Fib/B within $\widehat{\mathbf{B}}$

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November 2002

Abstract
We show how the internal language of $\widehat{\mathbf{B}} = \mathbf{Set}^{\mathbf{B}^{\mathsf{op}}}$ can be used for speaking about fibrations over $\mathbf{B}$. The key point is that every (split) fibration over $\mathbf{B}$ appears as internal category in $\widehat{\mathbf{B}}$ where, of course, the category $\mathbf{Set}$ has to be chosen as large enough.
The objects of $\widehat{\mathbf{B}}$ have to be thought of as "big" sets, *i.e.* "classes", and representable presheaves play the role of "small" classes, *i.e.* "sets". Grothendieck's notion of *representable morphism* between presheaves provides the appropriate notion of family of small objects indexed by a possibly big object. It turns out that a fibration is locally small iff for the corresponding category internal to $\widehat{\mathbf{B}}$ its family of morphisms is representable.
Viewing fibrations over $\mathbf{B}$ as categories internal to $\widehat{\mathbf{B}}$ allows us to define an appropriate notion of *distributor between fibrations.* We show that distributors between categories $\mathbf{X}$ and $\mathbf{Y}$ internal to $\widehat{\mathbf{B}}$ correspond to cartesian functors from $\mathbf{Y}^{\mathsf{op}}\times\mathbf{X}$ to $\mathbf{Set}(\mathbf{B})$, *i.e.* $\widehat{\mathbf{B}}$ fibred over $\mathbf{B}$.

1 Introduction

The theory of fibred categories has been developed by the first named author since about thirty years to an extent that category theory can be developed over fairly general base categories $\mathbf{B}$ instead of only $\mathbf{Set}$. For a large part of elementary category theory it suffices that $\mathbf{B}$ has finite limits though sometimes it turns out as necessary to assume that $\mathbf{B}$ is for example regular or even an elementary topos.

In this fibrational approach to category theory over base categories $\mathbf{B}$ the rôle of categories is played by fibrations over $\mathbf{B}$, the role of functors is played by cartesian functors between fibrations over $\mathbf{B}$ and the role of natural transformations is played by cartesian natural transformations between cartesian functors over $\mathbf{B}$ (*i.e.* natural transformations all whose components are vertical arrows). This way fibrations over $\mathbf{B}$ organise into a 2-category denoted as $\mathbf{Fib}/\mathbf{B}$. According to this view properties of categories in elementary category theory like (local) smallness, completeness, well-poweredness *etc.* have to be reformulated as corresponding properties of fibrations over $\mathbf{B}$ (coinciding, of course, with the original ones if $\mathbf{B}$ equals $\mathbf{1}$, the terminal category with one object and one morphism).

Although this programme has turned out as working quite smoothly it still suffers from the drawback that such properties of fibrations (and cartesian functors and natural transformations) have to be formulated *externally*. Following the practice of elementary category theory (over $\mathbf{Set}$) one would rather like to formulate these properties within the internal language of the base category $\mathbf{B}$. Actually, already for $\mathbf{B} = \mathbf{Set}$ this is not literally possible when the category $\mathbf{C}$ under consideration is not small. The commonly accepted way out of this dilemma is to employ Grothendieck universes, *i.e.* to assume that $\mathbf{C}$ is a category internal to some Grothendieck universe $\mathcal{U}$ inside $\mathbf{Set}$. It is not clear at first sight how to do something similar for base categories $\mathbf{B}$ different from $\mathbf{Set}$.

However, as happens very often, the Yoneda lemma tells us how to overcome this problem: we simply replace base $\mathbf{B}$ by $\widehat{\mathbf{B}} = \mathbf{Set}^{\mathbf{B}^{\mathrm{op}}}$, the category of presheaves over $\mathbf{B}$, where, of course, the category $\mathbf{Set}$ has to be chosen large enough. An obvious advantage of such a procedure is that even if base $\mathbf{B}$ has moderately good properties the extended base $\widehat{\mathbf{B}}$ is a very simple topos and, therefore, has excellent logical properties.

Of course, some problems have to be solved for putting this idea to work.

Firstly, in Section 2 we show via the fibred Yoneda lemma how up to equivalence fibrations over $\mathbf{B}$ can be considered as internal categories in $\widehat{\mathbf{B}}$ and this way extend to split fibrations over $\widehat{\mathbf{B}}$.

