashv \Gamma_P$ can be described more concretely as follows.*
The left adjoint Δ_P acts by application of F to arrows and squares in $\mathbf{B}$. The fiber of Γ_P over $I \in \mathbf{B}$ is given by $\eta_I^* \circ U_{/I}$. The unit $\widetilde{\eta}_u$ for $u : I \to J$ is given by

$$\begin{tikzcd}
I \arrow[dr, "\widetilde{\eta}_u"] \arrow[drr, "\eta_I"] \arrow[ddr, "u"'] & & \\
& K \arrow[r, "q"] \arrow[d, "p"] \arrow[dr, phantom, "\lrcorner", very near start] & UFI \arrow[d, "UFu"] \\
& J \arrow[r, "\eta_J"'] & UFJ
\end{tikzcd}$$

and for $a : A \to FI$ the counit $\widetilde{\varepsilon}_a$ is given by $\varepsilon_A \circ Fq : Fp \to a$ where

$$\begin{tikzcd}
K \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & UA \arrow[d, "Ua"] \\
I \arrow[r, "\eta_I"'] & UFI
\end{tikzcd}$$

Proof. Straightforward exercise when using the description of ε from the proof of Theorem 16.1 and the descriptions of $\widetilde{\eta}$ and $\widetilde{\varepsilon}$ from the proof of Theorem 16.3. $\square$

17 Fibrational Characterisation of Boundedness

Recall (e.g. from [Jo77]) that a geometric morphism $F \dashv U : \mathbf{E} \to \mathbf{S}$ between elementary toposes is called *bounded* iff there is an object $S \in \mathbf{E}$ such that for every $X \in \mathbf{E}$ there is an object $I \in \mathbf{S}$ such that X appears as a subquotient of $S \times FI$

$$\begin{tikzcd} C \arrow[r, tail] \arrow[d, two heads] & S \times FI \\ X & \end{tikzcd}$$

i.e. X appears as quotient of some subobject C of $S \times FI$. Such an S is called a *bound* for the geometric morphism $F \dashv U$. The importance of bounded geometric morphisms lies in the fact that they correspond to Grothendieck toposes over $\mathbf{S}$ (as shown e.g. in [Jo77]).

In this section we will show that a geometric morphism $F \dashv U : \mathbf{E} \to \mathbf{S}$ is bounded iff for its corresponding geometric fibration P_F there exists a generating family.

Lemma 17.1 *Let $\mathbf{B}$ have finite limits and $P : \mathbf{X} \to \mathbf{B}$ be a fibration of categories with finite limits with stable disjoint internal sums. Then a cocartesian arrow $\varphi : X \to Y$ is monic w.r.t. to vertical arrows, i.e. vertical arrows $\alpha_1, \alpha_2 : Z \to X$ are equal whenever $\varphi \circ \alpha_1 = \varphi \circ \alpha_2$.*

Proof. Let $\alpha_1, \alpha_2 : Z \to X$ be vertical arrows with $\varphi \circ \alpha_1 = \varphi \circ \alpha_2$. Then there is a unique morphism α with $\pi_i \circ \alpha = \alpha_i$ for $i = 1, 2$. Consider the pullback

$$\begin{tikzcd} \cdot \arrow[r, "\beta"] \arrow[d, "\psi"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\delta_\varphi"] \\ Z \arrow[r, "\alpha"'] & \cdot \end{tikzcd}$$

where δ_φ is the fiberwise diagonal. Notice that both α and δ_φ are above the same mono in $\mathbf{B}$. Thus, the map ψ lies above an isomorphism in the base (as P preserves pullbacks) and, moreover, it is cocartesian as it appears as pullback of the cocartesian arrow δ_φ. Thus, the arrow ψ is an isomorphism and we have $\alpha = \delta_\varphi \circ \beta \circ \psi^{-1}$ from which it follows that $\alpha_i = \beta \circ \psi^{-1}$ for i=$1, 2$. Thus, we have $\alpha_1 = \alpha_2$ as desired.

Alternatively, one may argue somewhat simpler as follows. For $i{=}1,2$ we have $\alpha_i \circ \psi = \pi_i \circ \alpha \circ \psi = \pi_i \circ \delta_\varphi \circ \beta = \beta$. Accordingly, we have $\alpha_1 \circ \psi = \alpha_2 \circ \psi$ from which it follows that $\alpha_1 = \alpha_2$ since ψ is cocartesian and the α_i are vertical. $\square$

For formulating the next lemma we have to recall the notion of collectively epic morphism as introduced in Theorem 10.5. If $P : \mathbf{X} \to \mathbf{B}$ is a fibration then a morphism $f : X \to Y$ in $\mathbf{X}$ is called *collectively epic* iff for all vertical arrows $\alpha_1, \alpha_2 : Y \to Z$ from $\alpha_1 \circ f = \alpha_2 \circ f$ it follows that $\alpha_1 = \alpha_2$. Notice that for a collectively epic morphism $f : X \to Y$ for maps $g_1, g_2 : Y \to Z$ with $P(g_1) = P(g_2)$ from $g_1 f = g_2 f$ it follows that $g_1 = g_2$ because if $g_i = \varphi\alpha_i$ with φ cartesian and α_i vertical then $\alpha_1 f = \alpha_2 f$ and thus $\alpha_1 = \alpha_2$ from which it follows that $g_1 = \varphi\alpha_1 = \varphi\alpha_2 = g_2$.

If P is $\mathrm{Fam}(\mathbf{C})$ for an ordinary category $\mathbf{C}$ then an arrow $f : X \to Y$ in the total category of $\mathrm{Fam}(\mathbf{C})$ over $u : I \to J$ is collectively epic iff for all $j \in J$ the family $(f_i : X_i \to Y_j)_{i \in u^{-1}(j)}$ is collectively epic in the usual sense of ordinary category theory. Thus, it would be more precise to say "family of collectively epic families" but as this formulation is too lengthy we prefer the somewhat inaccurate formulation "collectively epic".

Notice that for a bifibration $P : \mathbf{X} \to \mathbf{B}$ a morphism $f : X \to Y$ in $\mathbf{X}$ is collectively epic iff for a cocartesian/vertical factorisation $f = \alpha \circ \varphi$ the vertical arrow α is epic in its fiber.

Lemma 17.2 *Let $\mathbf{B}$ be a category with finite limits and $P : \mathbf{X} \to \mathbf{B}$ a geometric fibration which is locally small and well–powered. Moreover, suppose that collectively epic arrows in $\mathbf{X}$ are stable under pullbacks.*

Then for P there exists a generating family iff for P there exists a separator*, i.e. an object $S \in P(1_{\mathbf{B}})$ such that for every object $X \in P(1_{\mathbf{B}})$ there exist morphisms $\varphi : Y \to S$, $m : Z \to Y$ and $\psi : Z \to X$ with φ cartesian, m a vertical mono and ψ collectively epic.*

Proof. Let $P : \mathbf{X} \to \mathbf{B}$ be a fibration satisfying the conditions above.

Suppose that $G \in P(I)$ is a generating family for P. Let $\psi_0 : G \to S$ be a cocartesian arrow over $!_I : I \to 1$. Let $\psi_0 = \varphi_0 \circ \eta$ with φ_0 cartesian and η vertical. Notice that η is monic as by Lemma 17.1 the cocartesian ψ_0 is monic w.r.t. vertical arrows. We show that S is a separator for P. Let $X \in P(1_{\mathbf{B}})$. As G is a generating family for P and $\mathbf{B}$ has binary products by Theorem 10.5 there are morphisms $\theta : Z \to G$ and $\psi : Z \to X$ with θ cartesian and ψ collectively epic. Then consider the diagram

$$
\begin{tikzcd}
X & Z \arrow[l, "\psi"'] \arrow[r, "\theta"] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & G \arrow[d, tail, "\eta"'] \arrow[dr, "\psi_0"] & \\
 & Y \arrow[r, "\theta'"'] & I^*S \arrow[r, "\varphi_0"'] & S
\end{tikzcd}
$$

where θ' is cartesian over $P(\theta)$ and m is vertical. Thus, the middle square is a pullback and m is a vertical mono. Furthermore, $\varphi := \varphi_0 \circ \theta'$ is cartesian. Thus, we have constructed morphisms $\varphi : Y \to S$, $m : Z \to Y$ and $\psi : Z \to X$ with φ cartesian, m a vertical mono and ψ collectively epic as required.

Suppose that $S \in P(1_{\mathbf{B}})$ is a separator for P. By well–poweredness of P there exists a vertical mono $m_S : G \rightarrowtail \sigma_S^* S$ classifying families of subobjects of S. We show that G is a generating family for P.

