A Fibrational View of Geometric Morphisms

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Abstract In this short note we will give a *survey* of the fibrational aspects of (generalised) geometric morphisms. Almost all of these results and observations are due to Bénabou's work on fibrational category theory.

Contents

1 Minimal Structural Requirements for the Base of a Fibration

What are the minimal structural requirements on a category $\mathbb{B}$ in order to develop category theory over base $\mathbb{B}$? We always want to be able to consider the base $\mathbb{B}$ as fibred over itself. That means that we require that the functor $P_{\mathbb{B}} = \partial_1 : \mathbb{B}^{\mathbf{2}} \to \mathbb{B}$ is a fibration. This is equivalent to $\mathbb{B}$ having all pullbacks.

Furthermore in order to simply *state* that a bifibration $P : \mathbb{X} \to \mathbb{B}$ satisfies the Beck-Chevalley condition the base $\mathbb{B}$ must have pullbacks.

2 Change of Base and Geometric Morphisms

Let $F : \mathbb{A} \to \mathbb{B}$ be a functor and let $F^* : Fib(\mathbb{B}) \to Fib(\mathbb{A})$ be the functor *reindexing fibrations along* F, i.e. performing *change of base along* F.

Already in [Bén80] Jean Bénabou has identified elementary conditions on a functor F which are equivalent to the preservation of "good properties" of fibrations by change of base along F.

Theorem 2.1 (Bénabou)
Let $F : \mathbb{A} \to \mathbb{B}$ be an arbitrary functor. Then change of base along F preserves smallness of fibrations *iff F has a right adjoint.*

Proof. $\Rightarrow$: Suppose that change of base along F preserves smallness of fibrations. Then for any $X \in \mathbb{B}$ the fibration $\partial_0 : F/X \to \mathbb{A}$ is small it is obtained by change of base along F from the small fibration $\partial_0 : \mathbb{B}/X \to \mathbb{B}$. But if the discrete (and therefore split) fibration $\partial_0 : F/X \to \mathbb{A}$ is small then there exists an object $UX \in \mathbb{A}$ such that the fibrations $\partial_0 : F/X \to \mathbb{A}$ and $\partial_0 : \mathbb{A}/UX \to \mathbb{A}$ are isomorphic. As both fibrations are discrete and therefore split this means that $\mathbb{B}(F\_, X)$ and $\mathbb{A}(\_, UX)$ are naturally isomorphic. As such a UX exists for all $X \in \mathbb{B}$ it follows that F has a right adjoint U.
$\Leftarrow$: Suppose that F has a right adjoint U. Then U preserves all limits existing in $\mathbb{B}$ and maps any category C internal to $\mathbb{B}$ to a category $U(C)$ internal to $\mathbb{A}$. A small fibration P over $\mathbb{B}$ is by definition isomorphic to a fibration of the

form $cat(C)$, the *externalisation* of C, for some category C internal to $\mathbb{B}$ But as $F^*cat(C)$ is canonically isomorphic to $cat(U(C))$ we get that F^*P is isomorphic to $cat(U(C))$. Thus, F^*P is a small fibration. □

Theorem 2.2 *Let $\mathbb{A}$ and $\mathbb{B}$ be categories with pullbacks and $F : \mathbb{A} \to \mathbb{B}$. Then the following are equivalent*

1. *change of base along F preserves the property of* having internal sums satisfying the Beck-Chevalley condition
2. *the fibration $\partial_1 : \mathbb{B}/F \to \mathbb{A}$* has internal sums satisfying the Beck-Chevalley condition
3. F preserves pullbacks*.*

Proof. It is well known from [Bén80] that change of base along a pullback preserving functor between categories with pullbacks preserves the property of having internal sums satisfying the Beck-Chevalley condition.

If change of base along F preserves the property of having internal sums satisfying the Beck-Chevalley condition then $F^*P_{\mathbb{B}} = \ partial_1 : \mathbb{B}/F \to \mathbb{A}$ has internal sums satisfying the Beck-Chevalley condition as $P_{\mathbb{B}} = \partial_1 : \mathbb{B}/\mathbb{B} \to \mathbb{B}$ has internal sums satisfying the Beck-Chevalley condition.

For closing the circle it remains to show that F preserves pullbacks if $F^*P_{\mathbb{B}} = \partial_1 : \mathbb{B}/F \to \mathbb{A}$ has internal sums satisfying the Beck-Chevalley condition.

Let

$$\begin{array}{ccc} U & \xrightarrow{q} & Y \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

be a pullback in $\mathbb{A}$. We have to show that

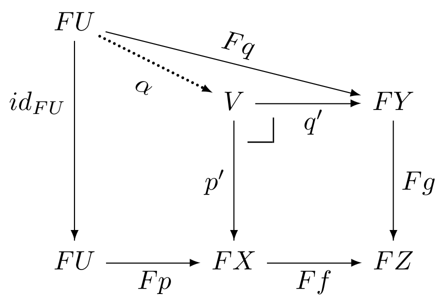
$$\begin{array}{ccc} FU & \xrightarrow{Fq} & FY \\ {\scriptstyle Fp}\downarrow & \lrcorner & \downarrow{\scriptstyle Fg} \\ FX & \xrightarrow[Ff]{} & FZ \end{array}$$

is a pullback in $\mathbb{B}$.

Consider the following diagram in $\mathbb{B}/F$

$$\begin{array}{ccccc} FU & \xrightarrow{Fq} & FY & \xrightarrow{id_{FY}} & FY \\ {\scriptstyle id_{FU}}\downarrow & & {\scriptstyle id_{FY}}\downarrow & & \downarrow{\scriptstyle Fg} \\ FU & \xrightarrow[Fq]{} & FY & \xrightarrow[Fg]{} & FZ \end{array}$$

where the left square is cartesian over q and the right square is cocartesian over g. Notice that the outer rectangle of the diagram above coincides with the outer rectangle of the following diagram



as $Ff \circ Fp = F(f \circ p) = F(g \circ q) = Fg \circ Fq$.

Due to the assumption that the fibration $\partial_1 : \mathbb{B}/F \to \mathbb{A}$ has sums satisfying the Beck-Chevalley condition we know that cocartesian arrows in $\mathbb{B}/F$ are stable under pullbacks along arbitrary cartesian arrows. As in the previous diagram the right hand square is cartesian over f the left hand square must be cocartesian over p, i.e. α must be an isomorphisms.

Therefore the square

$$\begin{array}{ccc} FU & \xrightarrow{Fq} & FY \\ {\scriptstyle Fp}\downarrow & \lrcorner & \downarrow{\scriptstyle Fq} \\ FX & \xrightarrow[Ff]{} & FZ \end{array}$$

is a pullback square.

Thus F preserves pullbacks. $\square$

Thus if $\mathbb{A}$ and $\mathbb{B}$ are categories with pullbacks then a *necessary* condition for F^* *preserving all good properties of fibrations under reindexing along* F is that F *preserves pullbacks and has a right adjoint.* It can be shown that this condition on F is also sufficient for reindexing along F to preserve all good properties of fibrations.

Thus a functor between categories with pullbacks preserves all good properties of fibrations under reindexing iff it preserves pullbacks and has a right adjoint.

If $\mathbb{A}$ and $\mathbb{B}$ are categories with pullbacks then a functor $F : \mathbb{A} \to \mathbb{B}$ that preserves pullbacks and has a right adjoint will be called *(the inverse image part of) a generalised geometric morphism.*

Thus a functor between categories with pullbacks preserves all good properties of fibrations under reindexing iff it is (the inverse image part of) a generalised geometric morphism.

If $F : \mathbb{A} \to \mathbb{B}$ is (the inverse image part of) a generalised geometric morphism then it is more general than an *ordinary geometric morphism* only in the aspect that it need not preserve *all* finite limits existing in $\mathbb{A}$ (but only those whose existence is guaranteed by $\mathbb{A}$ having pullbacks). *From the point of view that the categories involved are required only to have pullbacks it is more natural to require only preservation of pullbacks.*

A generalised geometric morphism F ($\dashv U$) will be called a *proper geometric morphism* or simply a *geometric morphism* iff F *preserves all finite limits existing in the source of* F. At first sight for lex categories $\mathbb{A}$ and $\mathbb{B}$ the adequate notion of 'geometric morphism' seems to that of a proper geometric morphism. That this definitely is not the case will be shown by the following examples.

If $\mathbb{A}$ and $\mathbb{B}$ have pullbacks and $\mathbb{A}$ furthermore has a terminal object then a generalised geometric morphism $F : \mathbb{A} \to \mathbb{B}$ need not preserve it. Actually there is a lot of important examples of generalised geometric morphisms that do not preserve terminal objects.

(i) If $f : J \to I$ is a morphism in $\mathbb{B}$ then $\Sigma_f : \mathbb{B}/J \to \mathbb{B}/I$ preserves pullbacks and has right adjoint $f^* : \mathbb{B}/I \to \mathbb{B}/J$.

Of course, the slice categories $\mathbb{B}/I$ and $\mathbb{B}/J$ have terminal objects *but* $\Sigma_f : \mathbb{B}/J \to \mathbb{B}/I$ preserves terminal objects *if and only if* the morphism f is an isomorphism in $\mathbb{B}$.

(ii) If furthermore $\mathbb{B}$ has binary products then for any object I in $\mathbb{B}$ the functor $\Sigma_I : \mathbb{B}/I \to \mathbb{B}$ preserves pullbacks and has right adjoint $I^* : \mathbb{B} \to \mathbb{B}/I$.

Of course, the category $\mathbb{B}/I$ has a terminal object but Σ_I preserves terminal objects *if and only if* the object I itself is terminal in $\mathbb{B}$.

The functors F considered in the previous examples are instances of so called *inclusions of localizations* meaning that the functor $F_1 : \mathbb{A} \to \mathbb{B}_{F1}$ is an isomorphism. Inclusions of localizations are precisely those functors $F : \mathbb{A} \to \mathbb{B}$ such that there is an isomorphism of categories $G : \mathbb{A} \tilde{\to} \mathbb{B}/F1$ such that $F = \Sigma_{F1} \circ G$.

If $\mathbb{A}$ has finite limits and $\mathbb{B}$ has pullbacks then a pullback preserving functor $F : \mathbb{A} \to \mathbb{B}$ can be factored as

$$F = \Sigma_{F1} \circ F_1$$

where $F_1 : \mathbb{A} \to \mathbb{B}/F1$ preserves all finite limits.

This factorization is *maximal* in the sense that whenever $F = H \circ G$ for some $H : \mathbb{C} \to \mathbb{B}$ and $G : \mathbb{A} \to \mathbb{C}$ where $\mathbb{C}$ has all finite limits and G preserves them and H preserves pullbacks then there is a unique functor $K : \mathbb{C} \to \mathbb{B}/F1$ such that $H = \Sigma_{F1} \circ K$ and $F = K \circ G$.

Thus any pullback preserving functor $F : \mathbb{A} \to \mathbb{B}$ uniquely up to isomorphism factors as a lex functor followed by an inclusion of a localization.

If $F : \mathbb{A} \to \mathbb{B}$ has a right adjoint U then the functor F_1 has right adjoint $\eta_1^* \circ U_{F1}$ where $\eta_1 : 1 \to UF1$ is the counit at 1 of the adjunction $F \dashv U$. Thus if F preserves pullbacks and has a right adjoint U then $F_1 \dashv \eta_1^* \circ U_{F1}$ is a proper geometric morphism from $\mathbb{B}/F1$ to $\mathbb{A}$ as $F_1 : \mathbb{A} \to \mathbb{B}/F1$ preserves pullbacks and the terminal object and therefore all finite limits. If we assume that $\mathbb{B}$ has *all binary products* then Σ_{F1} has also a right adjoint $(F1)^* : \mathbb{B} \to \mathbb{B}/F1$ sending $X \in \mathbb{B}$ to $\pi : F1 \times X \to F1$ in $\mathbb{B}/F1$. Notice that this does not require $\mathbb{B}$ to have a terminal object.

When restricted to lex categories our notion of generalised geometric morphism is slightly more general than the usual one as the inverse image part need not preserve terminal objects. The typical examples of generalised geometric morphisms not preserving the terminal object are the inclusions of localisations. Those actually form the "ortho-complement" of the proper geometric morphisms as in the category of generalised geometric morphisms the *proper geometric morphisms* and the *inclusions of localisations* form a factorisation system (in the appropriate 2-categorical sense).

As a generalised geometric morphism $F \dashv U$ from $\mathbb{B}$ to $\mathbb{A}$ corresponds in a unique way to a proper geometric morphism $F_1 \dashv \eta_1^* \circ U_{F1}$ from $\mathbb{B}/F1$ to $\mathbb{A}$, i.e. a geometric morphism to $\mathbb{A}$ with source $\mathbb{B}/F1$, generalised geometric morphisms are sometimes also called *partial geometric morphisms.* Then the *domain of definition* of the proper geometric morphism corresponding to a partial geometric morphism from $\mathbb{B}$ to $\mathbb{A}$ is the slice $\mathbb{B}/F1$ considered a part of $\mathbb{B}$ via the inclusion of localisation morphism $\Sigma_{F1} : \mathbb{B}/F1 \to \mathbb{B}$.

We conclude this section by the observation that for a pullback preserving functor $F : \mathbb{A} \to \mathbb{B}$ where $\mathbb{A}$ has finite limits and $\mathbb{B}$ has pullbacks the following are equivalent : F preserves binary products iff $F1$ is subterminal. Let $F : \mathbb{A} \to \mathbb{B}$ preserve pullbacks and binary products then $\langle id_{F1}, id_{F1} \rangle = F(\langle id_1, id_1 \rangle)$ is an isomorphism as F preserves isomorphism and $\langle id_1, id_1 \rangle$ is an isomorphism. Thus $F1$ is a *subterminal* in $\mathbb{B}$, i.e. any two morphisms to $F1$ are equal. On the other hand if $F : \mathbb{A} \to \mathbb{B}$ preserves pullbacks and $F1$ is a subterminal then F preserves all binary products. The reason is that for objects $A, B \in \mathbb{A}$ any pullback cone for $F(A \to 1)$ and $F(B \to 1)$ in $\mathbb{B}$ is a product cone as $F1$ is subterminal by assumption. As F preserves pullbacks the morphisms $F(\pi_1 : A \times B \to A)$ and $F(\pi_2 : A \times B \to B)$ form a pullback cone for $F(A \to 1)$ and $F(B \to 1)$ and therefore give rise to a product cone for FA and FB in $\mathbb{B}$.