Secondly, in Section 3 we show how the Yoneda embedding of $\mathbf{B}$ into $\widehat{\mathbf{B}}$ can be lifted to the fibrational level by identifying a full subfibration of $P_{\widehat{\mathbf{B}}} = \partial_1 : \widehat{\mathbf{B}}^{\mathbf{2}} \to \widehat{\mathbf{B}}$ as given by the *representable morphisms* in $\widehat{\mathbf{B}}$ in the sense of Grothendieck. Representable morphisms in $\widehat{\mathbf{B}}$ may be thought of families of *small*, *i.e.* representable, objects indexed by a big object, *i.e.* an arbitrary presheaf over $\mathbf{B}$. It turns out that this subfibration is equivalent to the one obtained by externalising the internal category in $\widehat{\mathbf{B}}$ obtained from the fundamental fibration $P_{\mathbf{B}} = \partial_1 : \mathbf{B}^{\mathbf{2}} \to \mathbf{B}$ for $\mathbf{B}$ in the way as described in Section 2.

In Section 5 we define a notion of distributor between fibrations over $\mathbf{B}$. As from Section 2 we know that fibrations over $\mathbf{B}$ can be considered as internal categories in $\widehat{\mathbf{B}}$ in a canonical way this just amounts to explicitating the well-established notion of distributor between internal categories (see *e.g.* Chapter 3 of [Joh]) in some category $\mathbb{C}$ with finite limits for the particularly simple case where $\mathbb{C} = \widehat{\mathbf{B}}$.

In Section 4 we identify a split fibration $\mathbf{Set}(\mathbf{B})$ equivalent over $\mathbf{B}$ to ${\mathsf{Y}_{\mathbf{B}}}^* P_{\widehat{\mathbf{B}}}$, the restriction of the fundamental fibration $P_{\widehat{\mathbf{B}}} : \widehat{\mathbf{B}}^{\mathbf{2}} \to \widehat{\mathbf{B}}$ for $\widehat{\mathbf{B}}$ along the Yoneda embedding $\mathsf{Y}_{\mathbf{B}}$. The split fibration $\mathbf{Set}(\mathbf{B})$ is motivated by the desire that the equivalence

$$\mathsf{Dist}_{\mathbf{B}}(\mathbf{X}, \mathbf{Y}) \cong \mathsf{Func}_{\mathbf{B}}(\mathbf{Y}^{\mathsf{op}} \times \mathbf{X}, \mathbf{Set}(\mathbf{B})) \cong \mathsf{Func}_{\mathbf{B}}(\mathbf{X}, \mathbf{Set}(\mathbf{B})^{\mathbf{Y}^{\mathsf{op}}})$$

holds for all categories $\mathbf{X}$ and $\mathbf{Y}$ internal to $\widehat{\mathbf{B}}$ in analogy to the theory of distributors over $\mathbf{Set}$. As in the latter we have $\mathsf{Dist}(\mathbf{1}, \mathbf{1}) \cong \mathbf{Set}$ the analogy forces us to define $\mathbf{Set}(\mathbf{B})$ as a split fibration isomorphic to $\mathsf{Dist}_{\mathbf{B}}(\mathbf{1}, \mathbf{1})$. Explicitating the construction of $\mathsf{Dist}_{\mathbf{B}}(\mathbf{1}, \mathbf{1})$ from Section 5 suggests to define $\mathbf{Set}(\mathbf{B})$ as

$$\mathbf{Set}(\mathbf{B})(-) = \mathbf{Set}^{(\mathbf{B}/-)^{\mathsf{op}}}$$

i.e. $\mathbf{Set}(\mathbf{B})(I) = \widehat{\mathbf{B}/I}$ for $I \in \mathbf{B}$ and $\mathbf{Set}(\mathbf{B})(\alpha)$ is change of base along $\mathbf{B}/\alpha = \Sigma_\alpha : \mathbf{B}/I \to \mathbf{B}/J$ for morphisms $\alpha : J \to I$ in $\mathbf{B}$. Evidently $\mathbf{Set}(\mathbf{B})$ is a split fibration over $\mathbf{B}$ which is too big to be considered as a category internal to $\widehat{\mathbf{B}}$. However, there is a canonical equivalence between $\widehat{\mathbf{B}}/X$ and $\mathsf{Cart}_{\mathbf{B}}(X, \mathbf{Set}(\mathbf{B}))$ natural in $X \in \widehat{\mathbf{B}}$ in which sense $\mathbf{Set}(\mathbf{B})$ may be considered as an *internalisation* of $\widehat{\mathbf{B}}$ fibred over itself.

2 Fib/B within $\widehat{\mathbf{B}}$

The aim of this section is to show that $\widehat{\mathbf{B}} = \mathbf{Set}^{\mathbf{B}^{\mathsf{op}}}$ provides a suitable internal language for speaking about $\mathbf{B}$ and fibrations over $\mathbf{B}$ when choosing $\mathbf{Set}$ large enough.

For this purpose we first recall a useful fact entailing that presheaf toposes are closed under slicing.