Suppose $X \in P(I)$. Let $\theta_0 : X \to X_0$ be a cocartesian arrow over $!_I : I \to 1$. As S is a separator there exist morphisms $\varphi_0 : Y_0 \to S$, $m_0 : Z_0 \to Y_0$ and $\psi_0 : Z_0 \to X_0$ with φ_0 cartesian, m_0 a vertical mono and ψ_0 collectively epic. Consider the pullback

$$
\begin{tikzcd}
Z \arrow[r, "\theta"] \arrow[d, "\psi"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z_0 \arrow[d, "\psi_0"] \\
X \arrow[r, "\theta_0"'] & X_0
\end{tikzcd}
$$

where ψ is collectively epic and θ is cocartesian since these classes of arrows are stable under pullbacks. Consider further the diagram

$$
\begin{tikzcd}
Z \arrow[dr, "\eta"] \arrow[drr, "\theta"] \arrow[ddr, "m"'] & & \\
 & \cdot \arrow[r, "\varphi'"] \arrow[d, tail, "m'"] \arrow[dr, phantom, "\lrcorner", very near start] & Z_0 \arrow[d, tail, "m_0"] \\
 & Y \arrow[r, "\varphi_1"'] & Y_0
\end{tikzcd}
$$

where φ_1 and φ' are cartesian over $P(\theta)$ and m' and η are vertical. The inner square is a pullback and thus m' is monic as it appears as pullback of the monic arrow m_0. The arrow η is a vertical mono as by Lemma 17.1 θ is monic w.r.t. vertical arrows. Thus $m = m' \circ \eta$ is a vertical mono, too. Moreover, $\varphi_0 \circ \varphi_1 : Y \to S$ is cartesian. Thus, the mono $m : Z \to Y$ is a family of subobjects of S and, accordingly, we have

$$\begin{tikzcd} Z \arrow[r, "\varphi"] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & G \arrow[d, tail, "m_S"] \\ Y \arrow[r, "\widetilde{\varphi}"'] & \sigma_S^* S \end{tikzcd}$$

for some cartesian arrows φ and $\widetilde{\varphi}$. Thus, we have morphisms $\varphi : Z \to G$ and $\psi : Z \to X$ with φ cartesian and ψ collectively epic.

Thus, by Theorem 10.5 it follows that G is a generating family for P. $\square$

Suppose $F : \mathbf{B} \to \mathbf{C}$ is a finite limit preserving functor between categories with finite limits. One easily checks that an arrow

$$\begin{tikzcd} B \arrow[r, "e"] \arrow[d, "b"'] \arrow[dr, phantom, "f"] & A \arrow[d, "a"] \\ FJ \arrow[r, "Fu"'] & FI \\ J \arrow[r, "u"'] & I \end{tikzcd}$$

in $\mathbf{C}{\downarrow}F$ is collectively epic (w.r.t. the fibration $P_F = \partial_1 : \mathbf{C}{\downarrow}F \to \mathbf{B}$) iff the map e is epic in $\mathbf{C}$. Apparently, this condition is sufficient. On the other hand if f is collectively epic then e is epic in $\mathbf{C}$ which can be seen as follows: suppose $g_1, g_2 : A \to C$ with $g_1 e = g_2 e$ then the maps

$$\begin{tikzcd} A \arrow[r, "g_i"] \arrow[d, "a"'] \arrow[dr, phantom, "\alpha_i"] & C \arrow[d] \\ FI \arrow[r, "F!_I"'] & F1 \\ I \arrow[r, "!_I"'] & 1 \end{tikzcd}$$

are both above $I \to 1$ and satisfy $\alpha_1 f = \alpha_2 f$ from which it follows – since f is collectively epic – that $\alpha_1 = \alpha_2$ and thus $g_1 = g_2$.

Thus, if in $\mathbf{C}$ epimorphisms are stable under pullbacks along arbitrary morphisms then in $\mathbf{C}{\downarrow}F$ collectively epic maps are stable under pullbacks along arbitrary morphisms.

Theorem 17.3 *A geometric morphism $F \dashv U : \mathbf{E} \to \mathbf{S}$ between toposes is bounded iff for the corresponding geometric fibration P_F there exists a generating family.*

Proof. Let $F \dashv U : \mathbf{E} \to \mathbf{S}$ be a geometric morphism between toposes. Then the corresponding geometric fibration P_F is locally small and well-powered.

As for P_F reindexing preserves the topos structure and in toposes epis are stable under pullbacks vertical epis are stable under pullbacks. Thus, collectively epic arrows are stable under pullbacks as both vertical epis and cocartesian arrows are stable under pullbacks. Alternatively, this follows from the observations immediately preceding the current theorem and pullback stability of epimorphisms in toposes.

Thus, since the assumptions of Lemma 17.2 are satisfied for P_F there exists a generating family for P_F iff there exists a separator for P_F which, obviously, is equivalent to the requirement that the geometric morphism $F \dashv U$ is bounded. $\square$

From inspection of the proof of Lemma 17.2 it follows[16] in particular

[16] In more concrete terms for the fibration $P_F = F^* P_{\mathbf{E}}$ this can be seen as follows. Suppose $a : A \to F(I)$ is a map in $\mathbf{E}$. As S is a bound there exists $J \in \mathbf{S}$ and $e : C \twoheadrightarrow A$ with $n : C \rightarrowtail F(J) \times S$. Consider the diagram

$$\begin{tikzcd}
A \arrow[dd, "a"'] & C \arrow[l, two heads, "e"'] \arrow[d, tail, "m"'] \arrow[dr, tail, "n"] & \\
& F(I\times J)\times S \arrow[r, "F(\pi')\times S"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & F(J)\times S \arrow[d, "\pi"] \\
F(I) & F(I\times J) \arrow[l, "F(\pi)"] \arrow[r, "F(\pi')"'] & F(J)
\end{tikzcd}$$

(where $F(\pi)$ and $F(\pi')$ form a product cone because F preserves finite limits and π and π' form a product cone) and notice that $\pi \circ m$ appears as pullback of g_S along $F(\rho)$ where $\rho : I \times J \to U\mathcal{P}(S)$ is the unique map classifying m, i.e. $((\varepsilon_{\mathcal{P}(S)} \circ F(\rho)) \times S)^* \ni_S \;\cong m$.

that if $S \in \mathbf{E}$ is a bound for a geometric morphism $F \dashv U : \mathbf{E} \to \mathbf{S}$ between toposes then $g_S = \pi \circ m_S : G_S \to F(U\mathcal{P}(S))$

$$\begin{tikzcd}
G_S \arrow[rr] \arrow[d, tail, "m_S"'] \arrow[dr, phantom, "\lrcorner"] & & \ni_S \arrow[d, tail, "\ni_S"] \\
FU\mathcal{P}(S){\times}S \arrow[rr, "\varepsilon_{\mathcal{P}(S)}\times S"] \arrow[d, "\pi"'] \arrow[drr, phantom, "\lrcorner"] & & \mathcal{P}(S){\times}S \arrow[d, "\pi"] \\
FU\mathcal{P}(S) \arrow[rr, "\varepsilon_{\mathcal{P}(S)}"'] & & \mathcal{P}(S)
\end{tikzcd}$$

is a generating family for P_F. This condition, however, also implies that S is a bound for $F \dashv U$ since if $g_S = \pi \circ m_S$ is a generating family for P_F then for every $A \in \mathbf{E}$ there is a map $u : I \to U\mathcal{P}(S)$ in $\mathbf{S}$ and an epi $e : u^*G_S \twoheadrightarrow A$ such that

$$\begin{tikzcd}
A \arrow[ddd] & u^*G_S \arrow[l, two heads, "e"'] \arrow[r] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner"] & G_S \arrow[r] \arrow[d, tail, "m_S"'] \arrow[dr, phantom, "\lrcorner"] & \ni_S \arrow[d, tail, "\ni_S"] \\
 & FI{\times}S \arrow[r, "Fu\times S"] \arrow[dd, "\pi"'] \arrow[ddr, phantom, "\lrcorner"] & FU\mathcal{P}(S){\times}S \arrow[r, "\varepsilon_{\mathcal{P}(S)}\times S"] \arrow[dd, "\pi"'] \arrow[ddr, phantom, "\lrcorner"] & \mathcal{P}(S){\times}S \arrow[dd, "\pi"] \\
 & & & \\
F1 & FI \arrow[l, "F!_I"] \arrow[r, "Fu"'] & FU\mathcal{P}(S) \arrow[r, "\varepsilon_{\mathcal{P}(S)}"'] & \mathcal{P}(S)
\end{tikzcd}$$

from which it follows that A appears as quotient of a subobject of some $FI{\times}S$.

Thus S is a bound for a geometric morphism $F \dashv U : \mathbf{E} \to \mathbf{S}$ between toposes iff $g_S = \pi \circ m_S : G_S \to FU\mathcal{P}(S)$ is a generating family for P_F. In case $\mathbf{S}$ is **Set** this amounts to the usual requirement that the family of subobjects of S is generating for the topos $\mathbf{E}$.

One can characterize boundedness of geometric morphisms in terms of preservation properties as follows.