For such a functor $F : \mathbb{A} \to \mathbb{B}$ preserving pullbacks and binary products we obviously have the following equivalence : $F1$ is terminal in $\mathbb{B}$ iff for any $\mathbb{X} \in \mathbb{B}$ there is a morphism $X \to F1$.

3 Partial Lex Categories and Other Partial Doctrines

As already seen in the first section the minimum requirement for a base $\mathbb{B}$ is that $\mathbb{B}$ has pullbacks. That means that *each slice* $\mathbb{B}/I$ *has finite limits*. An equivalent condition is that for any morphism f in $\mathbb{B}$ the functor Σ_f has a right adjoint f^*. This especially guarantees that for any morphism $f : J \to I$ in $\mathbb{B}$ change of base along $\Sigma_f : \mathbb{B}/J \to \mathbb{B}/I$ preserves all good properties of fibrations.

Now for a category $\mathbb{B}$ with pullbacks any functor $\Sigma_I : \mathbb{B}/I \to \mathbb{B}$ preserves pullbacks. Change of base along Σ_I preserves all good properties of fibrations iff Σ_I has a right adjoint I^*. Equivalently, products of I and X exist for all objects $X \in \mathbb{B}$. Thus change of base along Σ_I preserves all good properties of fibrations for all $I \in \mathbb{B}$ iff $\mathbb{B}$ has all binary products. Again it is not necessary that $\mathbb{B}$ has a terminal object.

Thus a category $\mathbb{B}$ has pullbacks and binary products iff functors of the form Σ_f or Σ_I have right adjoints f^* and I^*, respectively. Equivalently the category $\mathbb{B}$ has lex slices and change of base along functors of the form Σ_f or Σ_I preserves all good properties of fibrations.

This motivates the definition of a *partial lex category* as a category having pullbacks and binary products, i.e. a category where all *finite nonempty diagrams* have a limit.

Accordingly, a category $\mathbb{B}$ is a *partial locally cartesian closed category (a partial lccc)* iff $\mathbb{B}$ is partial lex and for any morphism f in $\mathbb{B}$ there is a string of adjoints $\Sigma_f \dashv f^* \dashv \Pi_f$. If, additionally, for any object I of $\mathbb{B}$ one would require the existence of a string of adjoints $\Sigma_I \dashv I^* \dashv \Pi_I$ then $\mathbb{B}$ would be a locally cartesian closed category provided $\mathbb{B}$ is nonempty as $\Pi_I id_I$ would give rise to a terminal object in $\mathbb{B}$ (as id_I is terminal in $\mathbb{B}/I$ and Π_I being a right adjoint preserves terminal objects).

Similarly one can define the notion of a *partial topos*. It is well known that a category $\mathbb{B}$ is a(n elementary) topos iff $\mathbb{B}$ has finite limits and the fundamental fibration $\partial_1 : \mathbb{B}^{\mathbf{2}} \to \mathbb{B}$ of $\mathbb{B}$ is *well-powered*. Equivalently, $\mathbb{B}$ is a(n elementary) topos iff $\mathbb{B}$ has finite limits and any slice of $\mathbb{B}$ is a(n elementary) topos.

Now one might define a partial topos as a category $\mathbb{B}$ such that the fundamental fibration of $\mathbb{B}$ is well-powered. The existence of the fundamental fibration of $\mathbb{B}$ is equivalent to $\mathbb{B}$ having pullbacks (cf. sction 1) and the well-poweredness of the fundamental fibration of $\mathbb{B}$ is equivalent to any slice of $\mathbb{B}$ being a(n elementary) topos. But as any groupoid has pullbacks and all its slices are (elementary) toposes (as they are equivalent to the terminal category) the definition seems to be too general.

Therefore we define a category $\mathbb{B}$ to be a *partial topos* iff $\mathbb{B}$ is partial lex and any slice is a(n elementary) topos. Equivalently, $\mathbb{B}$ is a partial topos iff $\mathbb{B}$ is partial lex and its fundamental fibration is well-powered.

Examples of partial toposes :

1. the category of topological spaces and local homeomorphisms

2. the category of small categories and discrete fibrations
3. the full subcategory $\mathbb{E}_{\mathcal{F}}$ of a topos $\mathbb{E}$ consisting of those objects X such that $supp\,X \in \mathcal{F}$ where $\mathcal{F}$ is a downward closed subset of $Sub_{\mathbb{E}}(1)$ and $supp\,X = im(X \to 1)$
4. more generally if $\mathbb{B}$ is already a partial topos (including the case that it is an ordinary topos) and $\mathbb{X}$ is a subclass of the class of objects of $\mathbb{A}$ then let $\mathbb{A}_{\mathbb{X}}$ be the full subcategory of $\mathbb{A}$ on those objects Z having a morphism to some $X \in \mathbb{X}$; without any assumptions on $\mathbb{X}$ every slice of $\mathbb{A}_{\mathbb{X}}/Z = \mathbb{A}/Z$ is an elementary topos and if $\mathbb{X}$ is assumed to be closed under binary products then $\mathbb{A}/\mathbb{X}$ has also binary products (as e.g. a downward closed class of subterminals) and therefore is a partial topos;
5. the category of finite trees considered as a full subcategory of the category $Tree = Set^{\omega^{op}}$ (notice that the terminal objects is an infinite tree)

We finish this section by some remarks on pullback preserving functors and generalised geometric morphims $F : \mathbb{A} \to \mathbb{B}$ where $\mathbb{A}$ has pullbacks and $\mathbb{B}$ is partial lex.

Now firstly assume that $F : \mathbb{A} \to \mathbb{B}$ preserves pullbacks. As $\mathbb{A}$ is not required to have a terminal object we cannot factor F as "$F = \Sigma_{F1} \circ F_1$" since the right hand side of the equation is meaningless as 1 need not exist in $\mathbb{A}$.

Nevertheless for any object I of $\mathbb{A}$ we have that

$$F \circ \Sigma_I = \Sigma_{FI} \circ F_I$$

where, obviously, $F_I : \mathbb{A}/I \to \mathbb{B}/FI$ is a lex functor between lex categories. That means that though for the functor F itself we do not have a canonical factorisation as a lex functor followed by an inclusion of a localisation - *we do have such a factorisation for each restriction of F along the inclusion Σ_I of the localisation $\mathbb{A}/I$ into $\mathbb{A}$.*

Now if, moreover, the pullback preserving functor $F : \mathbb{A} \to \mathbb{B}$ has a right adjoint U then for every object I of $\mathbb{A}$ the lex functor F_I has right adjoint $\eta_I^* \circ U_{FI}$ where $\eta_I : I \to UFI$ is the counit of the adjunction $F \dashv U$ at I. As $\mathbb{B}$ is partial lex the functor Σ_{FI} has a right adjoint $(FI)^*$. Therefore the functor $F \circ \Sigma_I = \Sigma_{FI} \circ F_I$ has right adjoint $\eta_I^* \circ U_{FI} \circ (FI)^*$ and factors canonically as the inverse image part of geometric morphism followed by an inclusion of a localisation where the latter is even the inverse image part of a generalised geometric morphism.

Notice that for $F \circ \Sigma_I = \Sigma_{FI} \circ F_I$ having a right adjoint it would also be sufficient that $\mathbb{A}$ is partial lex as then Σ_I has a right adjoint I^* and therefore $F \circ \Sigma_I \dashv I^* \circ U$. But as the requirement that $\mathbb{A}$ is partial lex in general does not entail $\mathbb{B}$ to be partial lex as well it is appropriate to require that both $\mathbb{A}$ and B are partial lex. In that case we have that both $I^* \circ U$ and $\eta_I^* \circ U_{FI} \circ (FI)^*$ are right adjoint to $F \circ \Sigma_I = \Sigma_{FI} \circ F_I$ and therefore isomorphic.

One might wonder whether a pullback preserving functor $F : \mathbb{A} \to \mathbb{B}$ between partial toposes has a right adjoint already if each $F_I : \mathbb{A}/I \to \mathbb{B}/FI$ has a right adjoint $U_{(I)}$. We have seen that this definitely is the case if $\mathbb{A}$ has a terminal object. The following example demonstrates that if $\mathbb{A}$ is properly partial, i.e. does not have a terminal object, then each F_I may have a right adjoint though F itself does not, i.e. the "local right adjoints" (of the F_I's) cannot be "glued together" to a "global right adjoint" of F.

Let $\mathbb{E}$ be the topos $Set^{\mathbb{N}}$ whose lattice of subobjects of $1_{\mathbb{E}}$ is (isomorphic to) $\mathcal{P}(\mathbb{N})$. We consider the following (nonempty) downward closed subsets of $\mathcal{P}(\mathbb{N}) : \mathcal{F}_0 = \mathcal{P}_{fin}(\mathbb{N})$, $\mathcal{F}_1 = \{\, \{n\} \mid n \in \mathbb{N} \,\}$. The inclusion F of $\mathbb{E}_{\mathcal{F}_1}$ into $\mathbb{E}_{\mathcal{F}_0}$ preserves pullbacks (and binary products !) and any F_I is an isomorphism and therefore has a right adjoint. But F does not have a right adjoint U as there are objects A, B in $\mathbb{E}_{\mathcal{F}_1}$ which are unbounded in $\mathbb{E}_{\mathcal{F}_1}$, i.e. there are no morphisms $f : A \to C$ and $g : B \to C$ in $\mathbb{E}_{\mathcal{F}_1}$, although they are bounded in $\mathbb{E}_{\mathcal{F}_0}$, i.e. there do exist $f : A \to C$ and $g : B \to C$ in $\mathbb{E}_{\mathcal{F}_0}$. Take e.g. for A and B objects in $\mathbb{E}_{\mathcal{F}_1}$ s.t. A_0 and B_1 are nonempty.

We finally give an example of a generalised geometric morphism between partial toposes neither of which contains a terminal object. Again we employ the topos $\mathbb{E} = Set^{\mathbb{N}}$ from above. Let $\mathcal{F}_2 = \{\, \{2n\} \mid n \in \mathbb{N} \,\}$ and consider the inclusion F of $\mathcal{F}_2$ into $\mathcal{F}_1$. Both partial toposes surely do not have a terminal object and the inclusion preserves pullbacks (and binary products). The inclusion functor F has a right adjoint U which sends objects A with $suppA$ containing an even number to A and all other objects to the initial object O (i.e. $O_n = \emptyset$ for all $n \in \mathbb{N}$). The functor U sends a morphism $f : A \to B$ to f if $suppA$ does contain an even number and to id_O otherwise. Though this kind of example is generic it nevertheless would be interesting to find (and study) examples of generalised geometric morphism between proper partial toposes that are not (full) inclusions.

4 Partial Lex Categories and Partial Toposes with Support

The treatment of the various questions about partial lex categories/toposes is facilitated by assuming that any such category "has supports".

That means that for any object A there is a least subterminal object U admitting a morphism $A \to U$. An object U of a category $\mathbb{A}$ is *subterminal* iff any morphisms $f, g : A \to U$ in $\mathbb{A}$ are equal. If $\mathbb{A}$ has a terminal object then U is subterminal iff $U \to 1$ is a mono, i.e. U is a subobject of 1 which justifies the name "subterminal". Let $subT(\mathbb{A})$ denote the full subcategory of $\mathbb{A}$ on subterminal objects. The category $subT(\mathbb{A})$ actually is a quasi-order. We say that $\mathbb{A}$ *has supports* iff the inclusion $i : subT(\mathbb{A}) \to \mathbb{A}$ has a left adjoint $supp \dashv i$. Thus $supp(A) \to U$ iff $A \to i(U)$, i.e. $supp(A)$ is a least subterminal U with $A \to U$. Of course, $\mathbb{A}$ has a terminal object iff $subT(\mathbb{A})$ has a greatest element which then is a terminal object.

From now on let us assume that $\mathbb{A}$ has pullbacks. If $\mathbb{A}$ has supports then there exist binary products of objects whose supports are bounded in $subT(\mathbb{A})$.

Furthermore one can show that $\mathbb{A}$ has binary products iff $subT(\mathbb{A})$ has binary infima. For $U, V \in subT(A)$ let $U \wedge V$ denote an infimum of U and V in $subT(A)$. Let A and B be objects in $\mathbb{A}$ then their product is obtained by taking the product $A' \times B'$ where A' and B' are obtained by pulling back $A \to supp(A)$ and $B \to supp(B)$ along the inclusions of $supp(A) \wedge supp(B)$ into $supp(A)$ and $supp(B)$, respectively. The product of A' and B' exists as their supports are bounded by $supp(A) \wedge supp(B)$. The projections for the product of A and B are obtained by composing the projections of the product of A' and B' with the inclusions $A' \to A$ and $B' \to B$, respectively, which are obtained from the pullback construction of A' and B', respectively. On the other hand if $\mathbb{A}$ has binary products then $supp(A)$ has binary infima as subterminals are closed under binary products in $\mathbb{A}$ and such are infima in $supp(\mathbb{A})$.

If $\mathbb{A}$ has binary products then an object $X \in \mathbb{A}$ is subterminal iff $\delta_X = \langle id_X, id_X \rangle : \mathbb{A} \to \mathbb{A} \times \mathbb{A}$ is an isomorphism.

A further benefit of the property of having supports is the following. If $\mathbb{A}$ is a partial topos having supports then $\mathbb{A}$ is guaranteed to have all coequalisers as any pair of arrows $f, g : A \to B$ is coequalised by $B \to supp(B)$. Now take the coeqaliser of $f, g : (A \to supp(B)) \to (B \to supp(B))$ in the topos $\mathbb{A}/supp(B)$ and then apply the colimit preserving $\Sigma_{supp(B)} : \mathbb{A}/supp(B) \to \mathbb{A}$ which gives the coequaliser in $\mathbb{A}$.

Next we take a closer look at pullback preserving functors between partial lex categories having supports. Let $\mathbb{A}$ and $\mathbb{B}$ be partial lex categories having supports and let $F : \mathbb{A} \to \mathbb{B}$ preserve pullbacks. The functor F preserves monos as F by assumption preserves pullbacks. Therefore F restricts to a functor $F_{|subT(\mathbb{A})} : subT(\mathbb{A}) \to Mono(\mathbb{B})$.

