The World's Simplest Proof of Moens's Lemma

Let $\mathbb{B}$ be a category with finite limits and $P : \mathbb{X} \to \mathbb{B}$ be a fibration of finite limit categories which is also a cofibration. For $I \in \mathbb{B}$ let $\varphi_I : 1_I \to \Delta(I)$ be a cocartesian arrow from 1_I (terminal object in the fibre $\mathbb{X}_I$) over the terminal projection $I \to 1$ in $\mathbb{B}$. For $u : J \to I$ in $\mathbb{B}$ let $\Delta(u) : \Delta(J) \to \Delta(I)$ be the unique vertical arrow making the diagram

$$\begin{tikzcd} 1_J \arrow[r, "\varphi_J"] \arrow[d, "1_u"'] \arrow[dr, phantom, "(1)"] & \Delta(J) \arrow[d, "\Delta(u)"] \\ 1_I \arrow[r, "\varphi_I"'] & \Delta(I) \end{tikzcd}$$

commute. This gives rise to a functor $\Delta : \mathbb{B} \to \mathbb{X}_1$.

A fibration P as above is called *internally extensive* iff every commuting diagram with α and β vertical

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & U \arrow[d, "\beta"] \\ 1_I \arrow[r, "\varphi_I"'] & \Delta(I) \end{tikzcd}$$

is a pullback iff φ is cocartesian.[1]

Obviously, for internally extensive fibrations P the functor $\varphi_I^* : \mathbb{X}_1/\Delta(I) \to \mathbb{X}_I/1_I \cong \mathbb{X}_I$ is an equivalence and thus, since (1) commutes, the diagram

$$\begin{tikzcd} \mathbb{X}_J & \mathbb{X}_1/\Delta(J) \arrow[l, "\varphi_J^*"', "\simeq"] \\ \mathbb{X}_I \arrow[u, "u^*"] & \mathbb{X}_1/\Delta(I) \arrow[l, "\simeq"', "\varphi_I^*"] \arrow[u, "\Delta(u)^*"'] \end{tikzcd}$$

commutes up to isomorphism for all $u : J \to I$ in $\mathbb{B}$ and thus $P \simeq P_\Delta = \Delta^* P_{\mathbb{X}_1}$ (where $P_{\mathbb{X}_1} = \partial_1 : \mathbb{X}_1^{\mathbf{2}} \to \mathbb{X}_1$ is the fundamental fibration of $\mathbb{X}_1$).

Obviously, the functor Δ preserves 1. Recall[2] that a functor $F : \mathbb{B} \to \mathbb{C}$ between categories with pullbacks preserves pullbacks iff $P_F = F^* P_{\mathbb{C}}$ has internal sums, i.e. cocartesian arrows are stable under pullbacks along cartesian arrows. Thus, the fibration $P \simeq P_\Delta = \Delta^* P_{\mathbb{X}_1}$ has internal sums iff Δ preserves pullbacks (and thus all finite limits).

[1] Equivalently one may require that (1) cocartesian arrows are stable under pullbacks along vertical arrows and (2) from φ and $\varphi\alpha$ cocartesian and α vertical it follows that α is an iso, i.e. $\coprod_u$ reflects isos for all u in $\mathbb{B}$.

[2] see Lemma 13.2 (on pp.46-47) of my notes on *Fibred Categories*