Theorem 17.4 *A geometric morphism $F \dashv U : \mathbf{E} \to \mathbf{S}$ between toposes is bounded if and only if change of base along F preserves existence of small generating families for geometric fibrations of toposes.*

Thus, a terminal object preserving functor $F : \mathbf{S} \to \mathbf{E}$ between toposes is the inverse image part of a bounded geometric morphism iff $F^*P_{\mathbf{E}}$ is is a *geometric fibration with a small generating family iff change of base along F preserves the property of being a geometric fibration of toposes with a small generating family.*

Proof. Suppose F^* preserves existence of small generating families for geometric fibrations of toposes. Then since $P_{\mathbf{E}} : \mathbf{E}^{\mathbf{2}} \to \mathbf{E}$ has a small generating family so does $F^*P_{\mathbf{E}} = P_F$ and thus by Theorem 17.3 the geometric morphism $F \dashv U$ is bounded.

On the other hand if a geometric morphism $F \dashv U$ is bounded and P is a geometric fibration of toposes over $\mathbf{E}$ with a small generating family then by Theorem 17.3 P is equivalent to $G^*P_{\mathbf{F}} = P_G$ for some bounded geometric morphism $G \dashv V : \mathbf{F} \to \mathbf{E}$ and thus F^*P is equivalent to $F^*G^*P_{\mathbf{F}} \simeq (GF)^*P_{\mathbf{F}} = P_{GF}$ which has a small generating family by Theorem 17.3 since $GF \dashv UV$ is a bounded geometric morphism (as by [Jo77] bounded geometric morphisms are closed under composition).

For the second claim first recall that by Lemma 13.2 and Lemma 16.2 a terminal object preserving functor $F : \mathbf{S} \to \mathbf{E}$ between toposes is the inverse image part of a geometric morphism iff $F^*P_{\mathbf{E}}$ is a geometric fibration of toposes iff change of base along F preserves geometric fibrations of toposes.

Thus, if F is the inverse image part of a bounded geometric morphism then $F^*P_{\mathbf{E}}$ is a geometric fibration with a small generating family and change of base along F preserves geometric fibrations with a small generating family.

Suppose $F : \mathbf{S} \to \mathbf{E}$ preserves terminal objects. Thus, if $F^*P_{\mathbf{E}}$ is a geometric fibration with a small generating family then F is the inverse image part of a geometric morphism which by Theorem 17.3 is also bounded. If F^* preserves geometric fibrations with a small generating family then $F^*P_{\mathbf{E}}$ is a geometric fibration with a small generating family and thus F is the inverse image part of a bounded geometric morphism. $\square$

Thus we may observe that for (bounded) geometric morphisms $F \dashv U$ change of base along F for geometric fibrations of toposes (with a small generating family) corresponds to postcomposition with $F \dashv U$ for (bounded) geometric morphisms.[17]

By Theorem 17.4 change of base along inverse image parts of unbounded geometric morphisms does not preserve existence of small generating families. From [Jo77] we recall the following example of an unbounded geometric

[17] A consequence of this observation is that change of base along inverse image parts of geometric morphisms for geometric fibrations of toposes reflects the property of having a small generating family since as observed in [Jo77] for geometric morphisms f and g from fg bounded it follows that g is bounded.

morphism. Let $\mathbf{E}$ be the full subcategory of $\widehat{\mathbb{Z}} = \mathbf{Set}^{\mathbb{Z}^{\mathrm{op}}}$ on those objects A such that $\forall a{\in}A.A(n)(a) = a$ for some $n \in \mathbb{N}$, i.e. there is a finite bound on the size of all orbits of the action A. One easily sees that $\mathbf{E}$ is a topos and $\Delta \dashv \Gamma : \mathbf{E} \to \mathbf{Set}$ is a geometric morphism which, however, is not bounded (as otherwise there were an *a priori* bound on the size of all orbits of objects of $\widehat{\mathbb{Z}}$). Notice, however, that $\mathbf{E}$ admits a (countable) generating family in the sense of ordinary category theory, namely the family $(\mathbb{Z}_n)_{n\in\mathbb{N}}$ (of all finite orbits up to isomorphism), whose sum, however, does not exist in $\mathbf{E}$.

Johnstone's example also demonstrates that toposes $\mathbf{E}$ over $\mathbf{Set}$ need not be cocomplete in the sense of ordinary category theory, i.e. do not have all small sums, although the associated fibration P_Δ certainly has internal sums.[18] Apparently there is a difference between *internal* and *external* families of objects in $\mathbf{E}$ where a family $(X_i)_{i\in I}$ in $\mathbf{E}$ is internal if there is a map $f : Y \to \Delta(I)$ in $\mathbf{E}$ with $X_i \cong \mathrm{in}_i^* f$ for all $i \in I$. Of course, every internal family gives rise to an external one whereas e.g. $(\mathbb{Z}_n)_{n\in\mathbb{N}}$ is an external family in the topos $\mathbf{E}$ which is not internal. It is an easy exercise to show that a family $(X_i)_{i\in I}$ in a topos $\mathbf{E}$ over $\mathbf{Set}$ is internal if and only if the family $(X_i)_{i\in I}$ is *bounded* in the sense that there exists an object $X \in \mathbf{E}$ such that all X_i appear as subobjects of X.[19]

Notice that due to Giraud's Theorem (see [Jo77]) toposes bounded over

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[18] Thus, the fibrations $P_\Delta = \Delta^* P_{\mathbf{E}}$ and $\mathrm{Fam}(\mathbf{E})$ over $\mathbf{S}$ are not equivalent because P_Δ has internal sums whereas $\mathrm{Fam}(\mathbf{E})$ doesn't!

Consider also the following somewhat weaker counterexample. Let $\mathcal{A}$ be a *partial combinatory algebra*, $\mathbf{RT}[\mathcal{A}]$ the realizability topos over $\mathcal{A}$ (see e.g. [vOo]) and $\Gamma \dashv \nabla : \mathbf{Set} \to \mathbf{RT}[\mathcal{A}]$ the geometric morphism where $\Gamma = \mathbf{RT}[\mathcal{A}](1, -)$ is the global elements functor. Then $P_\nabla = \nabla^* P_{\mathbf{RT}[\mathcal{A}]}$ is a fibration with stable and disjoint internal sums over $\mathbf{Set}$ although for **nontrivial** $\mathcal{A}$ in the realizability topos $\mathbf{RT}[\mathcal{A}]$ the sum $\coprod_{|\mathcal{A}|} 1$ does not exist for cardinality reasons.

Moreover, for nontrivial $\mathcal{A}$ internal sums w.r.t. the fibration P_∇ in general do not coincide with the corresponding external sums (if they exists): consider e.g. $\coprod_2 1$ w.r.t. P_∇, i.e. $\nabla(2)$, which is not isomorphic to $1 + 1$ in $\mathbf{RT}(\mathcal{A})$. Thus $\nabla(1+1) \not\cong \nabla(1) + \nabla(1)$ from which it follows that ∇ does not have a right adjoint. Accordingly, the fibration P_∇ over $\mathbf{Set}$ does not have small global elements.

[19] In the example $P_\nabla = \partial_1 : \mathbf{RT}[\mathcal{A}]{\downarrow}\nabla \to \mathbf{Set}$ every $X \to \nabla(I)$ in $\mathbf{RT}[\mathcal{A}]$ may be understood as a family of I-indexed subobjects of X but the ensuing cartesian functor (over $\mathbf{Set}$) from P_∇ to $\mathrm{Fam}(\mathbf{RT}[\mathcal{A}])$ is far from being an equivalence.

Firstly, it does not reflect isos (in each fiber) since $id_{\nabla(2)}$ and $\eta_2 : 2 \to \nabla\Gamma(2) \cong \nabla(2)$ are not isomorphic in the slice over $\nabla(2)$ but both give rise to $(1)_{i\in 2}$ in $\mathrm{Fam}(\mathbf{RT}[\mathcal{A}])(2)$. Thus, different internal families (over 2 already) may give rise to the same external family.

Secondly, there are external $\mathbb{N}$-indexed families $(X_n)_{n\in\mathbb{N}}$ in $\mathbf{RT}[\mathcal{A}]$ which do not arise from a morphism $X \to \nabla(\mathbb{N})$ because any such family would have to be isomorphic to a family $(X'_n)_{n\in\mathbb{N}}$ for which symmetry and transitivity are realized by $e_1, e_2 \in \mathcal{A}$ independently from $n \in \mathbb{N}$. It is left as an exercise to the reader to give a concrete counterexample.

Set are precisely the Grothendieck toposes and, therefore, do have all small sums. Actually, one may see this more directly as follows. Suppose S is a bound for the geometric morphism $\Delta \dashv \Gamma : \mathbf{E} \to \mathbf{Set}$. Then $\mathbf{E}$ has all small copowers $\coprod_{i\in I} X \cong \Delta(I) \times X$. Suppose $(X_i)_{i\in I}$ is a family in $\mathbf{E}$. Then for every $i \in I$ there is a set J_i such that X_i is a subquotient of $\Delta(J_i) \times S$. Thus, all X_i are subobjects of $\mathcal{P}(\Delta(J)\times S)$ via some mono m_i where $J = \bigcup_{i\in I} J_i$ (since $\Delta(J_i) \times S$ is a subobject of $\Delta(J) \times S$). Let χ_i classify the subobject m_i for $i \in I$ and $\chi : \coprod_{i\in I} \mathcal{P}(\Delta(J)\times S) \to \Omega$ be the source tupling of the χ_i. Then the sum $\coprod_{i\in I} X_i$ appears as the subobject of the copower $\coprod_{i\in I} \mathcal{P}(\Delta(J)\times S) \cong \Delta(I)\times\mathcal{P}(\Delta(J)\times S)$ classified by χ in $\mathbf{E}$.