If furthermore F preserves binary products then F maps subterminals of $\mathbb{A}$ to subterminals of $\mathbb{B}$ as F preserves diagonals and isomorphisms. On the other hand the requirement on a pullback preserving functor to preserve also subterminal objects is not sufficient for guaranteeing the preservation of binary products as exhibited by the embedding of the category of shape V into the category of shape Y discussed in Section 5. (The intuitive explanation is that the fictitious $F1$ is not subterminal !)

But we have that F preserves binary products iff F preserves binary products of subterminals (and therefore also preserves subterminals).

As right adjoints to functors between partial lex categories having supports preserves all limits they also preserve subterminal objects. Thus, if we have a functor $F : \mathbb{A} \to \mathbb{B}$ between partial lex categories having supports such that F preserves pullbacks and binary products and has a right adjoint U then by restriction to subterminal objects we get an adjunction between $subT(\mathbb{A})$ and $subT(\mathbb{B})$.

5 Geometric Morphisms as Fibrations

Let $\mathbb{B}$ be a category with pullbacks.

A fibration $P : \mathbb{X} \to \mathbb{B}$ has *sums* iff P is also a cofibration and pullbacks of cocartesian arrows along cartesian arrows exist and are cocartesian again.

A *fibration of categories with pullbacks over* $\mathbb{B}$ is a fibration $P : \mathbb{X} \to \mathbb{B}$ such that $\mathbb{X}$ is has pullbacks and P preserves them. Such a fibration P is a *fibration of lex categories* iff moreover any fiber of P has a terminal object and those are preserved by reindexing.

For any morphism $\varphi : Y \to X$ let δ_φ be the unique mediating arrow from the pair id_Y, id_Y to the kernel pair of φ.

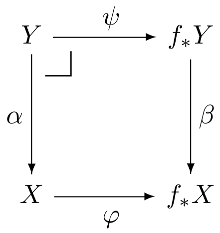
Let P be a fibration of lex categories over $\mathbb{B}$. The fibration P has *stable sums* iff P is also a cofibration and cocartesian arrows are preserved by pullbacks along *arbitrary* morphisms in $\mathbb{X}$ and P has *disjoint sums* iff P has sums and δ_φ is cocartesian whenever φ is cocartesian.

A *pre-geometric fibration over* $\mathbb{B}$ is a *fibration of lex categories over* $\mathbb{B}$ *having stable disjoint sums.*

For pre-geometric fibrations it holds that for any $f : I \to J$ in $\mathbb{B}$ and $X \in P(I) = \mathbb{X}_I$ there is a canonical equivalence between $\mathbb{X}_I/X$ and $\mathbb{X}_J/f_*X$.

In one direction this equivalence is given by sending a vertical arrow $\alpha : Y \to X$ in $\mathbb{X}_I/X$ to the vertical arrow $\beta = f_*\alpha$. The reverse direction is given by sending a vertical arrow $\beta : Z \to f_*X$ to the vertical arrow $\alpha = \varphi^*\beta$ where $\varphi : X \to f_*X$ is some cocartesian arrow from X over f.

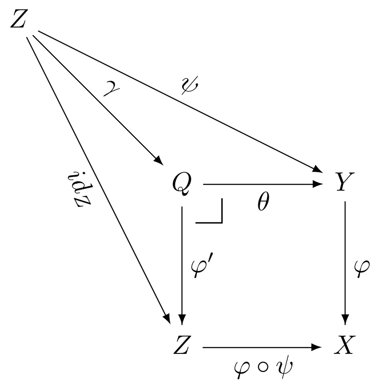
That this actually is an equivalence is ensured by Moens' Theorem saying that



provided it commutes in $\mathbb{X}$ and φ and ψ cocartesian and α and β vertical.

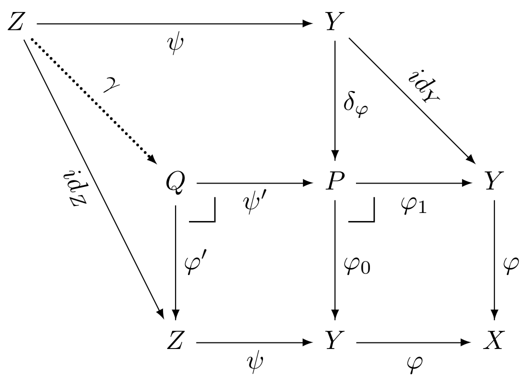
But, before proving Moens' Theorem we need two auxiliary lemmas.

Lemma 5.1 *Let* $\varphi : Y \to X$ *and* $\psi : Z \to Y$ *be maps in* $\mathbb{X}$. *If* φ *is cocartesian then the mediating map* $\gamma : Z \to Q$ *in*



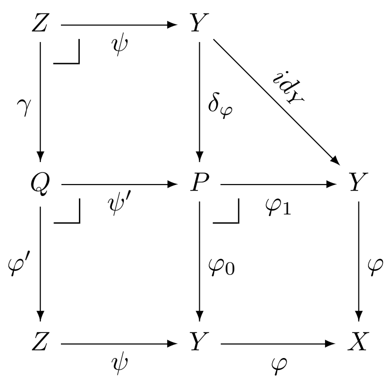
is cocartesian.

Proof. First consider the diagram



where $\varphi_i \circ \delta_\varphi = id_Y$ for $i = 0, 1$.

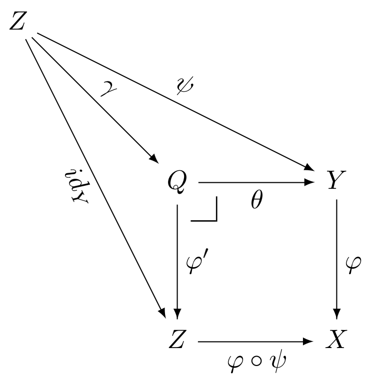
Thus we have



with $\varphi' \circ \gamma = id_Z$.

As P has disjoint sums δ_φ is cocartesian and as P has stable sums γ is cocartesian as it arises as pullback of δ_φ along ψ'.

Thus with $\theta := \varphi_1 \circ \psi'$ we have

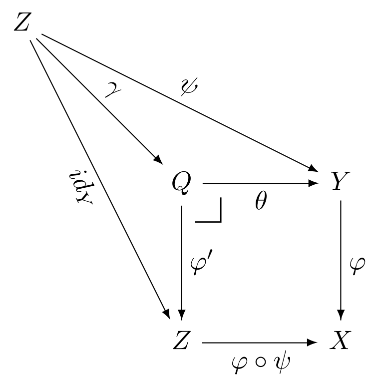


with γ cocartesian. $\square$

From this we get the following useful fact.

Lemma 5.2 *Let $P : \mathbb{X} \to \mathbb{B}$ be a pre-geometric fibration. Let $\varphi : Y \to X$ and $\psi : Z \to Y$ be maps in $\mathbb{X}$. Then ψ is cocartesian if φ and $\varphi \circ \psi$ are cocartesian.*

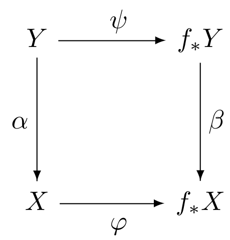
Proof. Consider the diagram



where γ is cocartesian by Lemma 5.1 and θ is cocartesian as it arises as pullback of cocartesian $\varphi \circ \psi$ along φ. Thus, $\psi = \theta \circ \gamma$ is cocartesian as it arises as composition of two cocartesian arrows. $\square$

Now we are ready to prove Moens' Theorem.

Theorem 5.3 *Let $P : \mathbb{X} \to \mathbb{B}$ be a pre-geometric fibration. Then any commuting square*

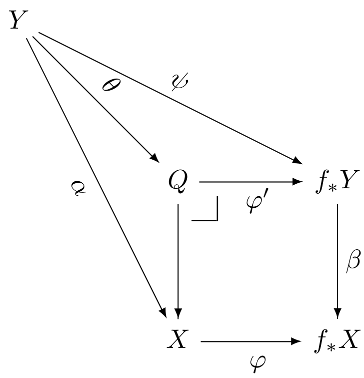


in $\mathbb{X}$ with φ and ψ cocartesian and α and β vertical is already a pullback square in $\mathbb{X}$.

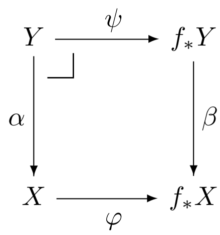
Therefore, for any $f : I \to J$ in $\mathbb{B}$ and $X \in P(I) = \mathbb{X}_I$ it holds that $\mathbb{X}_I/X$ is equivalent *to $\mathbb{X}_J/f_*X$ by sending a vertical arrow $\alpha : Y \to X$ in $\mathbb{X}_I/X$ to the vertical arrow $\beta = f_*\alpha$ and sending a vertical arrow $\beta : Z \to f_*X$ to the vertical arrow $\alpha = \varphi^*\beta$.*

Proof. We will give two variants of proof. The first is very short by using Lemma 5.2 whereas the second variant is Moens' original proof.

1. Variant.
Consider the diagram



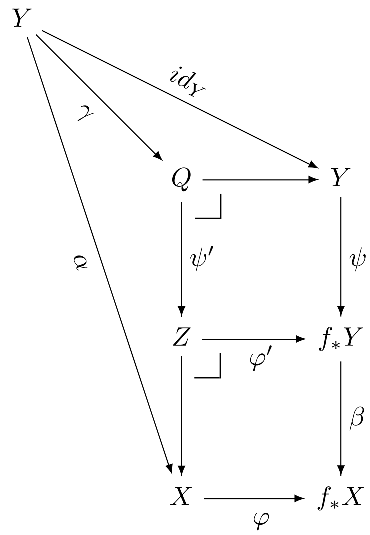
where θ is cocartesian by Lemma 5.2 as $\psi = \varphi' \circ \theta$ is cocartesian by assumption and φ' is cocartesian as it arises as pullback of the cocartesian arrow φ along β. As θ is cocartesian over an isomorphisms it follows that θ itself is an isomorphism. Thus,



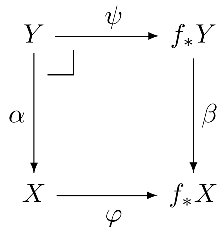
is a pullback diagram.

2. Variant.

As $\varphi \circ \alpha = \beta \circ \psi$ we have the diagramm



where γ is cocartesian by Lemma 5.1 (which is applicable as $\varphi \circ \alpha = \beta \circ \psi$). The morphism ψ' is cocartesian as it appears as pullback of the cocartesian morphism ψ. Therefore $\psi' \circ \gamma$ is cocartesian over an isomorphism. Thus, $\psi' \circ \gamma$ is itself an isomorphism from which it follows immediately that



is a pullback diagram. □

Choosing $X = 1_I$ we now get that $\mathbb{X}_I$ is equivalent to $\mathbb{X}_J/f_*1_I$. Now if $\mathbb{B}$ has a terminal object 1 then by choosing $J = 1$ we finally get that $\mathbb{X}_I$ is equivalent to $\mathbb{X}_1/!_{I*}1_I$.

This gives rise to the following generalisation of a result of Moens in his Thesis (1982). Namely, that for categories $\mathbb{B}$ *with finite limits* pre-geometric fibrations over $\mathbb{B}$ are precisely those fibrations over $\mathbb{B}$ which are *equivalent* to a fibration $\partial_1 : \mathbb{E}/F \to \mathbb{B}$ for some lex functor $F : \mathbb{B} \to \mathbb{E}$ between lex categories. The lex functor F is determined *uniquely up to equivalence* by the fibration P.

Notice that for this generalisation of Moens' result one definitely needs that the base category $\mathbb{B}$ has a terminal object because when given a fibration $P : \mathbb{X} \to \mathbb{B}$ the $\mathbb{E}$ of the functor $F : \mathbb{B} \to \mathbb{E}$ with P equivalent to $\partial_1 : \mathbb{E}/F \to \mathbb{B}$ is equivalent to the fibre of P over the terminal object in $\mathbb{B}$.

It is well known that a functor $F : \mathbb{B} \to \mathbb{E}$ between lex categories has a right adjoint iff the fibration $\partial_1 : \mathbb{E}/F \to \mathbb{B}$ *has small global sections* .

Therefore a fibration over a lex category will be called *geometric* iff it is *pre-geometric and has small global sections.* In the light of the the discussion above a geometric fibration over $\mathbb{B}$ is a fibration equivalent to one of the form $\partial_1 : \mathbb{E}/F \to \mathbb{B}$ for some geometric morphism to $\mathbb{B}$ with inverse image $F : \mathbb{B} \to \mathbb{E}$.

Some years later in 1990 M. Jibladze has shown that for an elementary topos $\mathbb{B}$ cocomplete fibred toposes over $\mathbb{B}$ are precisely those fibrations *equivalent* to fibrations $\partial_1 : \mathbb{E}/F \to \mathbb{B}$ where $F : \mathbb{B} \to \mathbb{E}$ is a lex functor between toposes.

Already from the appendix of PTJ's *Topos Theory* one knows that a functor $F : \mathbb{B} \to \mathbb{E}$ between toposes has a right adjoint if and only if the fibration $\partial_1 : \mathbb{E}/F \to \mathbb{B}$ is locally small.

Thus Jibaldze's result says that locally small, cocomplete fibred toposes over a base topos $\mathbb{B}$ are up to equivalence precisely those fibrations $\partial_1 : \mathbb{E}/F \to \mathbb{B}$ where F is the inverse image part of a geometric morphism from $\mathbb{E}$ to $\mathbb{B}$.

This analogy can be extended in as far as for a geometric morphism $F \vdash U : \mathbb{E} \to \mathbb{B}$ the fibration $\partial_1 : \mathbb{E}/F \to \mathbb{B}$ has a *(small) generating family* iff the geometric morphism is *bounded*, i.e. there exists an object S in $\mathbb{E}$ such that any object X in $\mathbb{E}$ is a subquotient of $FI \times S$ for some object I in $\mathbb{B}$.