Theorem 2.1 *Let* $\mathbf{B}$ *be a category and* $P : \mathbf{X} \to \mathbf{B}$ *be a discrete fibration over* $\mathbf{B}$*. Then* $(\mathbf{Fib}/\mathbf{B})/P$ *is* 2*-isomorphic to* $\mathbf{Fib}/\mathbf{X}$*. Moreover, this isomorphism preserves discreteness of fibrations.*

Proof: First we show that a functor $F : \mathbf{Y} \to \mathbf{X}$ is a fibration iff $Q = P \circ F$ is a fibration and $F \in \mathsf{Cart}_{\mathbf{B}}(Q, P)$. The implication from left to right is easy. For the reverse direction assume that $F \in \mathsf{Cart}_{\mathbf{B}}(Q, P)$. For showing that F is a fibration over $\mathbf{X}$ suppose that $X \in \mathbf{Y}$ and $f : A \to F(X)$ in $\mathbf{X}$. Let $\varphi : Y \to X$ be a Q-cartesian arrow over $P(f)$. We claim that φ is F-cartesian arrow over f. As $F(\varphi)$ and f have the same codomain and are both above $F(f)$ it follows from discreteness of the fibration Q that $F(\varphi) = f$. Thus, the arrow φ is over f. For F-cartesianness of φ suppose that $\psi : Z \to X$ and $g : B \to A$ with $F(\psi) = f \circ g$. As f is Q-cartesian there exists a $\theta : Z \to Y$ over $P(g)$ with $\psi = \varphi \circ \theta$. As $P(g) = Q(\theta) = P(F(\theta))$ and both g and $F(\theta)$ have the same codomain A it follows by discreteness of P that $g = F(\theta)$. Uniqueness of θ is immediate from Q-cartesianness of φ as if $\theta' : Z \to Y$ with $\varphi \circ \theta'$ and $F(\theta') = g$ then $Q(\theta') = P(F(\theta')) = P(g)$ and thus $\theta = \theta'$.

Actually, we have shown that a morphism φ in $\mathbf{Y}$ is cartesian w.r.t. F iff it is cartesian w.r.t. $P \circ F$. Moreover, as P is a discrete fibration it reflects identities and, therefore, a morphism v in $\mathbf{Y}$ is vertical w.r.t. F iff it is vertical w.r.t. $P \circ F$.

Thus, if G is a fibration over $\mathbf{X}$ and $U : G \to F$ in $\mathbf{Cat}/\mathbf{X}$ then U is a cartesian functor from G to F over $\mathbf{X}$ iff U is a cartesian functor from $P \circ G$ to $P \circ F$. If U and U' are morphisms from G to F in $\mathbf{Cat}/\mathbf{X}$ and τ is a natural transformation from U to U' then τ is cartesian over $\mathbf{X}$ iff it is cartesian over $\mathbf{B}$.

As P reflects identities it follows that a fibration $F : \mathbf{Y} \to \mathbf{X}$ is discrete iff $P \circ F$ is discrete. □

Using that for arbitrary $\mathbf{B}$ the presheaf category $\widehat{\mathbf{B}}$ is equivalent to $\mathbf{Fib}_d/\mathbf{B}$, the category of discrete fibrations over $\mathbf{B}$, we get as an immediate consequence of Theorem 2.1 the following corollary.

Corollary 2.1 *For every* $X \in \widehat{\mathbf{B}}$ *we have* $\widehat{\mathbf{B}}/X \simeq \widehat{\mathbf{El}(X)}$ *where* $P_X = \partial_0 : \mathbf{El}(X) = \mathsf{Y}_{\mathbf{B}}{\downarrow}X \to \mathbf{B}$ *is the discrete fibration obtained from* X *via the Grothendieck construction.*

Proof: By Theorem 2.1 we have $(\mathbf{Fib}_d/\mathbf{B})/P_X \cong \mathbf{Fib}_d/\mathbf{El}(X)$. The claim follows as $\widehat{\mathbf{B}}/X \simeq (\mathbf{Fib}_d/\mathbf{B})/P_X$ and $\mathbf{Fib}_d/\mathbf{El}(X) \simeq \widehat{\mathbf{El}(X)}$. $\square$

In the sequel we will often tacitly use Corollary 2.1 for constructing maps to X by exhibiting the corresponding presheaf over $\mathbf{El}(X)$.