But there exist toposes over **Set** which, in the sense of ordinary category theory, are cocomplete but do not admit a small generating family. A typical such example (due to Peter Freyd) is the topos $\mathcal{G}$ whose objects are pairs (A, f) where A is a set and f is a family of bijections of A indexed over the class of all sets such that the class $\mathsf{supp}(A, f) = \{s \mid f_s \neq id_A\}$ is a set and whose morphisms from (A, f) to (B, g) are the maps $h : A \to B$ with $h(f_s(a)) = g_s(h(a))$ for all $a \in A$ and all sets s. The construction of this category can be rephrased as follows. Let $\mathbb{G}$ be the free group generated by the class of all sets. Then $\mathcal{G}$ is isomorphic to the full subcategory of $\widehat{\mathbb{G}}$ on those objects A where $\{s \mid A(s) \neq id_{A(*)}\}$ is a set. The proof that $\mathcal{G}$ is a topos is analogous to the proof that for every group G the presheaf category $\mathbf{Set}^{G^{\mathrm{op}}}$ is a boolean topos. Moreover $\mathcal{G}$ has all small limits and colimits (which are constructed pointwise). Suppose $(G_i, g^{(i)})_{i\in I}$ were a small generating family for $\mathcal{G}$. Let $J = \bigcup_{i\in I} \mathsf{supp}(G_i, g^{(i)})$ and s_0 be a set with $s_0 \notin J$. Now let (A, f) be the object of $\mathcal{G}$ where $A = \{0, 1\}$ and $f_s \neq id_A$ only for $s = s_0$. There cannot exist a morphism $h : (G_i, g^{(i)}) \to (A, f)$ unless G_i is empty as otherwise there is a $z \in G_i$ for which we have $h(z) = h(g^{(i)}_{s_0})(z)) = f_{s_0}(h(z))$ Obviously (A, f) has two different endomorphisms which, however, cannot be distinguished by morphisms of the form $h : (G_i, g^{(i)}) \to (A, f)$. Thus, there cannot exist a small generating family for the cocomplete boolean topos $\mathcal{G}$.

One easily shows that for a cocomplete topos $\mathbf{E}$ the functor $\Delta : \mathbf{E} \to \mathbf{Set}$ preserves finite limits. Thus, for a locally small topos $\mathbf{E}$ it holds that

$$\mathbf{E} \text{ bounded over } \mathbf{Set} \Longrightarrow \mathbf{E} \text{ cocomplete } \Longrightarrow \mathbf{E} \text{ over } \mathbf{Set}$$

and the above counterexamples show that none of these implications can be reversed in general.[20] Freyd's counterexample shows that the first implica-

[20]The category $\mathbf{Set}^{\mathbf{Ord}^{\mathrm{op}}}$ of **Set**-valued presheaves over the large category **Ord** of ordi-

tion cannot be reversed in general. Johnstone's counterexample shows that the second implication cannot be reversed in general.

If **E** is a topos bounded over **Set** then for **E** there exists a generating family in the sense of ordinary category theory. However, as Johnstone's counterexample shows the reverse implication does not hold in general for toposes over **Set**. Freyd's counterexample shows there are toposes **E** over **Set** such that there does not even exist a generating family for **E** in the sense of ordinary category theory and that such toposes may even be cocomplete.

Notice that toposes **E** cocomplete in the sense of ordinary category theory are bounded over **Set** iff there exists a generating family for **E** in the sense of ordinary category theory. The reason is that if $(G_i)_{i\in I}$ is a generating family for **E** in the sense of ordinary category theory then $\coprod_{i\in I} !_{G_i} : \coprod_{i\in I} G_i \to \coprod_{i\in I} 1_{\mathbf{E}} = \Delta(I)$ is a generating family for the fibration $\Delta^* P_{\mathbf{E}} = P_\Delta$. Thus, a topos **E** is bounded over **Set** iff **E** is cocomplete and there exists a generating family for **E** in the sense of ordinary category theory. However, this characterisation does not generalise to arbitrary base toposes **S**. Formally, the fibrational characterisation of bounded toposes over **S** as cocomplete locally small fibered toposes over **S** with a generating family looks similar but as we have seen above cocomplete in the sense of fibered categories is weaker than cocomplete in the sense of ordinary category theory and generating family in the sense of fibered categories is stronger than in the sense of ordinary category theory.

Finally we observe that a topos over **Set** which in the sense of ordinary category theory is neither cocomplete nor has a small generating family can be obtained by combining the ideas of Freyd's and Johnstone's counterexamples, namely the full subcategory of Freyd's counterexample $\mathcal{G}$ on those objects (A, f) for which there exists an $n \in \mathbb{N}$ such that $(f_s)^n = id_A$ for all sets s.

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nals is an example of a cocomplete topos which, however, is not locally small since there are class many subterminals and thus Ω has class many global elements.

18 Properties of Geometric Morphisms

In this section we will characterise some of the most common properties of geometric morphisms $F \dashv U$ in terms of simple fibrational properties of the corresponding geometric fibration P_F. In the following we simply write $\Delta \dashv \Gamma$ for the fibered adjunction $\Delta_P \dashv \Gamma_P$ and the according unit and counit are denoted by $\widetilde{\eta}$ and $\widetilde{\varepsilon}$, respectively.

Theorem 18.1 *Let $F \dashv U : \mathbf{C} \to \mathbf{B}$ be a geometric morphism and P be the induced geometric fibration P_F. Then the following conditions are equivalent.*

(1) *The geometric morphism $F \dashv U$ is* injective*, i.e. U is full and faithful.*

(2) *The counit $\widetilde{\varepsilon}$ of $\Delta \dashv \Gamma$ is a natural isomorphism.*

(3) *For the counit ε of $1 \dashv G : \mathbf{C}{\downarrow}F \to \mathbf{B}$ it holds that ε_X is cocartesian for all objects $X \in \mathbf{C}{\downarrow}F$.*

Proof. Conditions (2) and (3) are equivalent as by Theorem 16.3 we have $\varepsilon_X = \widetilde{\varepsilon}_X \circ \varphi$ with $\varphi : 1_{GX} \to \Delta\Gamma X$ cocartesian over $P(\varepsilon_X)$.

Condition (2) says that all Γ_I are full and faithful. In particular, we have that $U \cong \Gamma_1$ is full and faithful. Thus (2) implies (1).

It remains to show that (1) entails (2). Condition (1) says that the counit ε of $F \dashv U$ is a natural isomorphism. But then for every $a : A \to FI$ in $\mathbf{C}{\downarrow}F$ we have

$$\begin{tikzcd}
FK \arrow[r, "Fq"] \arrow[d, "Fp"'] \arrow[dr, phantom, "\lrcorner", very near start] & FUA \arrow[r, "\varepsilon_A", "\cong"'] \arrow[d, "FUa"] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "a"] \\
FI \arrow[r, "F\eta_I"'] & FUFI \arrow[r, "\cong", "\varepsilon_{FI}"'] & FI
\end{tikzcd}$$

from which it follows by Theorem 16.4 that the map $\widetilde{\varepsilon}_a = \varepsilon_A \circ Fq$ is an isomorphism as it appears as pullback of the identity $id_{FI} = \varepsilon_{FI} \circ F(\eta_I)$. $\square$

Theorem 18.2 *Let $F \dashv U : \mathbf{C} \to \mathbf{B}$ be a geometric morphism and P be its induced geometric fibration P_F. Then the following conditions are equivalent.*[21]

[21] This holds without assuming that F has a right adjoint. It suffices that F preserves finite limits.

(1) *The geometric morphism $F \dashv U$ is* surjective*, i.e. F reflects isomorphisms.*

(2) *A morphism u in $\mathbf{B}$ is an isomorphism whenever 1_u is cocartesian.*

Proof. Obviously, the functor F reflects isomorphisms iff all Δ_I reflect isomorphisms.

For a morphism $u : w \to v$ in $\mathbf{B}/I$ (i.e. $w = v \circ u$) we have

$$\begin{tikzcd} 1_K \arrow[r, "\varphi_w"] \arrow[d, "1_u"'] & \Delta_I(w) \arrow[d, "\Delta_I(u)"] \\ 1_J \arrow[r, "\varphi_v"'] & \Delta_I(v) \end{tikzcd}$$

where φ_w and φ_v are cocartesian over $w : K \to I$ and $v : J \to I$, respectively, and $\Delta(u)$ is vertical over I. As internal sums in P_F are stable and disjoint it follows from Lemma 15.2 that $\Delta_I(u)$ is an isomorphism iff 1_u is cocartesian. Thus, the functor Δ_I reflects isomorphisms iff u is an isomorphism whenver 1_u is cocartesian.