This makes precise in which sense bounded geometric morphisms to a base topos $\mathbb{B}$ are considered (as representations of) *Grothendieck toposes over* $\mathbb{B}$ as bounded geometric morphisms to $\mathbb{B}$ correspond to locally small, cocomplete fibred toposes over $\mathbb{B}$ admitting a small family of generators.

6 Pullback Preserving Functors and Generalised Geometric Morphisms between Partial Lex Categories

In this section we study pullback preserving functors and generalised geometric morphisms between partial lex categories. In the first subsection we show how such $F : \mathbb{A} \to \mathbb{B}$ correspond to $\mathbb{A}$-indexed families of pullback preserving functors and in the second how they correspond to $\mathbb{A}$-indexed families of lex functors together with a natural family of Σ-embeddings. In the third subsection we show how they can be represented by certain fibrations over $1(\mathbb{A})$ (where $1(\mathbb{A})$ is obtained from $\mathbb{A}$ by freely adjoining a fresh terminal object).

6.1 Pullback Preserving Functors and Generalised Geometric Morphisms as Families of Generalised Geometric Morphisms

As partial lex categories and partial toposes are especially categories with pullbacks Bénabou's results as discussed in Section 2 are valid for them, i.e. a functor $F : \mathbb{A} \to \mathbb{B}$ between such categories preserves all good properties of fibrations by change of base along F iff F preserves pullbacks and has a right adjoint U.

Now if $\mathbb{A}$ has a terminal object 1 then change of base along F preserves all good properties of fibrations iff $F_1 : \mathbb{A}/1 \to \mathbb{B}/F1$ is the inverse image part of a geometric morphism. The reason is that F preserves pullbacks iff F_1 is lex and F has a right adjoint iff F_1 has a right adjoint. Of course, this argument does not at all require that $\mathbb{B}$ also has a terminal object.

Now if $\mathbb{A}$ does not have a terminal object then we still have that F preserves pullbacks iff $F_I : \mathbb{A}/I \to \mathbb{B}/FI$ is lex for all I in $\mathbb{A}$. But, alas, F having a right adjoint is not ensured by F locally having right adjoints, i.e. for all I in $\mathbb{A}$ the functor F_I may have a right adjoint although F itself does not have a right adjoint (globally).

A nice example for this situation is the embedding $F : \mathbb{E}_{\mathcal{F}} \to \mathbb{E}_{\mathcal{G}}$ where $\mathbb{E} = Set^{\mathbb{N}}$, $\mathcal{F} = \{\{n\} \mid n \in \mathbb{N}\}$ and $\mathcal{G} = \mathcal{P}_{fin}(\mathbb{N})$. This functor F between partial toposes clearly preserves pullbacks and binary products and nevertheless does not have a right adjoint.

Thus, though any adjunction $F \dashv U$ gives rise to an *induced family of generalised geometric morphisms*

$$F \circ \Sigma_I \dashv U \circ I^* \qquad (I \in \mathbb{A})$$

such that for any $f : J \to I$ in $\mathbb{A}$ we have

$$F \circ \Sigma_I \circ \Sigma_f = F \circ \Sigma_J$$

and therefore also

$$f^* \circ I^* \circ U^* \cong J^* \circ U$$

such data are not necessarily induced by (the inverse image part of) a generalised geometric morphism.

Nevertheless, one can reconstruct F from these data as the mediating functor from the (pseudo-)colimit cone of the pseudo-functor $\mathbb{A}/(-) : \mathbb{A} \to Cat$ (whose cocone is given by $\Sigma_I : \mathbb{A}/I \to \mathbb{A}$ $(I \in \mathbb{A})$ as $\mathbb{A}$ is partial lex). The point is only that such data do not ensure that the reconstructed F has a right adjoint. But, of course, it is a *property* of these data that the reconstructed F has a right adjoint because for the reconstruction of F one needs just these data and the right adjoint to F is fully determined by F.

There is a more intuitive characterisation of the situation that a family $(F_{(I)} : \mathbb{A}/I \to \mathbb{B} \mid I \in \mathbb{A}))$ of pullback preserving functors that is *natural in I*, i.e. $F_{(I)} \circ \Sigma_f = F_{(J)}$ for all $f : J \to I$ in $\mathbb{A}$, gives rise to a functor $F : \mathbb{A} \to \mathbb{B}$ having a right adjoint (where F is obtained as the mediating functor to the family $F_{(I)} \mid I \in \mathbb{A}))$ from the colimiting cocone $(\Sigma_I : \mathbb{A}_I \to \mathbb{A}))$. Namely this is the case when all $F_{(I)}$ have a right adjoint $U_{(I)}$ and the mediating functor $G : \mathbb{B} \to \bar{\mathbb{A}}$ from the cone $(U_{(I)} : \mathbb{B} \to \mathbb{A}/I \mid I \in \mathbb{A})$ to the limiting cone $(\tau_I : \mathbb{C} \to \mathbb{A}/I \mid I \in \mathbb{A})$ of the pseudo-functor $\mathbb{A}/(-) : \mathbb{A}^{op} \to Cat$ factors through the inclusion $Inc : \mathbb{A} \to \mathbb{C}$ sending A in $\mathbb{A}$ to $(\pi_{I,A} : I \times A \to I \mid I \in \mathbb{A})$ in $\mathbb{C}$ via a functor $U : \mathbb{B} \to \mathbb{A}$, i.e. $Inc \circ U = G$, where U is then the right adjoint of F.

6.2 Pullback Preserving Functors and Generalised Geometric Morphisms as Families of Proper Geometric Morphisms

Notice that for $I \in \mathbb{A}$ we have the factorisation $F \circ \Sigma_I = \Sigma_{FI} \circ F_I$ where $\Sigma_I \dashv I^*$ is a generalised geometric morphism and $F_I \dashv \eta_I^* \circ U_{FI}$ is a proper geometric morphism between the lex categories $\mathbb{A}/I$ and $\mathbb{B}/FI$. Thus $U \circ I^*$ and $\eta_I^* \circ U_{FI} \circ I^*$ are isomorphic as they are right adjoint to the same functor $F \circ \Sigma_I = \Sigma_{FI} \circ F_I$.

Thus we may consider a generalised geometric morphism as giving rise to the *induced family of proper geometric morphisms*

$$F_I \dashv \eta_I^* \circ U_{FI} : \mathbb{B}/FI \to \mathbb{A}/I \qquad (I \in \mathbb{A})$$

For any $f : J \to I$ in $\mathbb{A}$ we have that

$$F_I \circ \Sigma_f = \Sigma_{Ff} \circ F_J$$

and therefore also have the isomorphism

$$f^* \circ \eta_I^* \circ U_{FI} \;\cong\; \eta_J^* \circ U_{FJ} \circ (Ff)^*.$$

From Moens' work we know that such data amount to a *fibred geometric morphism* between the lex fibrations $\partial_1 : \mathbb{B}/F \to \mathbb{A}$ and $\partial_1 : \mathbb{A}/\mathbb{A} \to \mathbb{A}$ which – up to equivalence – are in a *1-1-correspondence with geometric fibrations over* $\mathbb{A}$.

Moreover for any $f : J \to I$ in $\mathbb{A}$ we have that

$$\Sigma_{FI} \circ \Sigma_{Ff} = \Sigma_{FJ}$$

and therefore also

$$(Ff)^* \circ (FI)^* \;\cong\; (FJ)^*$$

Given the families $(F_I \mid I \in \mathbb{A})$ and $(\Sigma_{FI} \mid I \in \mathbb{A})$ we can reconstruct F from the family $(\Sigma_{FI} \circ F_I \mid I \in \mathbb{A})$ (i.e. the family $(F \circ \Sigma_I \mid I \in \mathbb{A})$) as described in the previous subsection 5.1. Of course, the functor F cannot be reconstructed alone from the family $(F_I \mid I \in \mathbb{A})$. This problem arises already if $\mathbb{A}$ has a terminal object as in that case F also cannot be reconstructed from the functor $F_1 : \mathbb{A} \to \mathbb{B}/F1$. Even in that case one must know the functor $\Sigma_1 : \mathbb{B}/F1 \to \mathbb{B}$ as well.

The factorisation $F = \Sigma_{F1} \circ F_1$ in the case of $\mathbb{A}$ having a terminal object generalises to a *family of factorisations* $(\Sigma_{FI} \circ F_I \mid I \in \mathbb{A})$ if $\mathbb{A}$ does not have a terminal object.

The only difference to the case where $\mathbb{A}$ has a terminal object is that if $(F_I \mid I \in \mathbb{A})$ is a family of (inverse image parts of) geometric morphisms – as given e.g. by a geometric fibration over $\mathbb{A}$ – not any family of $(\Sigma_{FI} \mid I \in \mathbb{A})$ natural in I gives rise to a pullback preserving functor F having a right adjoint such that $F \circ \Sigma_I = \Sigma_{FI} \circ F_I$ for all $I \in \mathbb{A}$ (as already demonstrated in the previous subsection). But it is still the case that one gets a pullback preserving functor F from the data $(F_I \mid I \in \mathbb{A})$ and $(\Sigma_F I \mid I \in \mathbb{A})$.

Summarizing one can say that that a pullback preserving functor F from $\mathbb{A}$ to some partial lex category $\mathbb{B}$ is given – up to equivalence – by a pre-geometric fibration $P : \mathbb{X} \to \mathbb{A}$ together with a *natural family of* Σ*-embeddings of* P *into* $\mathbb{B}$, i.e. a family $(E_I : \mathbb{X}_I \to \mathbb{B} \mid I \in \mathbb{A})$ such that $(E_I)_{1_{\mathbb{X}_I}} : \mathbb{X}_I \to \mathbb{B}/E_I 1_{\mathbb{X}_I}$ is an isomorphisms for all I in $\mathbb{A}$ and $E_I \circ \Sigma_f = E_J$ for all morphisms $f : J \to I$ in $\mathbb{A}$. Accordingly, a pullback preserving functor $F : \mathbb{A} \to \mathbb{B}$ having a right adjoint, i.e. (the inverse image part of) a geometric morphism is given by a *geometric* fibration $P : \mathbb{X} \to \mathbb{A}$ together with a *natural family* $(E_I : \mathbb{X}_I \to \mathbb{B} \mid I \in \mathbb{A})$ *of* Σ*-embeddings of* P *into* $\mathbb{B}$ such that the functor F reconstructed from $(E_I \circ F_I \mid I \in \mathbb{A})$ has a right adjoint.

6.3 Pullback Preserving Functors between Partial Lex Categories as Fibrations

In subsection 6.2. we have seen that a pullback preserving functor from a partial lex category $\mathbb{A}$ to some partial lex category $\mathbb{B}$ can be given equivalently as a *pre-geometric fibration* $P : \mathbb{X} \to \mathbb{A}$ together with a *natural family* $(E_I : \mathbb{X}_I \to \mathbb{B})$ *of* Σ*-embeddings*. Such a pair (P, E) can more elegantly be represented by one single fibration $\overline{P} : \overline{\mathbb{X}} \to 1(\mathbb{A})$ (where $\mathbb{A}$ is obtained from $\mathbb{A}$ by adjoining a fresh terminal object 1) such that $\overline{P}$ satisfies the following requirements : $\overline{\mathbb{X}}_1 = \mathbb{B}$, the restriction of $\overline{P}$ along $Inc : \mathbb{A} \to 1(\mathbb{A})$ gives rise to the pre-geometric fibration $P : \mathbb{X} \to \mathbb{A}$ and $\overline{P}$ is a cofibration such that for $u : I \to 1$ in $\mathbb{A}$, $\varphi : X \to \Sigma_I X$ cocartesian over u and vertical $\alpha : Y \to \Sigma_I X$ the pullback of φ along α exists and is cocartesian, too, and any square

$$\begin{array}{ccc} Y & \xrightarrow{\psi} & \Sigma_I Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g = \Sigma_I f} \\ X & \xrightarrow[\varphi]{} & \Sigma_I X \end{array}$$

is a pullback provided $\psi : Y \to u_* Y$ is cocartesian and f and g are vertical. The latter condition establishes an equivalence $\Sigma_I/X : \overline{P}(I)/X \to \overline{P}(1)/\Sigma_I X$ whose adjoint is given by pullback along $\varphi : X \to \Sigma_I X$. Specializing X to the terminal object of $P(I)$ we get the equivalence

$$E_I = \Sigma_I/1_{\overline{P}(I)} : \overline{P}(I) = \overline{P}(I)/1_{\overline{P}(I)} \to \overline{P}(1)/\Sigma_I 1_{\overline{P}(I)}$$

embedding the fibre $P(I)$ as a slice category into $\mathbb{B} = \overline{P}(1)$.

Of course, even if $\mathbb{A}$ has a terminal object $\overline{P}$ cannot be recovered from $P = Inc^* \overline{P}$. Counterexamples are given by pullback preserving functors that do not preserve terminal objects.

7 Proper Lex Functors and Geometric Morphisms between Partial Lex Categories

There arises the question to which extent a pullback preserving functor $F : \mathbb{A} \to \mathbb{B}$ (possibly having a right adjoint U and thus constituting the inverse image of a generalsied geometric morphism) can be reconstructed alone from its induced family $(F_I : \mathbb{A}/I \to \mathbb{B}/FI \mid I \in \mathbb{A})$ of proper lex functors or proper geometric morphisms.

First of all our extension of Moens' Theorem tells us that even if categories $\mathbb{A}$ and $\mathbb{B}$ are lex then a pullback preserving functor $F : \mathbb{A} \to \mathbb{B}$ can be reconstructed from its induced family of geometric morphisms $(F_I \mid I \in \mathbb{A})$ only if F is actually lex. Otherwise, one simply "reconstructs" its "best approximating" lex functor $F_1 : \mathbb{A} \to \mathbb{B}/F1$.