In theorem 2.1 we have seen that X-indexed families of fibrations correspond to cartesian functors over $\mathbf{B}$ to $X \in \widehat{\mathbf{B}}$ considered as a discrete fibration over $\mathbf{B}$. The dual concept of cartesian functors from X to a fibration $P : \mathbf{X} \to \mathbf{B}$ can be understood as the notion of X-indexed "family of objects" of the category over $\mathbf{B}$ as given by P. More generally, we may associate with every $P \in \mathbf{Fib}/\mathbf{B}$ the functor $\mathbf{SP}(P) : \widehat{\mathbf{B}}^{\mathsf{op}} \to \mathbf{Cat}$ defined as

$$\mathbf{SP}(P)(X) = \mathsf{Cart}_{\mathbf{B}}(X, P) \qquad \mathbf{SP}(P)(f) = \mathsf{Cart}_{\mathbf{B}}(f, P)$$

where $X \in \widehat{\mathbf{B}}$ understood as a discrete fibration over $\mathbf{B}$ and $f : Y \to X$ is a morphism in $\widehat{\mathbf{B}}$ understood as a cartesian functor between discrete fibrations over $\mathbf{B}$. Notice, that $\mathbf{Sp}(P)$, the restriction of $\mathbf{SP}(P)$ along $\mathsf{Y}_{\mathbf{B}}$, is canonically equivalent to P according to the fibred Yoneda lemma (see [B80]). Notice that $\mathbf{Fib}_{\mathsf{sp}}(\mathbf{B})$, the 2-category of split fibrations over $\mathbf{B}$, is isomorphic to $\mathbf{cat}(\widehat{\mathbf{B}})$, the 2-category of internal categories in $\widehat{\mathbf{B}}$. As for all $X \in \widehat{\mathbf{B}}$ there is a canonical isomorphism

$$\mathsf{Cart}_{\mathbf{B}}(X, P) \cong \mathbf{Fib}_{\mathsf{sp}}(\mathbf{B})(X, \mathbf{Sp}(P))$$

natural in X we get that $\mathbf{SP}(P)$ is canonically isomorphic to the externalisation of $\mathbf{Sp}(P)$ considered as a category internal to $\widehat{\mathbf{B}}$.

3 B as a Universe of Small Objects in $\widehat{\mathbf{B}}$

One wants to think of the representable objects in $\widehat{\mathbf{B}}$ as "small" objects. Nonrepresentable presheaves are thought of as "big" objects in $\widehat{\mathbf{B}}$. As usual for arbitrary $X \in \widehat{\mathbf{B}}$ a family of possibly big objects indexed by X is simply a morphism $f : Y \to X$ in $\widehat{\mathbf{B}}$. However, there arises the question of what is a family of "small" objects indexed by a possibly big object. The answer to this question is provided by A. Grothendieck's notion of *representable morphism.*

Definition 3.1 *A morphism $f : Y \to X$ in $\widehat{\mathbf{B}}$ is called* representable *or* a family of small objects *iff for all pullbacks*

$$\begin{tikzcd} J \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\ I \arrow[r] & X \end{tikzcd}$$

the object J is representable whenever I is representable.

Notice that $\mathsf{Y}_{\mathbf{B}}(\alpha)$ is representable iff all pullbacks of α exists in $\mathbf{B}$ (as $\mathsf{Y}_{\mathbf{B}}$ preserves and reflects finite limits). Thus, under the (reasonable) assumption that $\mathbf{B}$ has pullbacks families of small objects indexed by small objects are precisely the maps in $\mathbf{B}$. Notice that $X \to 1$ is small iff $I \times X$ is representable for all $I \in \mathbf{B}$. Thus, for representable X the terminal projection $X \to 1$ need not be a representable morphism (unless $\mathbf{B}$ has binary products). Moreover, the terminal object in $\widehat{\mathbf{B}}$ need not be representable (unless $\mathbf{B}$ has a terminal object).

However, under the reasonable assumption that $\mathbf{B}$ has finite limits we have that an object X is representable iff its terminal projection $X \to 1$ is a representable morphism and that a morphism from X to a representable object I is representable iff X is a representable object. Thus, if $\mathbf{B}$ has finite limits a family is a family of small objects iff all its subfamilies indexed by a small object are families of small objects.

We now investigate some closure properties of representable maps relevant when viewing them as families of small objects.

Lemma 3.1 *Let $\mathbf{B}$ be a category and $\mathcal{S}$ the collection of representable maps in $\widehat{\mathbf{B}}$. Then we have*

(1) *$\mathcal{S}$ is stable under pullbacks along arbitrary morphisms in $\widehat{\mathbf{B}}$.*

(2) *$\mathcal{S}$ is a subcategory of $\mathcal{S}$ containing all isomorphism of $\widehat{\mathbf{B}}$.*

(3) *There exists a map $el : E \to \mathbf{set}(\mathbf{B})$ in $\mathcal{S}$ such that every map in $\mathcal{S}$ can be obtained as pullback of el.*

(4) *If $\mathbf{B}$ is locally cartesian closed then for all $f : Y \to X$ and $g : Z \to Y$ in $\mathcal{S}$ it holds that $\Pi_f(g) \in \mathcal{S}$, too.*