Thus, the functor F reflects isomorphisms iff it holds for all maps u in $\mathbf{B}$ that u is an isomorphism whenever 1_u is cocartesian. $\square$

A geometric morphisms $F \dashv U$ between toposes is known to be surjective iff F is faithful. One easily sees that a finite limit preserving functor $F : \mathbf{B} \to \mathbf{C}$ between categories with finite limits is faithful iff for the associated fibration P_F it holds for $u, v : J \to I$ that $u = v$ whenever $\varphi_I \circ 1_u = \varphi_I \circ 1_v$ where $\varphi_I : 1_I \to \coprod_I 1_I$ is cocartesian over $I \to 1$. But, of course, in this general case F being faithful does not imply that F reflects isos, e.g. if $\mathbf{B}$ is posetal then F is always faithful but in general does not reflect isos. However, if F reflects isos then it is also faithful since F preserves equalizers.

Theorem 18.3 *Let $F \dashv U : \mathbf{C} \to \mathbf{B}$ be a geometric morphism and P be its induced geometric fibration P_F. Then the following conditions are equivalent.*

(1) *The geometric morphism $F \dashv U$ is* connected*, i.e. F is full and faithful.*

(2) *The right adjoint G of $1 : \mathbf{B} \to \mathbf{C}{\downarrow}F$ sends cocartesian arrows to isomorphisms.*

(3) *The fibered functor* Γ *is cocartesian, i.e. preserves cocartesian arrows.*

Proof. Obviously, the functor F is full and faithful iff all Δ_I are full and faithful, i.e. all $\widetilde{\eta}_u$ are isomorphisms. Let $u : I \to J$ be a morphism in $\mathbf{B}$. Then we have

$$
\begin{tikzcd}
1_{G(1_I)} \arrow[r, equal, "\varepsilon_{1_I}"] \arrow[d, "1_{\widetilde{\eta}_u}"'] & 1_I \arrow[d, "\varphi_u"] \\
1_{G(\Delta(u))} \arrow[r, "\varepsilon_{1_{\Delta(u)}}"'] & \Delta(u)
\end{tikzcd}
$$

where $\varphi_u : 1_I \to \Delta(u)$ is cocartesian over u and, therefore, we have $\widetilde{\eta}_u = G(\varphi_u)$. Thus, the functor F is full and faithful iff $G(\varphi)$ is an isomorphism for cocartesian φ whose source is terminal in its fiber. But then G sends all cocartesian arrows to isomorphisms which can be seen as follows. Suppose $\varphi : X \to Y$ is cocartesian over $u : I \to J$. Let $\varphi_u : 1_I \to \Delta(u)$ be cocartesian over u. Then by Lemma 15.2 the commuting square

$$
\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\beta"] \\
1_I \arrow[r, "\varphi_u"'] & \Delta(u)
\end{tikzcd}
$$

with α and β vertical over I and J, respectively, is a pullback. As G is a right adjoint it preserves pullbacks and, therefore,

$$
\begin{tikzcd}
G(X) \arrow[r, "G(\varphi)"] \arrow[d, "G(\alpha)"'] \arrow[dr, phantom, "\lrcorner", very near start] & G(Y) \arrow[d, "G(\beta)"] \\
G(1_I) \arrow[r, "G(\varphi_u)"'] & G(\Delta(u))
\end{tikzcd}
$$

is a pullback, too, from which it follows that $G(\varphi)$ is an isomorphism as $G(\varphi_u)$ is an isomorphism by assumption. Thus, we have shown the equivalence of conditions (1) and (2).

The equivalence of conditions (2) and (3) can be seen as follows. From (inspection of) the proof of Theorem 16.1 we know that for $\varphi : X \to Y$ its image under Γ is given by

$$\begin{tikzcd}
G(X) \arrow[r, "G(\varphi)"] \arrow[d, "P(\varepsilon_X)"'] \arrow[dr, phantom, "\Gamma(\varphi)"] & G(Y) \arrow[d, "P(\varepsilon_Y)"] \\
P(X) \arrow[r, "P(\varphi)"'] & P(Y)
\end{tikzcd}$$

Thus $\Gamma(\varphi)$ is cocartesian iff $G(\varphi)$ is an isomorphism. Accordingly, the functor G sends all cocartesian arrows to isomorphisms iff Γ preserves cocartesianness of arrows. $\square$

Notice that condition (2) of Theorem 18.3 is equivalent to the requirement that G inverts just cocartesian arrows over terminal projections which can be seen as follows. Suppose $\varphi : X \to Y$ is cocartesian over $u : I \to J$. Let $\psi : Y \to \coprod_J Y$ be a cocartesian arrows over $!_J : J \to 1$. Then $\psi \circ \varphi$ is cocartesian over $!_I : I \to 1$. As $G(\psi \circ \varphi) = G(\psi) \circ G(\varphi)$ and by assumption $G(\psi \circ \varphi)$ and $G(\psi)$ are isomorphisms it follows immediately that $G(\varphi)$ is an isomorphism, too. Moreover, one easily sees that G inverts cocartesian arrows over terminal projections if and only if G inverts cocartesian arrows above terminal projections whose source is terminal in its fiber. Of course, this condition is necessary. For the reverse direction suppose that $\varphi : X \to Y$ is cocartesian over $!_I : I \to 1$. Then as P is a geometric fibration we have

$$\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[d, "\alpha"'] \arrow[dr, phantom, very near start, "\lrcorner\text{cocart}"] & Y \arrow[d, "\beta"] \\
1_I \arrow[r, "\text{cocart}", "\varphi_I"'] & \Delta(I)
\end{tikzcd}$$

where φ_I is cocartesian over $I \to 1$ and α and β are the unique vertical arrows making the diagram commute. As G is a right adjoint it preserves pullbacks and, therefore, we have

$$\begin{tikzcd}
G(X) \arrow[r, "G(\varphi)"] \arrow[d, "G(\alpha)"'] \arrow[dr, phantom, very near start, "\lrcorner"] & G(Y) \arrow[d, "G(\beta)"] \\
G(1_I) \arrow[r, "G(\varphi_I)"'] & G(\Delta(I))
\end{tikzcd}$$

from which it follows that $G(\varphi)$ is an isomorphism as $G(\varphi_I)$ is an isomorphism by assumption. Thus, a geometric fibration P is connected if and only if G inverts all cocartesian arrows over terminal projections which start from a fiberwise terminal object, i.e. if for all $\sigma : 1_J \to \Delta(I)$ there exists a unique $u : J \to I$ with $\sigma = \varphi_I \circ 1_u$ as gets immediate from the following diagram

$$\begin{tikzcd}
1_{G1_I} \arrow[r, equal, "\varepsilon_{1_I}"] \arrow[d, "1_{G\varphi_I}"', "\cong"] & 1_I \arrow[d, "\varphi_I"] \\
1_{G\Delta(I)} \arrow[r, "\varepsilon_{\Delta(I)}"'] & \Delta(I)
\end{tikzcd}$$

with $\varphi_I : 1_I \to \Delta(I)$ cocartesian over $I \to 1$. Analogously, faithfulness of F is equivalent to the requirement that $u = v$ whenever $\varphi_I \circ 1_u = \varphi_I \circ 1_v$ providing an alternative characterisation of surjectivity for geometric morphisms between toposes (as $F \dashv U$ is surjective iff F is faithful).

Obviously, condition (1) of Theorem 18.3 is equivalent to the requirement that $\eta : Id_{\mathbf{B}} \to UF$ is a natural isomorphism. For the particular case of a geometric morphism $\Delta \dashv \Gamma : \mathbf{E} \to \mathbf{Set}$ where $\mathbf{E}$ is a topos one easily sees that $\eta_I : I \to \Gamma\Delta I$ (sending $i \in I$ to the injection $\mathrm{in}_i : 1 \to \coprod_{i \in I} 1$) is a bijection for all sets I iff the terminal object of $\mathbf{E}$ is *indecomposable* in the sense that for all subterminals U and V with $U{+}V \cong 1_{\mathbf{E}}$ either U or V is isomorphic to $0_{\mathbf{E}}$.

Theorem 18.4 *Let $F \dashv U : \mathbf{C} \to \mathbf{B}$ be a geometric morphism and P be the induced geometric fibration P_F. Then the following conditions are equivalent.*

(1) *The geometric morphism $F \dashv U$ is* hyperconnected*, i.e. connected and all components of the counit of the adjunction are monic.*

(2) *The right adjoint G of $1 : \mathbf{B} \to \mathbf{C}{\downarrow}F$ inverts cocartesian arrows and α is monic whenever $\varepsilon_X = \alpha \circ \varphi$ with α vertical and φ cocartesian.*

(3) *The fibered functor Γ preserves cocartesian arrows and all counits $\widetilde{\varepsilon}_X : \Delta\Gamma X \to X$ are vertical monos.*

For geometric morphisms $\Delta \dashv \Gamma : \mathbf{E} \to \mathbf{Set}$ the counit $\varepsilon_X : \Delta\Gamma X \to X$ at $X \in \mathbf{E}$ is monic iff distinct global elements of X are disjoint. The above Th. 18.4 generalizes this condition to arbitrary bases.