One cannot avoid this "defect" by only requiring that F preserves also binary products as this only enforces that F preserves subterminals (as F then preserves diagonals and an object is subterminal iff its diagonal is an iso). Thus if $\mathbb{X}$ is a lex category and U is a nontrivial subobject of $1_{\mathbb{X}}$ then $\Sigma_U : \mathbb{X}/U \to \mathbb{X}$ preserves pullbacks and binary products (and even has a right adjoint) but it does not preserve terminal objects as $\Sigma_U(id_U) = U$ which by assumption is not terminal in $\mathbb{X}$. Thus if $F : \mathbb{A} \to \mathbb{B}$ is a pullback preserving functor between lex categories such that $F(1_{\mathbb{A}}) \to 1_{\mathbb{B}}$ is a monomorphism then F preserves pullbacks as $F = \Sigma_{F(1_{\mathbb{A}})} \circ F_{1_{\mathbb{A}}}$ and both $\Sigma_{F(1_{\mathbb{A}})}$ and $F_{1_{\mathbb{A}}}$ preserves binary products. Thus we get that a pullback preserving functor $F : \mathbb{A} \to \mathbb{B}$ between lex categories *preserves binary products iff* $F(1)$ *is a subterminal.*

This problem remains even if $\mathbb{A}$ and $\mathbb{B}$ do not have terminal objects as shown by the following example. Let $\mathbb{E} = Set^{\mathbb{N}}$, $\mathcal{F} = \{ \{2n\} \mid n \in \mathbb{N}\}$ and $\mathcal{G} = \{ \{n\} \mid n \in \mathbb{N}\}$ then the embedding of partial toposes $F : \mathbb{E}_{\mathcal{F}} \to \mathbb{E}_{\mathcal{G}}$ preserves pullbacks and binary products and has a right adjoint but its *associated geometric fibration* $P_F = \partial_1 : \mathbb{E}_{\mathcal{G}}/F \to \mathbb{E}_{\mathcal{F}}$ is isomorphic to $P_{Id_{\mathbb{E}_{\mathcal{F}}}} = \partial_1 : \mathbb{E}_{\mathcal{F}}/Id_{\mathbb{E}_{\mathcal{F}}} \to \mathbb{E}_{\mathcal{F}}$, i.e. the fundamental fibration of $\mathbb{E}_{\mathcal{F}}$.

Our extension of Moens' Theorem shows that – up to equivalence – pre-geometric fibrations over *lex* $\mathbb{A}$ correspond to *lex* functors $F : \mathbb{A} \to \mathbb{B}$ for some *lex* category $\mathbb{B}$. By analogy, in order to characterise pre-geometric fibrations over a *partial* lex base $\mathbb{A}$ as corresponding – up to equivalence – to a certain class of functors from $\mathbb{A}$ to some partial lex category $\mathbb{B}$ we have to clarify first what it means for a functor $F : \mathbb{A} \to \mathbb{B}$ between partial lex categories to be "proper lex".

A possible solution which avoids the notorious problem of reconstructing the functor $F : \mathbb{A} \to \mathbb{B}$ from its associated fibration $P_F = \partial_1 : \mathbb{B}/F \to \mathbb{A}$ is to augment the properly partial lex category $\mathbb{A}$ with a fresh terminal object. More explicitely, we consider the functor $1(\_)$ left adjoint to the inclusion of Cat_{term}, the category of categories with a terminal object and functors preserving terminal objects, into the category Cat of all categories and all functors between them. More explicitely, $1(\mathbb{A})$ is the category whose objects are those of $\mathbb{A}$ together with a fresh terminal object 1. The arrows of $1(\mathbb{A})$ are those of $\mathbb{A}$

together with a unique morphism $A \to 1$ for every $A \in \mathbb{A}$ and $id_1 : 1 \to 1$, the identity on 1. Notice that in $1(\mathbb{A})$ any morphism $f : 1 \to X$ is equal to id_1, i.e. any *global element* in $1(\mathbb{A})$ is the identity on 1 or, equivalently, when an object in $1(\mathbb{A})$ has a global element then this object is already (the) terminal object. If $F : \mathbb{A} \to \mathbb{B}$ is a functor in Cat then $1(F) : 1(\mathbb{A}) \to 1(\mathbb{B})$ is the unique terminal object preserving functor whose restriction to $\mathbb{A}$ is F.

If $F : \mathbb{A} \to \mathbb{B}$ is an arbitrary functor between arbitrary categories $\mathbb{A}$ and $\mathbb{B}$ then F has a right adjoint iff $1(F)$ has a right adjoint that reflects the terminal object. If $F \dashv U$ then obviously $1(F) \dashv 1(U)$ and $1(U)$ reflects the terminal object. On the other hand if $1(F) \dashv G$ then G reflects the terminal object as if $1 \to G(Y)$ then $1(F)(1) \to Y$ and as $1 = 1(F)(1)$ the object Y in $1(\mathbb{B})$ has a global element and therefore $Y = 1$ in $1(\mathbb{B})$. Thus G reflects 1 and therefore its restriction U to the subcategory $\mathbb{B}$ provides a right adjoint to F.

Obviously, a category $\mathbb{A}$ is partial lex iff $1(\mathbb{A})$ is lex, i.e. has all finite limits (as one can see easily that nontrivial pullbacks over 1 in $1(\mathbb{A})$ correspond to binary cartesian products in $\mathbb{A}$). Furthermore it is immediate that a functor $F : \mathbb{A} \to \mathbb{B}$ between partial lex categories preserves pullbacks and binary products iff $1(F)$ is lex, i.e. preserves all finite limits.

In the light of these considerations one might be inclined to say that a functor $F : \mathbb{A} \to \mathbb{B}$ between partial lex categories $\mathbb{A}$ and $\mathbb{B}$ is proper lex iff $1(F) : 1(\mathbb{A}) \to 1(\mathbb{B})$ is lex, i.e. iff F preserves pullbacks and binary products, and it is (the inverse image part of) a *proper geometric morphism* iff $1(F)$ is (the inverse image part of) a geometric morphism, i.e. iff F *preserves pullbacks and binary products and has a right adjoint.*

Alas, this tentative definition is *not conservative* w.r.t. the case where both $\mathbb{A}$ and $\mathbb{B}$ are proper lex as there are functors $F : \mathbb{A} \to \mathbb{B}$ between proper lex categories that preserve pullbacks and binary products but do not preserve the terminal object. A typical example is a functor $\Sigma_U : \mathbb{B}/U \to \mathbb{B}$ for a proper subterminal object U which preserves pullbacks and binary products but does not respect the terminal object. Of course, when $\mathbb{A}$ and $\mathbb{B}$ lack a terminal object one cannot require that $F : \mathbb{A} \to \mathbb{B}$ preserves this non-existent terminal object. Thus we have to replace this property by an equivalent one that does not mention terminal objects.

If both $\mathbb{A}$ and $\mathbb{B}$ do have terminal objects and $F : \mathbb{A} \to \mathbb{B}$ preserves them then for any $B \in \mathbb{B}$ there is some morphism $B \to FA$ for some object $A \in \mathbb{A}$ (as one may take for A the terminal object of $\mathbb{A}$ and the unique morphism $B \to FA$ which exists as FA is terminal due to the assumption that F preserves terminal objects and A is a terminal object. If $\mathbb{A}$ and $\mathbb{B}$ are lex and $F : \mathbb{A} \to \mathbb{B}$ preserves pullbacks and binary products then the condition above actually implies that it preserves terminal objects. If for any B there is a morphism $f : B \to FA$ then there is also the morphism $F(!_{\mathbb{A}}) \circ f : B \to F(1_{\mathbb{A}})$. As F preserves binary products as well we know that $F(1_{\mathbb{A}})$ is subterminal. Thus for $B = 1_{\mathbb{B}}$ then there is a unique morphism $t : 1_{\mathbb{B}} \to F(1_{\mathbb{A}})$. Let $s =!_{F(1_{\mathbb{A}})} : F(1_{\mathbb{A}}) \to 1_{\mathbb{B}}$ then we have $s \circ t = id_{1_{\mathbb{B}}}$ and $t \circ s = id_{F(1_{\mathbb{A}})}$ as $F(1_{\mathbb{A}})$ is subterminal. Thus we get that $F(1_{\mathbb{A}})$ isomorphic to $1_{\mathbb{B}}$ and therefore terminal itself.

Thus a functor $F : \mathbb{A} \to \mathbb{B}$ between partial lex categories is called *proper lex* iff F preserves pullbacks and binary products and for any $B \in \mathbb{B}$ the comma category B/F is non-empty. We will show that the proper lex functors are precisely those functors F that can be reconstructed from their induced fibration $P_F = \partial_1 : \mathbb{B}/F \to \mathbb{A}$.

Let $\mathbb{A}$ and $\mathbb{B}$ be partial lex categories which both lack a terminal object. We have seen above that the functors $F : \mathbb{A} \to \mathbb{B}$ preserving pullbacks and binary products are in a 1-1-correspondence with the lex functors from $1(\mathbb{A})$ to $1(\mathbb{B})$ which reflect 1. By (our extension of) Moens' Theorem the latter are – up to equivalence – in a 1-1-correspondance with the pre-geometric fibrations P over $1(\mathbb{A})$ such that in $P(1)$, the fiber of P over 1, the only objects having a global element are the terminal objects and furthermore in the total category of P any cocartesian arrow whose codomain is terminal in $P(1)$ is already an isomorphism. As for lex fibrations over $1(\mathbb{A})$ the terminal objects of the total category coincide with the terminal objects in the fiber over 1 this additional condition on a pre-geometric fibration can be expressed more concisely as the requirement that in the total category of the fibration any global element and any cocartesian arrow to a terminal object is an isomorphism. Obviously, such pre-geometric fibrations correspond to generalised geometric morphisms whose inverse image part also preserves binary products iff they are *geometric* fibrations, i.e. also have small global sections.

Notice that for a pre-geometric fibration $P : \mathbb{X} \to 1(\mathbb{A})$ where any global element and any cocartesian arrow to a terminal object is an isomorphism it still may happen that there is a lot of objects in the fiber over 1 which are neither terminal nor appear as codomains of cocartesian arrows which are not isomorphisms, i.e. start from an object which is not in the fibre over 1 and therefore can be considered as disjoint sums in a nontrivial way. Up to equivalence it are those pre-geometric fibrations over $1(\mathbb{A})$ which are induced by a lex $F : 1(\mathbb{A}) \to 1(\mathbb{B})$ that reflect 1 but where not any $B \in \mathbb{B}$ lies above an FA via some morphism $B \to FA$. Such pre-geometric fibrations – even up to equivalence – are not uniquely determined by $P_{|\mathbb{A}}$, the restriction of P to $\mathbb{A}$, because they are equivalent to $P_G = \partial_1 : \mathbb{C}/G \to \mathbb{A}$ where $\mathbb{C}$ is the full subcategory of $\mathbb{B}$ on those objects B having a morphism to some FA and $G : \mathbb{A} \to \mathbb{C}$ is obtained from F by restricting its codomain to $\mathbb{C}$ (which still has domain $\mathbb{A}$ as for any $A \in \mathbb{A}$ we have $id_{FA} : FA \to FA$).

Thus we get the following extension of Moens' Theorem: the proper lex functors from a partial lex category $\mathbb{A}$ to some partial lex category $\mathbb{B}$ are – up to equivalence – in a 1-1-correspondence with those pre-geometric fibrations over $1(\mathbb{A})$ where *any global section and any cocartesian arrow to a terminal object (in the total category of the fibration) is an isomorphism* and *any non-terminal object in the fibre over* 1 *appears as the codomain of a non-isomorphic cocartesian arrow*.

For such P the fibre over 1 can be reconstructed from $P_{|\mathbb{A}}$ by taking the (pseudo-)colimit of the (covariant) pseudo-functor corresponding to the cofibration $P_{|\mathbb{A}}$ and adding a fresh terminal object. Notice that the (pseudo-)colimit of a cofibration $C : \mathbb{X} \to \mathbb{A}$ is obtained as $Colim\, C = \mathbb{X}[coCart(C)^{-1}]$ employing the calculus of fractions. The cofibration P itself – and therefore also the fibration P – can be reconstructed accordingly from the colimit cone for C.

Notice that this reconstruction of $\mathbb{B}$ from $\partial_1 : \mathbb{B}/F \to \mathbb{A}$ does work due to the assumption that $F : \mathbb{A} \to \mathbb{B}$ preserves binary products as then we have for morphisms $f : Y \to X,\ x_1 : X \to F(A_1), x_2 : X \to F(A_2)$ in $\mathbb{B}$ that there exists a (unique) morphism $x : X \to F(A_1 \times A_2)$ with $F(\pi_i) \circ x = x_i$ for $i = 1, 2$ and therefore $\mathbb{B}/F(\pi_i)(f : x \circ f \to x) = f : x_i \circ f \to x_i$ for $i = 1, 2$, i.e. $f : x_1 \circ f \to x_1$ and $f : x_2 \circ f \to x_2$ get identified in the colimit.

Thus we may characterise proper lex functors $F : \mathbb{A} \to \mathbb{B}$ from $\mathbb{A}$ to some partial lex category $\mathbb{B}$ as those pre-geometric fibrations $P : \mathbb{X} \to \mathbb{A}$ such that their "completion" $\overline{P} : \overline{\mathbb{X}} \to 1(\mathbb{A})$ are pre-geometric fibrations (where $\overline{\mathbb{X}}_1 = \overline{P}(1)$ is obtained by freely adjoining a terminal object to the pseudo-colimit of the cofibration P). These in turn correspond – up to equivalence – to those pre-geometric fibrations over $1(\mathbb{A})$ such that all global element and all cocartesian arrows to a terminal object (in the total category of the fibration) are isomorphism.

On can prove that for any pre-geometric fibration $P : \mathbb{X} \to \mathbb{A}$ its "completion" $\overline{P} : \overline{\mathbb{X}} \to 1(\mathbb{A})$ (as described above) is pre-geometric, too. Thus we get that for a partial lex $\mathbb{A}$ there is – up to equivalence – a 1-1-correspondence between proper lex functors from $\mathbb{A}$ to some partial lex $\mathbb{B}$ and pre-geometric fibrations over $\mathbb{A}$.

Alas, such a characterisation is not available anymore for pullback preserving functors $F : \mathbb{A} \to \mathbb{B}$ between partial lex categories if F does not preserve binary products – even if for any $B \in \mathbb{B}$ there is a morphism to some FA.

Firstly, for such F the induced fibration $P_F = \partial_1 : 1(\mathbb{B})/1(F) \to 1(\mathbb{A})$ – though having all (internal) sums – will not satisfy the Beck condition for sums over nontrivial pullbacks over 1 because this is equivalent to F preserving binary products.