Proof: It is obvious from Definition 3.1 that $f : Y \to X$ is in $\mathcal{S}$ iff for every $s : I \to X$ with I representable it holds that s^*f is a map between representable objects. From this it is clear that $\mathcal{S}$ is stable under pullbacks along arbitrary morphisms in $\widehat{\mathbf{B}}$ and that $\mathcal{S}$ contains all isos. For closure under composition suppose $f : Y \to X$ and $g : Z \to Y$ are in $\mathcal{S}$. Let $s : I \to X$ with I representable. Consider the pullback

$$\begin{tikzcd}
K \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "g"] \\
J \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\
I \arrow[r, "s"'] & X
\end{tikzcd}$$

As $f \in \mathcal{S}$ it follows that J is representable and, therefore, as $g \in \mathcal{S}$ it follows that K is representable, too.
ad (3) : According to Lemma 2.1 morphisms in $\widehat{\mathbf{B}}$ to $I \in \mathbf{B}$ correspond to presheaves over $\mathbf{B}/I$, the category of elements of $\mathsf{Y}_{\mathbf{B}}(I)$. One easily sees that A is a representable presheaf over $\mathbf{B}/I$ iff the source of the corresponding morphism to I in $\widehat{\mathbf{B}}$ is a representable presheaf over $\mathbf{B}$. Thus, a presheaf $A : (\mathbf{B}/I)^{\mathsf{op}} \to \mathbf{Set}$ corresponds to a small map to I iff for all $\alpha : J \to I$ in $\mathbf{B}$ the presheaf $(\Sigma_\alpha)^*A = A \circ (\Sigma_\alpha)^{\mathsf{op}} : (\mathbf{B}/J)^{\mathsf{op}} \to \mathbf{Set}$ is representable. Such presheaves A we call “stably representable” and they organise into a presheaf $\mathbf{set}(\mathbf{B}) : \mathbf{B}^{\mathsf{op}} \to \mathbf{Set}$ putting

$$\mathbf{set}(\mathbf{B})(I) = \{A \in \widehat{\mathbf{B}/I} \mid A \text{ stably representable}\} \quad \text{and} \quad U(\alpha) = \Sigma_\alpha^* \; .$$

Now we describe the generic map $el : E \to \mathbf{set}(\mathbf{B})$ in terms of its corresponding presheaf (also denoted as E) on $\mathbf{El}(\mathbf{set}(\mathbf{B}))$: if $A \in \mathbf{set}(\mathbf{B})(I)$ then $E(A) = A(id_I)$ and for $\alpha : \alpha^*A \to A$ in $\mathbf{El}(\mathbf{set}(\mathbf{B}))$ we define $E(\alpha) = A(\alpha \xrightarrow{\alpha} id_I)$. One readily checks that for $a : I \to \mathbf{set}(\mathbf{B})$ the map $el(a) = a^*el$ is isomorphic to the small map whose corresponding representable presheaf is $A = a(id_I)$. Thus, the map el is small. It is generic for small maps as every $f : Y \to X$ in $\mathcal{S}$ is isomorphic to χ^*el where $\chi : X \to \mathbf{set}(\mathbf{B})$ is defined as follows: for $x \in X(I)$ the presheaf $\chi_I(x) : (\mathbf{B}/I)^{\mathsf{op}} \to \mathbf{Set}$ is given by

$$\chi_I(x)(\alpha) = \{y \mid fy = x\alpha\} \qquad \text{and} \qquad \chi_I(\beta : \alpha\beta \to \alpha)(y) = y\beta \; .$$

ad (4) : Suppose $f : Y \to X$ and $g : Z \to Y$ are in $\mathcal{S}$. We have to show that for $s : I \to X$ with I representable the map $s^*\Pi_f(g) \cong \Pi_\alpha(\beta)$ is representable, too, where

$$\begin{tikzcd}
K \arrow[r] \arrow[d, "\beta"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "g"] \\
J \arrow[r] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\
I \arrow[r] & Z
\end{tikzcd}$$

but $\Pi_\alpha(\beta)$ is in $\mathbf{B}$ as $\mathsf{Y}_\mathbf{B}$ preserves locally cartesian closed structure. $\square$

As representable objects are never initial for every $X \in \widehat{\mathbf{B}}$ the morphism $0 \to X$ is not small unless X is initial. Thus, if $f : Y \to X$ is in $\mathcal{S}$ and $m : Z \rightarrowtail Y$ the composite $f \circ m$ will in general not be contained in $\mathcal{S}$. However, if $\mathbf{B}$ has finite limits then for $h_1, h_2 : g \to f$ with $f, g \in \mathcal{S}$ it holds that $g \circ e \in \mathcal{S}$ where e is the equalizer of h_1 and h_2 in $\widehat{\mathbf{B}}$.