But there is a different characterization of hyperconnected geometric morphisms between elementary toposes whose fibrational analogue might appear as more intuitive.[22]

Theorem 18.5 *Let $F \dashv U : \mathbf{C} \to \mathbf{B}$ be a geometric morphism and P be the induced geometric fibration P_F. Then the following conditions are equivalent.*

(1) *The geometric morphism $F \dashv U$ is hyperconnected, i.e. connected and for all $I \in \mathbf{B}$ the functor $F_{/I} : \mathbf{B}/I \to \mathbf{C}/FI$ restricts to an equivalence between* $\mathrm{Sub}_{\mathbf{B}}(I)$ *and* $\mathrm{Sub}_{\mathbf{C}}(FI)$.

(2) *The right adjoint G of $1 : \mathbf{B} \to \mathbf{C}{\downarrow}F$ inverts cocartesian arrows and every vertical subobject m of 1_I is isomorphic to $\Delta_I(G(m))$.*

(3) *The fibered functor Γ preserves cocartesian arrows and $\Delta : P_{\mathbf{B}} \to P_F$ restricts to a fibered equivalence between* $\mathrm{Sub}_{\mathbf{B}}$ *and* $F^*\mathrm{Sub}_{\mathbf{C}}$ *considered as subfibrations of $P_{\mathbf{B}}$ and P_F, respectively.*

The intuitive meaning of the second part of condition (3) in the above theorem is that the fibered adjunction $\Delta \dashv \Gamma$ between $P_{\mathbf{B}}$ and P_F restricts to a fibered equivalence between their subterminal parts.

Of course, the conditions (3) of theorems 18.4 and 18.5, respectively, are in general not equivalent for geometric fibrations where $\mathbf{B}$ is not a topos and P_F is not a fibration of toposes (i.e. $\mathbf{C}$ is not a topos).

Theorem 18.6 *Let $F \dashv U : \mathbf{C} \to \mathbf{B}$ be a geometric morphism and P be the induced geometric fibration P_F. Then the following conditions are equivalent.*

(1) *The geometric morphism $F \dashv U$ is* local, *i.e. F is full and faithful and U has a right adjoint.*

(2) *The fibered functor Γ has a fibered right adjoint ∇.*

Proof. First we show that (2) implies (1). If Γ has a fibered right adjoint ∇ then Γ preserves cocartesian arrows as it is a fibered left adjoint. Thus, by the previous Theorem 18.3 it follows that F is full and faithful. As $\Gamma_1 \dashv \nabla_1$ and $U \cong \Gamma_1$ it follows that U has a right adjoint.

[22] For the various characterizations of hyperconnected geometric morphisms between toposes see A.4.6 of [Jo02]. One of them is that U preserves subobject classifiers. Another one that F is full and faithful and the image of F is closed under subquotients.

Now we show that (1) implies (2). If F is full and faithful then the unit $\eta : Id_{\mathbf{B}} \to UF$ is an isomorphism. Therefore, the fibered functor Γ acts on objects and morphisms simply by applying the functor U and then postcomposing with the inverse of η, i.e. $\Gamma(a) = \eta_I^{-1} \circ U(a)$ for $a : A \to FI$ in $\mathbf{C}{\downarrow}F$. Then Γ has a fibered right adjoint ∇ with $\nabla(v) = \zeta_J^* R(v)$ for $v : K \to J$ in $\mathbf{C}{\downarrow}\mathbf{C}$ where $\zeta_J : FJ \to RJ$ is the transpose (w.r.t. $U \dashv R$) of $\eta_J^{-1} : UFJ \to J$ as follows from the natural 1-1-correspondence between

$$\begin{tikzcd} UA \arrow[rr, "w"] \arrow[d, "Ua"'] & & K \arrow[d, "v"] \\ UFI \arrow[r, "\eta_I^{-1}"'] & I \arrow[r, "u"'] & J \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[rr, "\check{w}"] \arrow[d, "a"'] & & RK \arrow[d, "Rv"] \\ FI \arrow[r, "Fu"'] & FJ \arrow[r, "\zeta_J"'] & RJ \end{tikzcd}$$

exploiting the fact that the transpose of $u \circ \eta_I^{-1} = \eta_J^{-1} \circ UFu$ is $\zeta_J \circ Fu$. □

Theorem 18.7 *Let $F \dashv U : \mathbf{C} \to \mathbf{B}$ be a geometric morphism and P be the induced geometric fibration P_F. Then the following conditions are equivalent.*

(1) *The geometric morphism $F \dashv U$ is* locally connected*, i.e. F has a left adjoint L such that*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "b"] \\ FI \arrow[r, "Fu"'] & FJ \end{tikzcd} \quad \textit{implies} \quad \begin{tikzcd} LA \arrow[r, "Lf"] \arrow[d, "\hat{a}"'] \arrow[dr, phantom, "\lrcorner", very near start] & LB \arrow[d, "\hat{b}"] \\ I \arrow[r, "u"'] & J \end{tikzcd}$$

where $\hat{a}$ and $\hat{b}$ are the upper transposes of a and b, respectively.

(2) *The fibered functor Δ has a fibered left adjoint Π.*

Proof. If $L \dashv F$ then Δ has an ordinary left adjoint Π_L sending

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ FI \arrow[r, "Fu"'] & FJ \end{tikzcd} \quad \text{to} \quad \begin{tikzcd} LA \arrow[r, "Lf"] \arrow[d, "\hat{a}"'] & LB \arrow[d, "\hat{b}"] \\ I \arrow[r, "u"'] & J \end{tikzcd}$$

and satisfying $P_{\mathbf{B}} \circ \Pi_L = P_F$. Obviously, this functor Π_L is cartesian iff L satisfies the requirement of condition (1). Thus, condition (1) entails condition (2).

On the other hand if Δ has a fibered left adjoint Π then $F \cong \Delta_1$ has an ordinary left adjoint $L \cong \Pi_1$ and as $\Pi \cong \Pi_L$ in the 2-category $\mathbf{Cat}{\downarrow}\mathbf{B}$ with vertical natural transformations as 2-cells (because both functors are left adjoints to Δ in this 2-category) it follows that Π_L is also cartesian and, therefore, the functor L satisfies the requirement of condition (1). Thus, condition (2) entails condition (1). $\square$

Moreover, it follows from the fibered version of the Special Adjoint Functor Theorem[23] that a geometric morphism $F \dashv U : \mathbf{E} \to \mathbf{S}$ between toposes is locally connected if and only if F preserves the locally cartesian closed structure as $\Delta : P_{\mathbf{S}} \to P_F$ preserves internal limits iff F preserves finite limits and dependent products.

Recall that a geometric morphism $F \dashv U : \mathbf{E} \to \mathbf{S}$ between toposes is called *atomic* iff $F : \mathbf{S} \to \mathbf{E}$ is logical. Thus, by the previous remark atomic geometric morphisms between toposes are locally connected. Atomic geometric morphisms can be characterised as those locally connected geometric morphisms $F \dashv U : \mathbf{E} \to \mathbf{S}$ where all monomorphisms m in $\mathbf{E}$ are $\mathbf{S}$–definable, i.e. satisfy

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & FLX \arrow[d, "FLm"] \\ Y \arrow[r, "\eta_Y"'] & FLY \end{tikzcd}$$

where η is the unit of $L \dashv F$. This can be seen as follows. Recall that for a locally connected geometric morphism $F \dashv U : \mathbf{E} \to \mathbf{S}$ the monomorphism $F(\top_{\mathbf{S}})$ classifies $\mathbf{S}$–definable monomorphisms. Now if F is logical then $F(\top_{\mathbf{S}})$ is a subobject classifier in $\mathbf{E}$ and, therefore, all monomorphisms in $\mathbf{E}$ are $\mathbf{S}$–definable. On the other hand, if all monomorphisms in $\mathbf{E}$ are $\mathbf{S}$–definable then $F(\top_{\mathbf{S}})$ is a subobject classifier (as it classifies all $\mathbf{S}$–definable monomorphisms) and thus F is logical.

[23] which applies as $P_{\mathbf{S}}$ has a small generating family and, therefore, also a small cogenerating family (as shown by R. Paré and D. Schumacher)

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A M. Jibladze's Theorem on Fibered Toposes

Let $\mathbf{B}$ be a category with finite limits. A *topos fibered over* $\mathbf{B}$ is a fibration $P : \mathbf{X} \to \mathbf{B}$ all whose fibers are toposes and all whose reindexing functors are logical. In [Jib] M. Jibladze has shown that if a fibered topos has internal sums then these sums are necessarily universal (i.e. pullback stable) and disjoint. Thus, by Moens's Theorem it follows that $P \simeq P_\Delta$ where $\Delta : \mathbf{B} \to \mathbf{E} = \mathbf{X}_1 : I \mapsto \coprod_I 1_I$.