(Notice that there is a difference between functors of the form $1(F)$ for some pullback preserving $F : \mathbb{A} \to \mathbb{B}$ and pullback preserving functors from $1(\mathbb{A})$ to $1(\mathbb{B})$ because the latter in general don't preserve 1 whereas the former in general don't preserve pullbacks over 1. Moreover, if $G : 1(\mathbb{A}) \to \mathbb{C}$ is a pullback preserving functor to a lex category $\mathbb{C}$ and G has a right adjoint V then V will not reflect terminal objects if $G(1)$ is not terminal as due to the counit $\eta_1 : 1 \to VG1$ the object $VG1$ is *equal* to 1 in $1(\mathbb{A})$ although – by assumption – $G1$ is not terminal.)

Secondly, for such $F : \mathbb{A} \to \mathbb{B}$ the category $\mathbb{B}$ cannot be reconstructed fromits induced fibration by taking its (pseudo-)colimit even if any $B \in \mathbb{B}$ admits an arrow to some $F(A)$. This can be seen from the following example.

Let $\mathbb{A}$ be the category arising from the poset with two maximal elements a and b and a least element $\perp$, i.e. its Hasse diagram has the shape of a V. Let $\mathbb{B}$ be the proper partial lex category arising from the poset which is obtained by adjoining to (the poset corresponding to) $\mathbb{A}$ a further element c which in $\mathbb{B}$ is the infimum of a and b but still is strictly greater than the least element $\perp$, i.e. $\mathbb{B}$ has the shape of a Y. Clearly both $\mathbb{A}$ and $\mathbb{B}$ are proper partial lex categories. The inclusion F from $\mathbb{A}$ into $\mathbb{B}$ preserves pullbacks but not binary products (as $\perp$, the infimum of a and b in $\mathbb{A}$, is sent by F to $\perp$ and not to c, the infimum of

a and b in $\mathbb{B}$). Obviously, F has also a right adjoint U sending a and b to itself and both c and $\bot$ to $\bot$. As all objects of $\mathbb{B}$ are below $a = Fa$ or below $b = Fb$ it also holds that for any $B \in \mathbb{B}$ there is a morphism to some FA.

When we now consider the colimit of the cofibration P_F then in $\mathbb{B}/F$ the c over a and the c over b will never get identified in the colimit as there is no cocartesian arrow in $\mathbb{B}/F$ from or to $c \to a$ or $c \to a$ which is not already an isomorphism. Actually the colimit of P_F is the poset $\mathbb{C}$ obtained by adjoining a new least element 0 to the sum $\mathbb{2} + \mathbb{2}$ (taken in the category of posets or categories).

Of course, there is also the cocone $(\Sigma_{FA} : \mathbb{B}/FA \to \mathbb{B} \mid A \in \mathbb{A})$ to $\mathbb{B}$ over the diagram in Cat corresponding to the (split) cofibration $\partial_1 : \mathbb{B}/F \to A$. The mediating functor $S : \mathbb{C} \to \mathbb{B}$ to this cocone from the colimiting cocone sends the object 0 to $\bot$, the top elements of the two copies of $\mathbb{2}$ to a and b, respectively, and the minimal elements c_a and c_b of the *two distinct* copies of $\mathbb{2}$ both to the *same* object c in $\mathbb{B}$. This illustrates how the colimiting process creates "two versions of c" namely (the isomorphism classes of) $c \to a$ and $c \to b$ (in $\mathbb{B}/F$).

Notice that this counterexample can be lifted from posets to partial toposes in the following way. Let $\mathbb{E}$ and $\mathbb{F}$ be the full subcategories of $\hat{\mathbb{A}}$ and $\hat{\mathbb{B}}$, respectivly, on those objects whose support is linearly ordered. (Notice that the support of a presheaf over a poset $\mathbb{P}$ is always a subobject of the terminal presheaf, i.e. a downward closed subset of $\mathbb{P}$.) Let F again denote the inclusion of $\mathbb{A}$ into $\mathbb{B}$ then the reindexing functor $F^* : \hat{\mathbb{B}} \to \hat{\mathbb{A}}$ restricts to a functor $V : \mathbb{F} \to \mathbb{E}$. The left Kan extension of F restricts to a functor $G : \mathbb{E} \to \mathbb{F}$ and is left adjoint to V. The functor G (can be chosen in a way such that it) sends $H \in \hat{\mathbb{A}}$ to the presheaf $K \in \hat{\mathbb{B}}$ with $K_{|\mathbb{A}} = H, K(c) = H(a) \cup H(b)$ and $K(c \to x)$ is the inclusion of $H(x)$ into $H(a) \cup H(b)$ for $x \in \{a, b\}$. Therefore G can be seen easily to preserve pullbacks (but not all binary products). Again, when taking the colimit of the cofibration $\mathbb{F}/G \to \mathbb{E}$ one can observe that if $\tau_1 : K \to K_1, \tau_2 : K \to K_2$ are morphisms in $\hat{\mathbb{B}}$ such that $K(c), K_1(a), K_2(b)$ are nonempty but $K(a) = \emptyset = K(b)$ then τ_1 and τ_2 considered as objects of $\mathbb{F}/G$ will not get identitified when taking the colimit of $\mathbb{F}/G \to \mathbb{E}$.

In a *fairly restricted sense* the factorisation of a pullback preserving functor $F : \mathbb{A} \to \mathbb{B}$ between lex categories as $F = \Sigma_{F1} \circ F_1$ can be generalised to pullback preserving functors between partial lex categories in the following way. Assume that $F : \mathbb{A} \to \mathbb{B}$ is a pullback preserving functor between partial lex categories. Now we may take the colimit $(M_A : \mathbb{B}/F(A) \to \mathbb{C} \mid A \in \mathbb{A})$ of $\mathbb{B}/F(\_)$ considered as a split opfibration. As $(\Sigma_{F(A)} : \mathbb{B}/F(A) \to \mathbb{B} \mid A \in \mathbb{A})$ is also a cocone there is a unique mediating functor $S : \mathbb{C} \to \mathbb{B}$ such that $S \circ M_A = \Sigma_{F(A)}$ for all $A \in \mathbb{A}$. If $\mathbb{A}$ has a terminal object then this functor S is Σ_{F1}.

As we know that $(\Sigma_A : \mathbb{A}/A \to \mathbb{A})_{A \in \mathbb{A}}$ is a colimit cone for the split cofibration $\mathbb{A}/\_$ we get a functor $F_\infty : \mathbb{A} \to \mathbb{C}$ defined as the mediating functor to the cocone $(M_A \circ F_A \mid A \in \mathbb{A})$ (this actually is a cocone as for any $f : J \to I$ in $\mathbb{A}$ we have $\Sigma_{Ff} \circ F_J = F_I \circ \Sigma_f$). We now get $F = S \circ F_\infty$ generalising the factorisation $F = \Sigma_{F(1)} \circ F_1$ as F_∞ preserves pullbacks and binary products. Furthermore, if F has a right adjoint then F_∞ has a right adjoint, too.

Alas, the functor S—though preserving pullbacks—need not have a right adjoint even if F has one. An example for this situation has been given already above: for $\mathbb{A}$ take the poset of shape V and for $\mathbb{B}$ take the poset of shape Y and let F be the inclusion of $\mathbb{A}$ into B which has a right adjoint; we have seen that the colimit $\mathbb{C}$ of $\mathbb{B}/F(-)$ is the poset which is obtained from $\mathbb{2}+\mathbb{2}$ by adding a new bottom element 0. The functor $S : \mathbb{C} \to \mathbb{B}$ cannot have a right adjoint T as we have $S(c_a) = c \to c$ and $S(c_b) = c \to c$ which implies that for a hypothetical right adjoint T it would hold that $c_a \to T(c)$ and $c_b \to T(c)$ which is impossible as c_a and c_b are unbounded in $\mathbb{C}$. Nevertheless the embedding functor $F_\infty : \mathbb{A} \to \mathbb{C}$ still has a right adjoint U_∞ which preserves minimal and maximal elements and sends both c_a and c_b to $\bot \in \mathbb{A}$.

A further remarkable aspect of the functor $S : \mathbb{C} \to \mathbb{B}$ is that all its slices $F_I : \mathbb{C}/I \to \mathbb{B}/SI$ are isomorphisms, i.e. for all $I \in \mathbb{C}$ the functor $E_I = \Sigma_{FI} \circ F/I : \mathbb{C}/I \to \mathbb{B}$ is an embedding as a slice category into $\mathbb{B}$, and nevertheless F does not have a right adjoint. Thus it serves as a most simple example demonstrating that a functor need not have a right adjoint even if all its localisations are embedding of slice categories (that even preserve binary products).

Finally we observe that *Moens' Theorem does* **not** *generalise to geometric fibrations over partial lex* $\mathbb{B}$.

If $\mathbb{B}$ is a non-empty partial lex category then $Id_{\mathbb{B}} : \mathbb{B} \to \mathbb{B}$ is a geometric fibration over $\mathbb{B}$ and by the considerations above it is canonically isomorphic to $\partial_1 : \mathbb{1}/\mathbb{F} \to \mathbb{B}$ where $F : \mathbb{B} \to \mathbb{1}$ is the unique functor form $\mathbb{B}$ to the the terminal category $\mathbb{1}$ (as $\mathbb{1} \cong \mathbb{B}[coCart(Id_{\mathbb{B}})^{-1}]$). The functor F preserves pullbacks and binary products and for any $X \in \mathbb{B}$ there is a morphism $X \to FI$ for some $I \in \mathbb{B}$. *But* F has a right adjoint U if and only if $\mathbb{B}$ has a terminal object (namely $U*$ where $*$ is the unique object in $\mathbb{1}$). Therefore, for any non-empty partial lex category $\mathbb{B}$ the geometric fibration $Id_{\mathbb{B}}$ over $\mathbb{B}$ corresponds to a partial geometric morphism *if and only if* $\mathbb{B}$ has a terminal object.

8 The Right Notion of Geometric Morphism between Partial Lex Categories

At the end of the last section we have seen that there is functor $F : \mathbb{A} \to \mathbb{B}$ between partial lex categories $\mathbb{A}$ and $\mathbb{B}$ which does not have a right adjoint *but* nevertheless F preserves binary products and $\partial_1 : \mathbb{B}/F \to \mathbb{A}$ is a geometric fibration. Thus, for partial lex categories *in general one cannot require that geometric morphisms have right adjoints*. We will demonstrate below that the condition of having a right adjoint has to be weakened to the condition that *each slice of F has a right adjoint*, i.e. that $F_{/I} : \mathbb{A}/I \to \mathbb{B}/FI$ has a right adjoint for all $I \in \mathbb{A}$.

Theorem 8.1 *For an arbitrary functor* $F : \mathbb{A} \to \mathbb{B}$ *the fibration* $F^*P_{\mathbb{B}} = \partial_1 : \mathbb{B}/F \to \mathbb{A}$ *has* small global sections *iff for all* $I \in \mathbb{A}$ *the functor* $F_{/I} : \mathbb{A}/I \to \mathbb{B}/FI$ has a right adjoint.

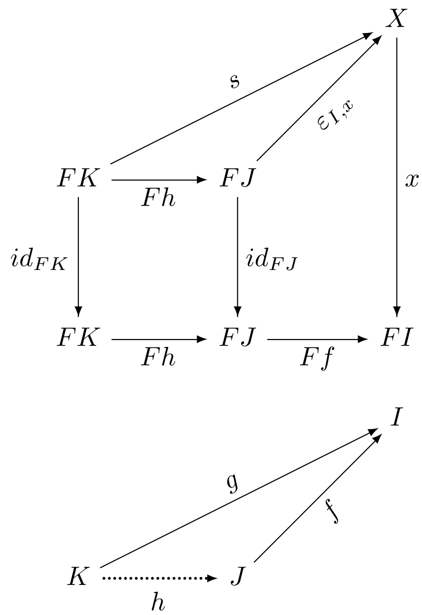
Proof. The fibration $\partial_1 : \mathbb{B}/F \to \mathbb{A}$ has small global sections iff for any $x : X \to FI$ there exists a morphism

$$\begin{array}{ccc} FJ & \xrightarrow{\varepsilon_{I,x}} & X \\ {\scriptstyle id_{FJ}}\downarrow & & \downarrow{\scriptstyle x} \\ FJ & \xrightarrow[Ff]{} & I \end{array}$$

over $f : J \to I$ such that for any morphism

$$\begin{array}{ccc} FK & \xrightarrow{s} & X \\ {\scriptstyle id_{FJ}}\downarrow & & \downarrow{\scriptstyle x} \\ FK & \xrightarrow[Fg]{} & I \end{array}$$

over $g : K \to I$ there exists a *unique* morphism $h : K \to J$ such that



But, obviously, this requirement is equivalent to the representability of the functor $\mathbb{B}/FI(F_{/I-}, x)$.

Therefore, $F^*P_{\mathbb{B}}$ has small global sections iff $F_{/I}$ has a right adjoint for all $I \in \mathbb{A}$. $\square$

Theorem 8.2 *Let $\mathbb{A}$ and $\mathbb{B}$ be categories with pullbacks. Then for a functor $F : \mathbb{A} \to \mathbb{B}$ the following are equivalent.*

1. *The fibration $F^*P_{\mathbb{B}} = \partial_1 : \mathbb{B}/F \to \mathbb{A}$ is* pre-geometric*.*
2. *The functor F* preserves pullbacks*.*
3. *For all $I \in \mathbb{A}$ the functor $F_{/I} : \mathbb{A}/I \to \mathbb{B}/FI$ is* lex*, i.e.* preserves finite limits*.*

Proof. Of course, the functor F preserves pullbacks iff for all $I \in \mathbb{A}$ the functor $F_{/I}$ preserves finite limits.

If F preserves pullbacks then the fibration $F^*P_{\mathbb{B}}$ is pre-geometric as as $P_{\mathbb{B}}$ is pre-geometric and change of base along F preserves this property.

By Theorem 2.2 the functor F preserves pullbacks if $F^*P_{\mathbb{B}}$ has internal sums satisfying the Beck-Chevalley condition. Thus the functor F preserves pullbacks especially if the fibration $F^*P_{\mathbb{B}}$ is pre-geometric. □

From these two theorem we get a characterisation of those functors $F : \mathbb{A} \to \mathbb{B}$ between categories with pullbacks that induce a geometric fibration.