In general $\Omega_{\widehat{\mathbf{B}}} \to 1$ will not be in $\mathcal{S}$. For example if $\mathbf{B}$ is a poset with a terminal object then $\Omega_{\widehat{\mathbf{B}}} \to 1$ is not small as $\Omega_{\widehat{\mathbf{B}}}$ is not representable (as it is not subterminal).

By Lemma 3.1(1) the collection $\mathcal{S}$ of representable morphisms in $\widehat{\mathbf{B}}$ determines a full subfibration of $P_{\widehat{\mathbf{B}}} : \widehat{\mathbf{B}}^{\mathbf{2}} \to \widehat{\mathbf{B}}$ which is definable in the sense of [B80].

Lemma 3.2 *For maps $f : Y \to X$ in $\widehat{\mathbf{B}}$ there exists a subobject $i : X_0 \hookrightarrow X$ such that for all $g : Z \to X$, $g^*f \in \mathcal{S}$ iff g factors through i.*

Proof: Define $X_0(I)$ as the set of all $s : I \to X$ such that s^*f is representable. Obviously, if $x \in X_0(I)$ and $\alpha : J \to I$ in $\mathbf{B}$ then $s\alpha \in X_0(J)$ as by Lemma 3.1 representable morphisms are stable under pullbacks in $\widehat{\mathbf{B}}$. Thus X_0 is a subpresheaf of X. Let i be the corresponding inclusion. Suppose $g : Z \to X$ in $\widehat{\mathbf{B}}$. Then g^*f is representable iff for all generalised elements $s : I \to Z$ the map $s^*g^*f \cong (gs)^*f$ is representable, *i.e.* iff $gs \in X_0(I)$ for all generalised elements $s : I \to Z$, *i.e.* if g factors through i. $\square$

That representable morphisms capture the right notion of smallness is supported by the following Theorem.

Theorem 3.1 *A fibration $P : \mathbf{X} \to \mathbf{B}$ is locally small iff for the corresponding internal category $C = \mathbf{Sp}(P)$ the "hom-family" $C_1 \to C_0 \times C_0$ is a representable morphism.*

Moreover, P is small iff P is locally small and C_0 is representable.

4 A Large Internal Category $\mathbf{Set}(\mathbf{B})$

It follows from the proof of Lemma 3.1(3) that if $\mathbf{B}$ has pullbacks then the fundamental fibration $P_{\mathbf{B}} = \partial_1 : \mathbf{B^2} \to \mathbf{B}$ is equivalent to the split fibration $\mathbf{set}(\mathbf{B})$ sending $I \in \mathbf{B}$ to the category $\mathbf{set}(\mathbf{B})(I)$ of *representable* presheaves over $\mathbf{B}/I$ and $\alpha : J \to I$ to $\mathbf{Set}^{(\Sigma_\alpha)^{\mathsf{op}}}$, *i.e.* change of base along Σ_α.[1]

The same constructions as in the proof of Lemma 3.1(3) can be performed when dropping the restriction to stably representable presheaves.

Definition 4.1 *For a category $\mathbf{B}$ let $\mathbf{Set}(\mathbf{B})$ be the presheaf of large categories over $\mathbf{B}$ with*

$$\mathbf{Set}(\mathbf{B})(I) = \mathbf{Set}^{(\mathbf{B}/I)^{\mathsf{op}}} \qquad \textit{and} \qquad \mathbf{Set}(\mathbf{B})(\alpha) = \mathbf{Set}^{(\Sigma_\alpha)^{\mathsf{op}}}$$

for objects I and morphisms α in $\mathbf{B}$.

Notice that the split fibration $\mathbf{Set}(\mathbf{B})$ is equivalent to $\mathsf{Y}_{\mathbf{B}}^* P_{\widehat{\mathbf{B}}}$ where $P_{\widehat{\mathbf{B}}} = \partial_1 : \widehat{\mathbf{B}}^{\mathbf{2}} \to \widehat{\mathbf{B}}$ is the fundamental fibration for $\widehat{\mathbf{B}}$.[2]

Again as in the proof of Lemma 3.1(3) we can construct a presheaf E over $\mathbf{El}(\mathbf{Set}(\mathbf{B}))$ sending $A \in \mathbf{Set}(\mathbf{B})(I)$ to $E(A) = A(id_I)$ and $\alpha : \alpha^* A \to A$ to $E(\alpha) = A(\alpha \stackrel{\alpha}{\to} id_I)$. This presheaf E is generic in the sense that for every presheaf $A : \mathbf{El}(X)^{\mathsf{op}} \to \mathbf{Set}$ we have $A \cong E \circ \widehat{A}$ where $\widehat{A} : \mathbf{El}(X) \to \mathbf{Set}(\mathbf{B})$ is the functor sending $x \in X(I)$ to $A \circ \widehat{x}^{\mathsf{op}}$ where $\widehat{x} : \mathbf{B}/I \to \mathbf{El}(X)$ is the cartesian functor (over $\mathbf{B}$) with $\widehat{x}(id_I) = x$. The discrete fibration corresponding to E is denoted as $El : E \to \mathbf{Set}(\mathbf{B})$.