As a preparation we need the following results about logical functors $F : \mathbf{E} \to \mathbf{F}$ between toposes. If $L \dashv F$ then (by A.2.4.8 of [Jo02]) functor L preserves monos and (by A.2.3.8 of [Jo02]) the following are quivalent

(1) $L_{/1} : \mathbf{F}_{/1} \to \mathbf{E}_{/L1}$ is an equivalence[24]

(2) L is faithful

(3) L preserves equalizers

(4) L preserves pullbacks.

One easily shows[25] that if L is full and faithful then $L1$ is subterminal from which it follows that all components of the counit ε are monos.

---

[24]the right adjoint of $L_{/1}$ is given by $\eta_1^* \circ F_{/L1}$

[25]Suppose $\Sigma_I : \mathbf{E}_{/I} \to \mathbf{E}$ is full and faithful. Then the unit η of $\Sigma_I \dashv I^*$ is an isomorphism. For $a : A \to I$ we have $\eta_a = \langle a, id_A \rangle : a \to I^*\Sigma_I a$ as depicted in

$$\begin{tikzcd}
A \arrow[dr, "\eta_a"] \arrow[drr, equal, bend left] \arrow[ddr, "a"', bend right] & & \\
& I \times A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d] \\
& I \arrow[r] & 1
\end{tikzcd}$$

Since η_a is an isomorphism the projection $\pi_2 : I \times A \to A$ is an isomorphism, too. Thus, for $a' : A \to I$ we have $\pi_2 \circ \langle a, id_A \rangle = id_A = \pi_2 \circ \langle a', id_A \rangle$ from which it follows that $a = a'$ since π_2 is an isomorphism. Thus I is subterminal. But then the counit ε_A at A is given by

$$\begin{tikzcd}
I^*A \arrow[r, tail, "\varepsilon_A"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d] \\
I \arrow[r, tail] & 1
\end{tikzcd}$$

and thus monic.

We are now ready to prove Jibladze's Theorem on Fibered Toposes.

Proof. First observe that for a mono $m : J \rightarrowtail I$ we have $m^* \coprod_m \cong Id_{P(J)}$ as follows from the Chevalley condition for internal sums at the pullback square

$$
\begin{tikzcd}
J \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & J \arrow[d, tail, "m"] \\
J \arrow[r, tail, "m"'] & I
\end{tikzcd}
$$

Since m^* is logical and $\coprod_m$ is full and faithful all components of the counit of $\coprod_m \dashv m^*$ are monic.

Next we show that for all $u : J \to I$ in $\mathbf{B}$ and $X \in P(J)$ the map $\eta_X : X \to u^* \coprod_u X$ is monic (where η is the counit of $\coprod_u \dashv u^*$). Recall that η_X is the unique vertical map such that

$$
\begin{tikzcd}
X \arrow[r, "\varphi_u(X)", "\text{cocart}"'] \arrow[d, "\eta_X"'] & \coprod\limits_u X \\
u^* \coprod\limits_u X \arrow[ur, "\text{cart}", "\theta_u(X)"'] &
\end{tikzcd}
$$

Let $k_0, k_1 : K \to J$ be a kernel pair of u in $\mathbf{B}$ and $d_u : J \to K$ with $k_0 d_u = id_J = k_1 d_u$. Consider the diagram

$$
\begin{tikzcd}
d_u^* k_1^* X \arrow[r, "\psi", "\text{cart}"'] \arrow[d, equal] & k_1^* X \arrow[r, "\theta", "\text{cart}"'] \arrow[d, "\varphi"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\varphi_u(X)"] \\
X \arrow[r, "\eta_X"'] & u^* \coprod\limits_u X \arrow[r, "\text{cart}", "\theta_u(X)"'] & \coprod\limits_u X
\end{tikzcd}
$$

with $\theta \circ \psi = id_X$. Notice that φ is cocartesian by the Chevalley condition for internal sums. Next consider the diagram

$$\begin{tikzcd}
d_u^* k_1^* X \arrow[d, "\widetilde{\varphi_1}"'] \arrow[drr, "\cong"] & & \\
\coprod_{d_u} d_u^* k_1^* X \arrow[rr, tail, "\varepsilon_{k_1^* X}"'] \arrow[d, "\widetilde{\varphi_2}"'] & & k_1^* X \arrow[d, "\varphi"] \\
\coprod_{k_0} \coprod_{d_u} d_u^* k_1^* X = X \arrow[rr, tail, "\eta_X"'] & & u^* \coprod_u X
\end{tikzcd}$$

where φ and $\widetilde{\varphi_2}$ are cocartesian over k_0 and $\widetilde{\varphi_1}$ is cocartesian over d_u. Since d_u is monic the map $\varepsilon_{k_1^* X}$ is monic. Since $\coprod_{k_0}$ is left adjoint to the logical functor k_0^* it preserves monos from which it follows that η_X is monic.

Now since all components of the counit η of $\coprod_u \dashv u^*$ are monic it follows that $\coprod_u$ is faithful. Since u^* is logical it follows that $\coprod_{u/1_J}$ is an equivalence. Recall that its right adjoint is given by $\eta_{1_J}^* \circ u^*/\coprod_u 1_J$, i.e. pullback along the cocartesian arrow $\varphi_u : 1_J \to \coprod_u 1_J$.

That the counit of the adjunction $\coprod_{u/1_J} \dashv \varphi_u^*$ is an isomorphism means that for vertical α in the pullback

$$\begin{tikzcd}
\varphi_u^* X \arrow[r, "\alpha^* \varphi_u"] \arrow[d, "\varphi_u^* \alpha"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\alpha"] \\
1_J \arrow[r, "\text{cocart}", "\varphi_u"'] & \coprod_u 1_J
\end{tikzcd}$$

the top arrow $\alpha^* \varphi_u$ is cocartesian. This is sufficient for showing that cocartesian arrows are stable under pullbacks along vertical arrows, i.e. that internal sums are universal (since by the Chevalley condition cocartesian arrows are stable under pullbacks along cartesian arrows anyway).

That the unit of the adjunction $\coprod_{u/1_J} \dashv \varphi_u^*$ is an isomorphism means that

$$\begin{tikzcd} X \arrow[r, "\text{cocart}"'] \arrow[d, "!_X"'] & \coprod_u X \arrow[d, "\coprod_u !_X"] \\ 1_J \arrow[r, "\text{cocart}", "\varphi_u"'] & \coprod_u 1_J \end{tikzcd}$$

is a pullback. From this it follows that all diagrams of the form

$$\begin{tikzcd} X \arrow[r, "\text{cocart}"'] \arrow[d, "\alpha"'] & \coprod_u X \arrow[d, "\coprod_u \alpha"] \\ Y \arrow[r, "\text{cocart}"] & \coprod_u Y \end{tikzcd}$$

are pullbacks. But (from the proof of Moens's Theorem) this is known to imply disjointness of internal sums.[26] $\square$

Though claimed otherwise in [Jo02] Jibladze's Theorem was not proved in Moens's Thése [Moe] from 1982. Johnstone claims that Moens proved in some other way that for a fibered topos internal sums are universal and disjoint. But this is not the case because he considered fibered variants of Giraud's Theorem where internal sums are *assumed* as universal and disjoint. The only known way of showing that for a fibered topos internal sums are universal and disjoint is via Jibladze's Theorem.

However, in Jibladze's original formulation he did not prove universality and disjointness for internal sums in a fibered topos. For him it was sufficient to show that all $\coprod_u$ are faithful because from this it follows that the adjunctions $\coprod_u \dashv u^*$ are (equivalent to ones) of the form $\Sigma_A \dashv A^*$ for some A in $\mathbf{X}_1$ and this is sufficient for showing that $P \simeq P_\Delta$.

[26] One can see this more easily as follows. Since $\coprod_{u/1_J}$ is an equivalence it follows that $\coprod_u$ reflects isomorphisms which is known (from the proof of Moens's Theorem) to entail that internal sums are disjoint provided they are universal.

B Descent and Stacks

Let $\mathfrak{C}$ be a 2-category, x a 0-cell in $\mathfrak{C}$ and $f : z \to y$ a 1-cell in $\mathfrak{C}$ then f is called a *descent* map w.r.t. x iff the functor

$$\mathfrak{C}(f, x) : \mathfrak{C}(y, x) \to \mathfrak{C}(z, x)$$

is an equivalence of (ordinary) categories. In particular this definition applies to 2-categories $\mathbf{Fib}(\mathbf{B})$ for arbitrary ordinary categories $\mathbf{B}$.

Definition B.1 *For $P \in \mathbf{Fib}(\mathbf{B})$ a* descent map w.r.t. P *is a cartesian functor $F : Q' \to Q$ over $\mathbf{B}$ such that*

$$\mathbf{Fib}(\mathbf{B})(F, P) : \mathbf{Fib}(\mathbf{B})(Q, P) \to \mathbf{Fib}(\mathbf{B})(Q', P)$$

is an equivalence of (ordinary) categories.