Theorem 8.3 *Let $F : \mathbb{A} \to \mathbb{B}$ be a functor between categories with pullbacks. Then the following are equivalent.*

1. *$F^*P_{\mathbb{B}}$ is a* geometric fibration*.*
2. *For all $I \in \mathbb{A}$ the functor $F_{/I} : \mathbb{A}/I \to \mathbb{B}/FI$ is (the inverse image part of) a geometric morphism, i.e. $F_{/I}$ preserves finite limits and has a right adjoint.*
3. *The functor F preserves pullbacks and each slice of F has a right adjoint, i.e. for all $I \in \mathbb{A}$ the functor $F_{/I} : \mathbb{A}/I \to \mathbb{B}/FI$ has a right adjoint.*

Proof. Immediate from Theorems 8.1 and 8.2. □

Therefore such functors will be called partial geometric morphisms between categories with pullbacks.

Definition 8.1 *A functor $F : \mathbb{A} \to \mathbb{B}$ between categories with pullbacks is called a* partial geometric morphism *iff F preserves pullbacks and each slice of F has a right adjoint, i.e. iff each slice of F preserves finite limits and has a right adjoint.*

Notice that this notion of partial geometric morphism is conservative w.r.t. the notion of partial geometric morphism between lex categories due to the following lemma.

Lemma 8.4 *Let $\mathbb{A}$ be a lex category and $\mathbb{B}$ be partially lex. Let $F : \mathbb{A} \to \mathbb{B}$ be an arbitrary functor.Then F has a right adjoint iff all slices of F have a right adjoint.*

Proof. If F has a right adjoint U then for all $I \in \mathbb{A}$ the slice functor $F_{/I} : \mathbb{A}/I \to \mathbb{B}/FI$ has the right adjoint $\eta_I^* \circ U_{/FI}$.

If all slices of F have a right adjoint then $F_{/1} : \mathbb{A} \cong \mathbb{A}/1 \to \mathbb{B}/F1$ has a right adjoint $U_{(1)}$. Then $F = \Sigma_{F1} \circ F_{/1} \dashv U_{(1)} \circ (F1)^*$. □

In the light of the previous section this suggest the following definition of *(total) geometric morphism* between partial lex categories.

Definition 8.2 *A functor $F : \mathbb{A} \to \mathbb{B}$ between partial lex categories is (the inverse image part of) a* (total) geometric morphism *iff the following conditions are satisfied*

1. *The functor $F : \mathbb{A} \to \mathbb{B}$ is (the inverse image part of) a* partial geometric morphism, *i.e. for all $I \in \mathbb{A}$ the functor $F_{/I} : \mathbb{A}/I \to \mathbb{B}/FI$ preserves finite limits and has a right adjoint.*
2. *The functor F* preserves binary products.
3. *The functor F is* cofinal, *i.e. for any $B \in \mathbb{B}$ there is a morphism $B \to FA$ in $\mathbb{B}$ for some $A \in \mathbb{A}$.*

Thus we get a 1-1-correspondence (up to equivalence) between geometric fibrations over $\mathbb{A}$ and total geometric morphisms from $\mathbb{A}$ to some partial lex $\mathbb{B}$.

We will show that the definition of total geometric morphism can be simplified considerably iff we assume that the partial lex catgeories involved have "enough subterminals".

Definition 8.3 *Let $\mathbb{C}$ be a category. An object $I \in \mathbb{C}$ will be called* subterminal *iff for any $X \in \mathbb{C}$ there exists* at most one *morphism $X \to I$. We write $st(\mathbb{C})$ for the class of subterminals of $\mathbb{C}$.*
The category $\mathbb{C}$ has enough subterminals *iff for any $X \in \mathbb{C}$ there is an $I \in st(\mathbb{C})$ with $X \to I$, i.e. iff $st(\mathbb{C})$ is* cofinal *in $\mathbb{C}$.*

Notice that any lex category has enough subterminals as it has a terminal object. But not any lex category needs to have supports as this would imply that all terminal projections admit an initial epi-mono-factorisation.

Notice that if a category $\mathbb{C}$ has pullbacks and enough subterminals then $\mathbb{C}$ has binary products if and only if products of subterminals exist in $\mathbb{C}$. For $I, J \in st(\mathbb{C})$ then $I \times J$ is also subterminal in $\mathbb{C}$ and therefore we write $I \cap J$ ("intersection of I and J") instead of $I \times J$.

Now we have the following lemma.

Lemma 8.5 *Let $\mathbb{A}$ and $\mathbb{B}$ be partial lex categories with enough subterminals and $F : \mathbb{A} \to \mathbb{B}$ preserve pullbacks. Then F preserves binary products iff F preserves subterminals and their intersections.*

Thus we get the following characterisation of total geometric morphisms between partial lex categories with enough subterminals.

Theorem 8.6 *Let $\mathbb{A}$ and $\mathbb{B}$ be partial lex categories with enough subterminals. A functor $F : \mathbb{A} \to \mathbb{B}$ is a total geometric morphism iff*

1. *The functor $F : \mathbb{A} \to \mathbb{B}$ is a* partial geometric morphism*.*
2. *The functor F* preserves subterminals and their intersections*.*
3. *The image of $st(\mathbb{A})$ under F is* cofinal *in $st(\mathbb{B})$, i.e. for any $J \in st(\mathbb{B})$ there exists an $I \in st(\mathbb{A})$ with $J \to FI$.*

This characterisation can be improved using the following lemma.

Lemma 8.7 *Let $F : \mathbb{A} \to \mathbb{B}$ be a functor between categories with pullbacks. If $\mathbb{A}$ has enough subterminals then*

1. *F preserves pullbacks iff $F_{/I}$ preserves finite limits for all subterminals I.*
2. *If F preserves pullbacks then all slices of F have right adjoints iff $F_{/I}$ has a right adjoint for all $I \in st(\mathbb{A})$.*

Proof. The implications from left to right are trivial.

For the reverse implications observe that for any $A \in \mathbb{A}$ there exists an $I \in st(\mathbb{A})$ with $t : A \to I$. Then the slice functor $F_{/A}$ is canonically isomorphic to the slice of $F_{/I}$ at $t : A \to I$.

If $F_{/I}$ preserves finite limits then all its slices preserve finite limits and therefore $F_{/A}$ preserves finite limits.

If $F_{/I}$ preserves finite limits and has a right adjoint then all its slices preserve finite limits and have a right adjoints. Thus $F_{/A}$ preserves finite limits and has a right adjoint.

As $\mathbb{A}$ is assumed to have enough subterminals we get the desired implications from right to left. □

So we get the following improved characterisations.

Theorem 8.8 *Let $\mathbb{A}$ and $\mathbb{B}$ be partial lex categories.*

If $\mathbb{A}$ has enough subterminals then a functor $F : \mathbb{A} \to \mathbb{B}$ is a partial geometric morphism iff for all $I \in st(A)$ the functor $F_{/I}$ preserves finite limits and has a right adjoint.

If both $\mathbb{A}$ and $\mathbb{B}$ have enough subterminals then a functor $F : \mathbb{A} \to \mathbb{B}$ is a total geometric morphism iff the following conditions hold :

1. *For all $I \in st(A)$ the functor $F_{/I}$ preserves finite limits and has a right adjoint.*
2. *The functor F* preserves subterminals and their intersections*.*
3. *The image of $st(\mathbb{A})$ under F is* cofinal *in $st(\mathbb{B})$, i.e. for any $J \in st(\mathbb{B})$ there exists an $I \in st(\mathbb{A})$ with $J \to FI$.*

Proof. The first claim follows is the same as the second claim of Lemma 8.7. From this the second claim follows by Theorem 8.6. □

The following lemma shows that the notion of total geometric morphisms is a conservative extension of the traditional notion of geometric morphism between categories with finite limits.

Lemma 8.9 *Let $F : \mathbb{A} \to \mathbb{B}$ be a functor between categories with finite limits. Then F is a total geometric morphism (in the sense of Definition 8.2) iff F preserves finite limits and has a right adjoint.*

Proof. Suppose that F is a total geometric morphisms. Then by Lemma 8.4 it has a right adjoint. By definition F preserves pullbacks.

It remains to show that F preserves the terminal object. By Theorem 8.6 there exists a subterminal $I \in st(\mathbb{A})$ and a morphism $h : 1 \to FI$. But then $F(!_I) \circ h : 1 \to F1$. As $F1$ is subterminal (by Theorem 8.6) we get that $F(!_I) \circ h$ is an isomorphism and therefore $F1$ is terminal.

Suppose that F preserves finite limits and has a right adjoint. By Lemma 8.4 all slices of F have a right adjoint. If F preserves finite limits then all slices of F preserve finite limits and F preserves binary products. As $F1$ is terminal the functor F is cofinal in $\mathbb{B}$. Thus F is a total geometric morphism in the sense of Definition 8.2. □

We finally give examples of *total geometric morphism between partial lex categories that do not have right adjoints.*

For any partial lex category $\mathbb{B}$ the functor $F : \mathbb{B} \to \mathbb{1}$ is a total geometric morphism. But F has a right adjoint if and only if $\mathbb{B}$ has a terminal object. Thus for all properly partial lex categories $\mathbb{B}$ the functor $F : \mathbb{B} \to \mathbb{1}$ is a total geometric morphism which does not have a right adjoint.

Let $\mathbb{E}$ be the full subcategory of $Tree = \hat{\omega} = Set^{\omega^{op}}$ on those objects $T \in \hat{\omega}$ with $T(n) = \emptyset$ for some $n \in \omega$. The category $\mathbb{E}$ is partial lex with enough subterminals.

Let $\pi : \hat{\omega} \to Set$ be the functor sending $T \in \hat{\omega}$ to $T(0)$ and $\alpha : T_1 \to T_2$ to α_0. We have $\pi \dashv \Delta \dashv \Gamma$ where $\Gamma = \hat{\omega}(1, \_)$ is the global sections functor. The functor π preserves limits and therefore π is the inverse image part of a geometric morphism. Let $F : \mathbb{E} \to Set$ be the restriction of π to the full subcategory $\mathbb{E}$. Therefore the functor F preserves all limits existing in $\mathbb{E}$. Moreover the image of $st(\mathbb{E})$ under F is cofinal in Set. For any $T \in \mathbb{E}$ the slice functor $F_{/T}$ is canonically isomorphic to the slice functor $\pi_{/T}$. As $\pi_{/T}$ has a right adjoint the slice functor $F_{/T}$ has a right adjoint as well.

Thus, by Theorem 8.6 the functor F is a total geometric morphism. But F does not have a right adjoint U as otherwise $U(1)$ were terminal in $\mathbb{E}$ which does not have a terminal object.

9 Pre-Geometric Fibrations are Induced by Glueing

In this section we prove that any pre-geometric fibration $P : \mathbb{X} \to \mathbb{B}$ over a partial lex $\mathbb{B}$ is equivalent to a pre-geometric fibration of the form $\partial_1 : \mathbb{C}/F \to \mathbb{B}$ for some partial lex functor $F : \mathbb{B} \to \mathbb{C}$ which is unique up to equivalence w.r.t. this property. [1]

We first observe that $\Sigma_P := coCart(P)$ is a *pullback congruence* in the sense of Bénabou's [Bén89]. This means that Σ_P satisfies the following three conditions

1. Σ contains all isomorphisms.
2. Σ is stable under arbitrary pullbacks.
3. Whenever φ and ψ are maps in $\mathbb{X}$ such that $\varphi \circ \psi$ is defined then the three maps φ, ψ and $\varphi \circ \psi$ are all in Σ whenever two of them are in Σ.

The first condition is obviously true. The second condition follows from stability of sums for P. For the third condition the only nontrivial case is covered by Lemma 5.2.

We now define $\mathbb{C}$ as $\mathbb{X}[\Sigma_P^{-1}]$. Recall that the objects of $\mathbb{C}$ are the objects of $\mathbb{X}$ and that $\mathbb{C}(X, Y)$ consists of equivalence classes of spans from X to Y where spans $\langle\varphi_0, \varphi_1\rangle$ and $\langle\psi_0, \psi_1\rangle$ are considered as equivalent iff there exist cocartesian arrows θ and ϑ with $\varphi_0 \circ \theta = \psi_0 \circ \vartheta$ and $\varphi_1 \circ \theta = \psi_1 \circ \vartheta$. We write $[\varphi_0, \varphi_1]$ for the equivalence class of the span $\langle\varphi_0, \varphi_1\rangle$.

Let $Q : \mathbb{X} \to \mathbb{C} = \mathbb{X}[\Sigma_P^{-1}]$ be the functor which is the identity on objects and sends $\varphi : X \to Y$ to the morphism $Q(\varphi) = [id_X, \varphi] : X \to Y$ in $\mathbb{C}$. Notice that Q is initial among the functors inverting all cocartesian arrows (i.e. sending all cocartesian arrows to isomorphisms).

As Σ_P is a pullback congruence on $\mathbb{X}$ (which has pullbacks) we can apply Theorem 1.4 from [Bén89] telling us that $\mathbb{C} = \mathbb{X}[\Sigma_P^{-1}]$ has pullbacks, $Q : \mathbb{X} \to \mathbb{C}$ preserves those finite limits which exists in $\mathbb{X}$ and that $\Sigma = Ker(Q)$ (i.e. that Σ contains all maps in $\mathbb{X}$ which Q_Σ maps to an isomorphisms). Thus, it follows immediately that $\mathbb{C} = \mathbb{X}[\Sigma_P^{-1}]$ has binary products as $\mathbb{X}$ has binary products, Q preserves them and Q is the identity on objects. Thus Q is a partial lex functor.

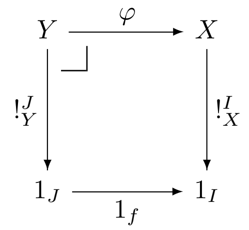
Let $1 : \mathbb{B} \to \mathbb{X}$ be a right adjoint right inverse to P picking a terminal object out of each fibre. As $1 : \mathbb{B} \to \mathbb{X}$ is a right adjoint it preserves pullbacks and binary products, i.e. 1 is partial lex. Thus, the functor $F := Q \circ 1 : \mathbb{B} \to \mathbb{C}$ is partial lex as both Q and 1 are partial lex. Furthermore, the image of F is cofinal in $\mathbb{C}$ as the terminals in the fibers of P are cofinal in $\mathbb{X}$ and Q is the identity on objects.