[1]If $\mathbf{B}$ has pullbacks then for all $\alpha : J \to I$ in $\mathbf{B}$ change of base along Σ_α preserves representability of presheaves because $(\Sigma_\alpha)^* \mathbf{B}/I(-, \beta) \cong \mathbf{B}/J(-, \alpha^*\beta)$.

[2]Notice that $\mathsf{Y}_{\mathbf{B}}^* P_{\widehat{\mathbf{B}}}$ is the fibration of discrete fibrations over $\mathbf{B}$ obtained as restriction of the fibration $\mathbf{Fib}(\mathbf{B}) {\to} \mathbf{B}$ of fibrations over $\mathbf{B}$ as discussed at the end of the first chapter of [B80] where $\mathbf{Fib}(\mathbf{B}) {\to} \mathbf{B}$ is constructed from the fibration $\mathbf{Fib} {\to} \mathbf{Cat}$ by change of base along $\mathsf{Y}_{\mathbf{B}}$.

One may consider the following category $\mathbf{E_B}$ fibered over (the domain of) $\mathbf{Fib}(\mathbf{B}) {\to} \mathbf{B}$ and thus over $\mathbf{B}$. Objects of $\mathbf{E_B}$ over I in $\mathbf{B}$ are pairs (P, S) where P is a fibration over $\mathbf{B}/I$ and S is a cartesian section of P. Morphisms over $u : J \to I$ from (Q, T) to (P, S)

are functors F rendering the following diagram

$$\begin{tikzcd}
\mathbf{B}/J \arrow[r, "\Sigma_u"] \arrow[d, "T"'] & \mathbf{B}/I \arrow[d, "S"] \\
\mathbf{Y} \arrow[r, "F"] \arrow[d, "Q"'] & \mathbf{X} \arrow[d, "P"] \\
\mathbf{B}/J \arrow[r, "\Sigma_u"'] & \mathbf{B}/I
\end{tikzcd}$$

commutative and which are cartesian as functors from Q to P over Σ_u, i.e. send Q-cartesian arrows to P-cartesian arrows. Such a morphism is cartesian iff the lower square is a pullback (in which case also the upper square is a pullpack). The category $\mathbf{E_B}$ is fibered (over the domain of) $\mathbf{Fib}(\mathbf{B}){\to}\mathbf{B}$ by first projection, i.e. sending (P,S) to P and $F:(Q,T)\to(P,S)$ to F.

The fibration $\mathbf{E_B}$ over $\mathbf{Fib}(\mathbf{B}){\to}\mathbf{B}$ weakly classifies fibrations over fibrations over $\mathbf{B}$ in the sense that for a fibration $P:\mathbf{X}\to\mathbf{B}$ fibrations over P (i.e. over $\mathbf{X}$) up to equivalence correspond to cartesian functors from P to $\mathbf{Fib}(\mathbf{B}){\to}\mathbf{B}$ via pullback of $\mathbf{E_B}$ over $\mathbf{Fib}(\mathbf{B}){\to}\mathbf{B}$ along them.

But there is a problem. This claim can hold only in a very weak sense for the following reason. Let $P:\mathbf{X}\to\mathbf{B}$ and $Q:\mathbf{Y}\to\mathbf{X}$ be fibrations. For a (split) cartesian functor $X:\underline{I}\to P$ we may consider the pullback

$$\begin{tikzcd}
\mathbf{Y}_X \arrow[r] \arrow[d, "Q_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Y} \arrow[d, "Q"] \\
\mathbf{B}/I \arrow[r, "X"'] & \mathbf{X}
\end{tikzcd}$$

which, of course, involves choice. For $u:J\to I$ in $\mathbf{B}$ we may now consider

$$\begin{tikzcd}
\mathbf{Y}_{u^*X} \arrow[r] \arrow[d, "Q_{u^*X}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Y}_X \arrow[r] \arrow[d, "Q_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Y} \arrow[d, "Q"] \\
\mathbf{B}/J \arrow[r, "\Sigma_u"'] & \mathbf{B}/I \arrow[r, "X"'] & \mathbf{X}
\end{tikzcd}$$

where u^*X stands for $X\circ\Sigma_u$. Obviously, these choices define a cartesian functor $Q_{(-)}:\mathbf{Sp}(P)\to[\mathbf{Fib}(\mathbf{B}){\to}\mathbf{B}]$ such that pulling back $\mathbf{E_B}\to[\mathbf{Fib}(\mathbf{B}){\to}\mathbf{B}]$ along it gives rise to the pullback of Q along $E_P:U(\mathbf{Sp}(P))\to P$, the counit of $U\dashv\mathbf{Sp}$ at P, which, alas, is only equivalent to Q in $\mathbf{Fib}$ over $\mathbf{Cat}$.