If $u : J \to I$ is a morphism in $\mathbf{B}$ we write K_u for the posetal groupoid fibered over $\mathbf{B}$ whose morphisms over K are pairs (v_1, v_2) of morphisms from K to J with $uv_1 = uv_2$ for which reindexing along $w : L \to K$ is given by (v_1w, v_2w). We write Q_u for the cartesian functor from K_u to $\underline{I}$ sending (v_1, v_2) to $uv_1 = uv_2$. This cartesian functor $Q_u : K_u \to \underline{I}$ factors through the discrete subfibration $i_u : S_u \hookrightarrow \underline{I}$ consisting of all maps of the form uv in $\mathbf{B}$ via a (unique) cartesian functor $E_u : K_u \to S_u$. Obviously $E_u : K_u \to S_u$ is an equivalence in the 2-category $\mathbf{Fib}(\mathbf{B})$ for which reason i_u is a descent map w.r.t. P, i.e. $\mathbf{Fib}(\mathbf{B})(i_u, P) : \mathbf{Fib}(\mathbf{B})(\underline{I}, P) \to \mathbf{Fib}(\mathbf{B})(S_u, P)$ is an equivalence of ordinary categories, iff $\mathbf{Fib}(\mathbf{B})(Q_u, P) : \mathbf{Fib}(\mathbf{B})(\underline{I}, P) \to \mathbf{Fib}(\mathbf{B})(K_u, P)$ is an equivalence of ordinary categories.

Traditionally, one writes $\mathsf{Des}_u(P)$ for $\mathbf{Fib}(\mathbf{B})(K_u, P)$ and calls it the category of "descent data for P w.r.t. u" and says that "u is is a descent map w.r.t. P" iff $\mathsf{Des}_u(P)$ is equivalent to $P(I) \simeq \mathbf{Fib}(\mathbf{B})(\underline{I}, P)$ via $\mathbf{Fib}(\mathbf{B})(Q_u, P)$.[27] Thus u is a descent map w.r.t. P in this traditional sense iff i_u is a descent map w.r.t. P in the sense of Def. B.1.

Using the notion of descent map one easily defines what is a $\mathfrak{J}$-stack for a Grothendieck topology $\mathfrak{J}$ on $\mathbf{B}$.

Definition B.2 *A* $\mathfrak{J}$-stack *is a fibration $P \in \mathbf{Fib}(\mathbf{B})$ such that for every $S \in \mathfrak{J}(I)$ the inclusion $i_S : S \hookrightarrow \underline{I}$ is a descent map w.r.t. P.*

---

[27] The celebrated Bénabou-Roubaud Theorem from 1970 characterizes descent maps for fibrations P with internal sums over a base category $\mathbf{B}$ with pullbacks as those maps $u : J \to I$ in $\mathbf{B}$ for which $u^* : P(I) \to P(J)$ is monadic.

Its proof is based on a lemma saying that $\mathsf{Des}_u(P)$ is equivalent to the category of algebras for the monad induced by the adjunction $\coprod_u \dashv u^*$.

Obviously, a discrete fibration over $\mathbf{B}$ is a $\mathfrak{J}$-stack iff the corresponding presheaf over $\mathbf{B}$ is a $\mathfrak{J}$-sheaf.

C Stably Precohesive Geometric Morphisms

In his work on "axiomatic cohesion" Lawvere calls a topos $\mathbf{E}$ over $\mathbf{Set}$ *precohesive* iff $\mathbf{E}$ is 2-valued, i.e. Γ preserves the subobject classifier, $\Delta : \mathbf{Set} \to \mathbf{E}$ has a left adjoint Π and $\Gamma : \mathbf{E} \to \mathbf{Set}$ has a right adjoint ∇.

Let $\mathbf{S}$ be an arbitrary base topos. By A.4.6 of [Jo02] a geometric morphism $F \dashv U : \mathbf{E} \to \mathbf{S}$ is hyperconnected iff U preserves subobject classifiers. Thus, it appears as natural to call a geometric morphism $F \dashv U : \mathbf{E} \to \mathbf{S}$ *precohesive* iff it is hyperconnected, locally connected and local. However, in order to stay in accordance with Lawvere's terminology we call such geometric morphisms *stably precohesive*.

Let $F \dashv U : \mathbf{E} \to \mathbf{S}$ be a connected geometric morphism. If F has a left adjoint L then the counit ε of this adjunction is a natural isomorphism. Thus a connected geometric morphism $F \dashv U : \mathbf{E} \to \mathbf{S}$ is locally connected iff F has a left adjoint L such that

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "b"'] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d, "a"] \\ FJ \arrow[r, "Fu"'] & FI \end{tikzcd} \quad \text{implies} \quad \begin{tikzcd} LB \arrow[r, "Lf"] \arrow[d, "Lb"'] \arrow[dr, phantom, "\lrcorner"] & LA \arrow[d, "La"] \\ LFJ \arrow[r, "LFu"'] & LFI \end{tikzcd}$$

since the counit ε of $L \dashv F$ is a natural isomorphism and thus the right hand square is a pullback iff

$$\begin{tikzcd} LB \arrow[r, "Lf"] \arrow[d, "\hat{b}"'] \arrow[dr, phantom, "\lrcorner"] & LA \arrow[d, "\hat{a}"] \\ J \arrow[r, "u"'] & I \end{tikzcd}$$

is a pullback (since $\hat{a} = \varepsilon_I \circ La$ and $\hat{b} = \varepsilon_J \circ Lb$). Thus, a geometric morphism $F \dashv U : \mathbf{E} \to \mathbf{S}$ is stably precohesive iff F has a left adjoint L and U has a right adjoint R such that U preserves subobject classifiers and L sends pullbacks of cospans where one of the legs is in the image of F to pullbacks.

As shown in [Jo11] for stably precohesive geometric morphisms $F \dashv U : \mathbf{E} \to \mathbf{S}$ the left adjoint L to F sends pullbacks of cospans whose common codomain is in the image of F to pullbacks, i.e. the fibered left adjoint Π to Δ preserves binary products in each fiber. Thus, a geometric morphism $F \dashv U : \mathbf{E} \to \mathbf{S}$ is stably precohesive iff it is hyperconnected local and F has a left adjoint L sending pullbacks of cospans with common codomain in the image of F to pullbacks, i.e. $L_{/FI} : \mathbf{E}/FI \to \mathbf{S}/LFI \simeq \mathbf{S}/I$ preserves binary products for all I in $\mathbf{S}$.

Since a geometric morphism is locally connected iff its inverse image part preserves dependent products a geometric morphism $F \dashv U : \mathbf{E} \to \mathbf{S}$ is stably precohesive iff F preserves dependent products, U preserves subobject classifiers and U has a right adjoint.

Moreover, one knows that a geometric morphism $F \dashv U : \mathbf{E} \to \mathbf{S}$ is locally connected iff the functor $F_{/I} : \mathbf{S}/I \to \mathbf{E}/FI$ preserves exponentials for all $I \in \mathbf{S}$. Lawvere calls a geometric morphism simply *precohesive* iff it is hyperconnected local and its inverse image part preserves exponentials. Thus, a geometric morphism $F \dashv U : \mathbf{E} \to \mathbf{S}$ is stably precohesive iff the geometric morphism $F_{/I} : \mathbf{S}/I \to \mathbf{E}/FI$ is precohesive for all I in $\mathbf{S}$.

It is an open question (raised by Lawvere and Menni in their 2015 TAC paper) whether all precohesive geometric morphisms are stably precohesive.[28] We expect the answer to this question to be negative although we have not been able to come up with a counterexample so far.[29] The reason for this expectation is that in non-wellpointed toposes a predicate on X need not be universally valid even if it holds for all global elements of X.

[28]The reason why Lawvere and Menni are interested in hyperconnected and local geometric morphisms $F \dashv U : \mathbf{E} \to \mathbf{S}$ such that F preserves exponentials and has a left adjoint is that such data correspond to a string of adjoints $L \dashv F \dashv U \dashv R : \mathbf{S} \to \mathbf{E}$ such that F (and thus also R) are full and faithful, L preserves finite products and the so called "Nullstellensatz" holds claiming that for every $X \in \mathbf{E}$ the unique map $\theta_X : UX \to LX$ with

$$\begin{tikzcd}
FUX \arrow[r, "\varepsilon_X"] \arrow[dr, "F\theta_X"'] & X \arrow[d, "\eta_X"] \\
& FLX
\end{tikzcd}$$

is epic. In intuitive terms the map θ_X sends every point of X to the piece in which it lies for which reason θ is called the points-to-pieces transform. Thus the "Nullstellensatz" may be read as the claim that "every piece of X contains a point" holds for all objects X in $\mathbf{E}$.

[29]However, R. Garner has recently come up with an example of a (bounded) local geometric morphism whose inverse image part preserves exponentials and has a finite product preserving left adjoint but which is not locally connected. Alas, this geometric morphism is not hyperconnected.