

# Observations on Fibered Toposes

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29th April 2017

## 1 Toposes Fibered over a Base Topos

Let  $\mathcal{S}$  be a (base) topos. A fibration  $P : \mathcal{X} \rightarrow \mathcal{S}$  is a fibration of toposes iff all fibers are toposes and all reindexing functors are logical.

**Theorem 1.1** *A fibration  $P : \mathcal{X} \rightarrow \mathcal{S}$  of toposes has internal sums iff  $\mathcal{X}$  is a topos and  $P$  is a logical functor.*

*Proof:* By a theorem of Jibladze from 1988 a fibered topos  $P : \mathcal{X} \rightarrow \mathcal{S}$  has internal sums iff  $P$  is equivalent to the fibration  $P_F = \partial_1 : \mathcal{E} \downarrow F \rightarrow \mathcal{S}$  for some finite limit preserving functor  $F : \mathcal{S} \rightarrow \mathcal{E}$  between toposes. Notice that  $F$  is unique up to isomorphism since  $FI = \coprod_I 1_I$ . From the theory of Artin glueing one knows that  $\mathcal{E} \downarrow F$  is a topos and  $P_F$  is logical.

Suppose that  $P : \mathcal{X} \rightarrow \mathcal{S}$  is a fibration of toposes with  $\mathcal{X}$  a topos and  $P$  logical (or just preserving the cartesian closed structure). Since  $P$  is in particular a fibration of finite limit categories it has a full and faithful right adjoint  $1$  picking terminal objects in each fiber. Thus  $P \vdash 1$  is an injective geometric morphism. Proposition A.4.5.1 of Johnstone's *Elephant* tells us that  $P$  is logical iff  $P \dashv 1$  is equivalent to  $U^* \dashv \Pi_U : \mathcal{X}/U \rightarrow \mathcal{X}$  for some subterminal  $U$ . Notice that  $U$  is isomorphic to  $\Sigma_U 1_{\mathcal{X}}$  where  $\Sigma_U \dashv U^*$ . From the theory of Artin glueing one knows that the open subtopos  $U^* \vdash \Pi_U$  has a closed complement  $j : \mathcal{E} \hookrightarrow \mathcal{X}$  induced by the closure operator  $U \vee (-)$  on  $\mathcal{X}$  and that the fibration  $U^* : \mathcal{X} \rightarrow \mathcal{X}/U$  is equivalent to the fibration  $P_F = \partial_1 : \mathcal{E}/F \rightarrow \mathcal{S}$  where  $F = j^* \Pi_U$ . But since  $F$  preserves finite limits the fibration  $P_F$  equivalent to  $P$  has internal sums.  $\square$

Since in the proof of the backwards direction we have just used that  $P$  preserves finite limits and exponentials we get the

**Corollary 1.1** *A fibration  $P : \mathcal{X} \rightarrow \mathcal{S}$  of toposes has internal sums iff  $\mathcal{X}$  is a topos and  $P$  preserves finite limits and exponentials.*

## 2 A topos $\mathcal{E}$ has small sums iff $\mathbf{Fam}(\mathcal{E})$ is a topos

For every topos  $\mathcal{E}$  the fibration  $Fam(\mathcal{E}) : \mathbf{Fam}(\mathcal{E}) \rightarrow \mathbf{Set}$  is a fibration of toposes.

**Theorem 2.1** *For an elementary topos  $\mathcal{E}$  the category  $\text{Fam}(\mathcal{E})$  is cartesian closed iff  $\mathcal{E}$  has small sums.*

*Proof:* If  $\mathcal{E}$  has small sums then  $\text{Fam}(\mathcal{E})$  is equivalent to  $\mathcal{E}\downarrow\Delta$  where  $\Delta(I) = \coprod_I 1_{\mathcal{E}}$ . Since  $\Delta : \mathbf{Set} \rightarrow \mathcal{E}$  preserves finite limits  $\mathcal{E}\downarrow\Delta$  is a topos and thus in particular cartesian closed.

Suppose  $\text{Fam}(\mathcal{E})$  is cartesian closed. Since it has a subobject classifier anyway it is also a topos. As observed by Peter Johnstone the fibration  $\text{Fam}(\mathcal{E})$  is equivalent to  $U^* : \text{Fam}(\mathcal{E}) \rightarrow \text{Fam}(\mathcal{E})/U$  for  $U = (1, (0))$  (the initial object in the fiber over 1) since in  $\mathcal{E}$  every morphism to 0 is an iso. Thus  $\text{Fam}(\mathcal{E})$  is logical since  $U^*$  is logical from which it follows by Theorem 1.1 that  $\text{Fam}(\mathcal{E})$  has internal sums, i.e.  $\mathcal{E}$  has small sums.  $\square$

Thus  $\text{Fam}(\mathcal{E})$  is not cartesian closed if  $\mathcal{E}$  is the free topos (with nno) or a nontrivial realizability topos though  $\text{Fam}(\mathcal{E})$  certainly is a fibered topos.