Notice that $\mathbf{set}(\mathbf{B})$ (as constructed in the proof of Lemma 3.1(3)) is the greatest subpresheaf of $\mathbf{Set}(\mathbf{B})$ such that the restriction of *El* to it gives rise to a small map.

For a category C in $\mathbf{B}$ the analogue of the category of set valued presheaves over C is given by the fibration $P_{\mathbf{B}}{}^{\underline{C}} \simeq \mathbf{set}(\mathbf{B})^{\underline{C}}$ over $\mathbf{B}$. Now as fibrations over $\mathbf{B}$ appear as internal categories in $\widehat{\mathbf{B}}$ the category of presheaves over P is given by $P_{\widehat{\mathbf{B}}}{}^{P} \simeq \mathbf{Set}(\mathbf{B})^{P}$.

5 Distributors between Fibrations over B

In [B73, Joh] it has been defined and investigated what are distributors between internal categories. As arbitrary fibrations over $\mathbf{B}$ may be considered a categories internal to $\widehat{\mathbf{B}}$ this opens up the possibility to define what is a distributor between arbitrary (non-small) fibrations. For internal categories A, B a distributor from A to B is given by a family $\Phi : F \to A_0 \times B_0$ together with an action of the morphisms of A and B. We will call a distributor *locally small* iff the map Φ is representable (in the sense of Def. 3.1).

References

[B71] J. Bénabou *Problèmes dans les topos* Lecture Notes by J.-R. Roisin of a Course from 1971, published as Tech. Rep. of Univ. Louvain-la-Neuve (1973).

[B73] J. Bénabou *Des Distributeurs* Tech. Rep. no. 33 of Univ. Louvain-la-Neuve (1973).

[B80] J. Bénabou *Des Catégories Fibrées* Lecture Notes by J.-R. Roisin of a Course at Univ. Louvain-la-Neuve (1980).

[Joh] Peter T. Johnstone *Topos Theory* Academic Press (1977).

6 Garbage

This can be achieved in a very pleasing way via the fibred Yoneda lemma giving rise to a full and faithful 2-functor $\mathbf{Sp} : \mathbf{Fib}/\mathbf{B} \to \mathbf{Fib}_{\mathbf{sp}}(\mathbf{B})$ where $\mathbf{Fib}_{\mathbf{sp}}(\mathbf{B})$ is the 2-category of split fibrations over $\mathbf{B}$. (Under axiom of choice $\mathbf{Sp}$ is even a 2-equivalence!) As split fibrations over $\mathbf{B}$ can be considered as internal categories in $\widehat{\mathbf{B}}$ an arbitrary fibration $P : \mathbf{X} \to \mathbf{B}$ gives rise to the split fibration $P_{\mathbf{Sp}(P)} : \underline{\mathbf{Sp}(P)} \to \widehat{\mathbf{B}}$ obtained by externalising the internal category $\mathbf{Sp}(P)$ in $\widehat{\mathbf{B}}$. It is easy to see that $\mathbf{Sp}(P)$ is isomorphic (over $\mathbf{B}$) to ${\mathsf{Y}_{\mathbf{B}}}^* P_{\mathbf{Sp}(P)}$, the restriction of the split fibration $P_{\mathbf{Sp}(P)}$ along the Yoneda embedding $\mathsf{Y}_{\mathbf{B}} : \mathbf{B} \hookrightarrow \widehat{\mathbf{B}}$. Thus, by the fibred Yoneda lemma the fibration P is equivalent over $\mathbf{B}$ to the split fibration ${\mathsf{Y}_{\mathbf{B}}}^* P_{\mathbf{Sp}(P)}$ isomorphic to $\mathbf{Sp}(P)$ over $\mathbf{B}$. Thus, as indicated in the diagram

$$
\begin{tikzcd}
\mathbf{X} \arrow[r, "\simeq"] \arrow[dr, "P"'] & \mathbf{Sp}(P) \arrow[r] \arrow[d, "\mathbf{Sp}(P)"] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathbf{Sp}(P)} \arrow[d, "P_{\mathbf{Sp}(P)}"] \\
& \mathbf{B} \arrow[r, hook, "\mathsf{Y}_{\mathbf{B}}"'] & \widehat{\mathbf{B}}
\end{tikzcd}
$$

$P_{\mathbf{Sp}(P)}$ provides the desired extension of P to a split fibration over $\widehat{\mathbf{B}}$.