For obtaining our desired result it remains to show that there is a *c*artesian equivalence $H : \mathbb{X} \to \mathbb{C}/F$ from P to $Gl(F) = \partial_1 : \mathbb{C}/F \to \mathbb{B}$. An object $X \in P(I)$ is sent by H to $Q(!_X^I) : X \to 1_I$ in $\mathbb{C}$ where $!_X^I : X \to 1_I$ is the unique

[1] A category is partial lex iff it has binary pullbacks and products and a functor between partial lex categories is called partial lex iff it preserves binary pullbacks and products.

vertical map in $\mathbb{X}$. A morphism $\varphi : X \to Y$ in $\mathbb{X}$ over $f = P(\varphi)$ is sent to the morphism $(Q(\varphi), f) : H(X) \to H(Y)$ in $\mathbb{C}/F$.

The functor $H : \mathbb{X} \to \mathbb{C}$ is cartesian as Q preserves pullbacks and

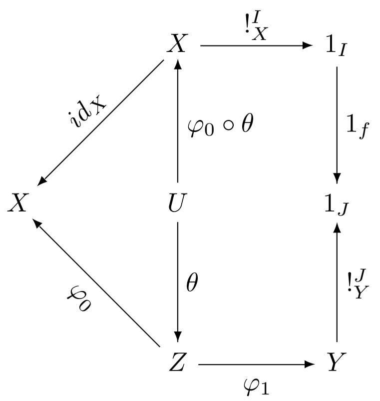


is a pullback if φ is cartesian over $f = P(\varphi) : J \to I$.

That H is faithful as a cartesian functor follows from the fact that whenever $\varphi, \psi : X \to Y$ are arrows in $\mathbb{X}$ with $P(\varphi) = P(\psi)$ and $Q(\varphi) = Q(\psi)$ then $\varphi = \psi$. This is the case as $Q(\varphi) = Q(\psi)$ means that there exists a cocartesian θ with $\varphi \circ \theta = \psi \circ \theta$ and this implies $\varphi = \psi$ as by the assumption $P(\varphi) = P(\psi)$.

Next we show that H is full as a cartesian functor.

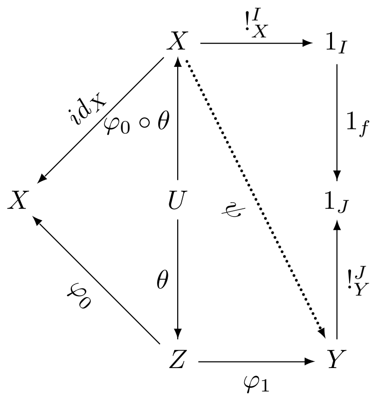
Suppose that $\langle[\varphi_0, \varphi_1], f\rangle : H(X) \to H(Y)$ in $\mathbb{C}/F$, i.e. $f : I \to J$ in $\mathbb{B}$, $X \in P(I)$, $Y \in P(J)$ and $[\varphi_0 : Z \to X, \varphi_1 : Z \to Y] : X \to Y$ in $\mathbb{C}$ with $Q(!^J_Y) \circ [\varphi_0, \varphi_1] = F(f) \circ Q(1^I_X)$ in $\mathbb{C}$. The latter equality means in explicit terms that there exists a cocartesian arrow $\theta : U \to Z$ in $\mathbb{X}$ such that



commutes, i.e. that $!^J_Y \circ \varphi_1 \circ \theta = 1_f \circ !^I_X \circ \varphi_0 \circ \theta$.

But then $P(\varphi_1 \circ \theta) = P(\varphi_1) \circ P(\theta) = f \circ P(\varphi_0) \circ P(\theta)$ and therefore there exists a unique arrow $\psi : X \to Y$ in $\mathbb{X}$ over f with $\psi \circ \varphi_0 \circ \theta = \varphi_1 \circ \theta$ as $\varphi_0 \circ \theta$

is cocartesian. This situation is illustrated by the following diagram



which commutes as $\varphi_0 \circ \theta$ is cocartesian and both $1_f \circ !^I_X$ and $!^J_Y \circ \psi$ are over f satisfying $1_f \circ !^I_X \circ \varphi_0 \circ \theta = !^J_Y \circ \varphi_1 \circ \theta = !^J_Y \circ \psi \circ \varphi_0 \circ \theta$.

Thus, $[\varphi_0, \varphi_1] = [id_X, \psi] = Q(\psi)$ and therefore $Q(!^J_Y) \circ Q(\psi) = F(f) \circ Q(!^I_X)$ holds in $\mathbb{C}$ demonstrating that $H(\psi : Y \to X) = \langle Q(\psi), f \rangle = \langle [\varphi_0, \varphi_1], f \rangle$.

Thus H is full as a cartesian functor.

It remains to show that any object $[\varphi_0 : Z \to Y, \varphi_1 : Z \to 1_I] : Y \to F(I)$ in $\mathbb{C}/F$ is isomorphic in the fiber over I to $H(X) = [id_X, !_X] : X \to F(I)$ for some $X \in P(I)$. Let $f = P(\varphi_1) : J \to I$ and $\psi : Z \to X$ be a cocartesian arrow over f. Then $[\varphi_0, \psi] : Y \to X$ is an isomorphism (with inverse $[\psi, \varphi_0]$) such that $[id_X, !_X] \circ [\varphi_0, \psi] = [\varphi_0, \varphi_1]$, i.e. $Q(!^I_X) \circ [\varphi_0, \psi] = [\varphi_0, \varphi_1]$, establishing that $H(X) = Q(!^I_X)$ is isomorphic to $[\varphi_0, \varphi_1] : Y \to F(I)$ in the fiber over I (via the isomorphism $[\varphi_0, \psi] : Y \to X$ in $\mathbb{C}$).

This finishes the proof that $H : \mathbb{X} \to \mathbb{C}/F$ establishes a cartesian equivalence between P and $Gl(F) = \partial_1 : \mathbb{C}/F \to \mathbb{B}$.

It is clear that each slice of F has a right adjoint if P is a geometric fibration as $Gl(F)$ is equivalent to P and the property of having small global sections is invariant under equivalence.

Notice that in the proof above we have never used the requirement that $\mathbb{B}$ has binary products but for showing that $\mathbb{X}[\Sigma_P^{-1}]$ has binary products and $F = Q \circ 1$ preserves them. Therefore, we also get a Moens' Theorem for categories $\mathbb{B}$ with pullbacks : any pre-geometric fibration $P : \mathbb{X} \to \mathbb{B}$ is equivalent to $Gl(F) = \partial_1 : \mathbb{C}/F \to \mathbb{B}$ for some category $\mathbb{C}$ with pullbacks and some pullback preserving functor $F : \mathbb{B} \to \mathbb{C}$ which are unique up to equivalence; furthermore, any slice of F has a right adjoint iff P is a geometric fibration.

If $G : \mathbb{B} \to \mathbb{D}$ is a pullback preserving functor between categories with pullbacks (and binary products) then one may apply the above construction to the pre-geometric fibration $Gl(G) = \partial_1 : \mathbb{D}/G \to \mathbb{B}$. By the above construction this gives rise to a functor $F : \mathbb{B} \to \mathbb{C}$ (with $\mathbb{C} = \mathbb{D}/G[\Sigma^{-1}_{Gl(G)}]$) between categories with pullbacks (and binary products) such that F preserves pullbacks (and binary products). Furthermore, by construction of F the pre-geometric fibrations $Gl(F)$ and $Gl(G)$ are equivalent.

Moreover, there is a functor $S : \mathbb{C} \to \mathbb{D}$ with $G = S \circ F$ sending a morphism

$$
\begin{array}{ccccc}
A & \xleftarrow[\cong]{i} & D & \xrightarrow{h} & B \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle b} \\
GI & \xleftarrow[Gf]{} & GK & \xrightarrow[Gg]{} & GJ
\end{array}
$$

in $\mathbb{C} = \mathbb{D}/G[\Sigma^{-1}_{Gl(F)}]$ (where i is an isomorphism as the left square is a cocartesian) to the morphism $h \circ i^{-1} : A \to B$ in $\mathbb{D}$. It can be shown that S preserves pullbacks (but in general need not preserve binary products!) and every slice of S has a right adjoint independent from whether every slice of G has a right adjoint or not! But as we have seen already, it need not be the case at all that S itself has a right adjoint.

Thus $S : \mathbb{C} \to \mathbb{D}$ generalises the functor $\Sigma_{G1} : \mathbb{D}/G1 \to \mathbb{D}$ to the case when $\mathbb{B}$ does not have a terminal object. The functor S need not preserve binary products even if $\mathbb{B}$ has a terminal object as $S = \Sigma_{G1} : \mathbb{D}/G1 \to \mathbb{D}$ preserves binary products iff $G1$ is subterminal.

10 Characterisation of Local Homeomorphisms

Let $\mathbb{C}$ be a category with finite limits and $\mathbb{B}$ a category with pullbacks. A functor $F : \mathbb{C} \to \mathbb{B}$ is a *local homeomorphism* iff $F_{/1} : \mathbb{C} \to \mathbb{B}_{/F1}$ is an equivalence and $\Sigma_{F1} : \mathbb{B}/F1 \to \mathbb{B}$ has a right adjoint $(F1)^*$. Thus, F is a local homeomorphism iff F is canonically equivalent to Σ_{F1} via $F_{/1}$ (as $F = \Sigma_{F1} \circ F_{/1}$) and the cartesian product $F1 \times B$ exists for all $B \in \mathbb{B}$. Actually, if $F_{/1}$ is an equivalence then F has a right adjoint iff Σ_{F1} has a right adjoint $(F1)^*$ (again as $F = \Sigma_{F1} \circ F_{/1}$).

Thus, every local homeomorphism $F : \mathbb{C} \to \mathbb{B}$ necessarily preserves pullbacks and has a right adjoint, i.e. necessarily is a partial geometric morphism.

A pullback preserving functor $F : \mathbb{C} \to \mathbb{B}$ with $F \dashv U$ is a local homeomorphism, i.e. $F_{/1}$ is an equivalence [2], iff the (fibred) geometric morphism $\Delta \dashv \Gamma : P_{F_{/1}} \to P_{\mathbb{C}}$ is an equivalence. Recall that $P_F \simeq P_{F_{/1}}$ for all pullback preserving functors $F : \mathbb{C} \to \mathbb{B}$. Thus, F is a local homeomorphism iff the fibred geometric morphism $\Delta \dashv \Gamma : P_F \to P_{\mathbb{C}}$ is an equivalence, i.e. iff $\Delta \dashv \Gamma$ is both injective and surjective as a (fibred) geometric morphism, i.e. iff the following two conditions [3] hold

1. Δ reflects isos (in the fiber over 1)

2. Γ is full and faithful (in the fiber over 1).

[2] If $F_{/1}$ is an equivalence Σ_{F1} has a right adjoint due to the assumption $F \dashv U$.

[3] As P_F is a geometric fibration these conditions hold for every fiber iff they hold in the fiber over 1.

Recall that the counit of $\Delta \dashv \Gamma$ at $b : B \to FI$ is given by $\varepsilon_b = \varepsilon_B \circ F(Ub)^* \eta_I$

$$\begin{array}{ccccc} FX & \xrightarrow{F(Ub)^*\eta_I} & FUB & \xrightarrow{\varepsilon_B} & B \\ {\scriptstyle F\Gamma_I b}\downarrow & & \downarrow{\scriptstyle FUb} & & \downarrow{\scriptstyle b} \\ FI & \xrightarrow[F\eta_I]{} & FUFI & \xrightarrow[\varepsilon_{FI}]{} & FI \end{array}$$

where ε_B and ε_{FI} are instances of the counit of $F \dashv U$ at B and

$$\begin{array}{ccc} X & \xrightarrow{(Ub)^*\eta_I} & UB \\ {\scriptstyle \Gamma_I b}\downarrow & & \downarrow{\scriptstyle Ub} \\ I & \xrightarrow[\eta_I]{} & UFI \end{array}$$

and $\varepsilon_{FI} \circ F\eta_I = id_{FI}$ as $F \dashv U$.

Thus, condition 2. can be reformulated as the requirement that $\varepsilon_B \circ F(Ub)^* \eta_1$ is an isomorphism for all $b : B \to F1$ (as a right adjoint is full and faithful iff its counit is an isomorphism). This latter condition on $F \dashv U$ has been introduced in Funk's Thesis [Fun90] under the name "Frobenius reciprocity" and therefore we refer to it as "Funk's Frobenius reciprocity" [4].

[4]The requirement that for all $b : B \to FI$ the morphism $\varepsilon_B \circ F(Ub)^* \eta_I$ is an isomorphism is equivalent to the requirement that $\hat{\alpha} = \varepsilon_X \circ F\alpha : FY \to X$ is an isomorphism whenever $b : B \to I$ and $c : C \to UI$ and

$$\begin{array}{ccccc} Y & \xrightarrow{\alpha} & UX & \xrightarrow{U\pi_2} & UB \\ \downarrow & & \downarrow{\scriptstyle U\pi_1} & & \downarrow{\scriptstyle Ub} \\ C & \xrightarrow[\eta_C]{} & UFC & \xrightarrow[U\hat{c}]{} & UI \end{array}$$

where

$$\begin{array}{ccc} X & \xrightarrow{\pi_2} & B \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle b} \\ FC & \xrightarrow[\hat{c}]{} & I \end{array}$$

and $\hat{c} = \varepsilon_I \circ Fc$. (The implication from left to right follows when instantiating I by C and b by π_1 and the implication from right to left follows when instantiating C by I, I by FI and c by η_I.)

Notice that the right hand condition states that for all $I \in \mathbb{B}$ the adjunction $\Sigma_{\varepsilon_I} \circ F_{/UI} \dashv U_{/I}$

Summarising we get the following elementary characterisation of local homeomorphisms.

Theorem 10.1 *If $\mathbb{B}$ and $\mathbb{C}$ are categories with pullbacks and $\mathbb{C}$ has a terminal object then $F : \mathbb{C} \to \mathbb{B}$ is a local homeomorphism, i.e. $F_{/1}$ is an equivalence and $\Sigma_{F1} : \mathbb{B}/F1 \to \mathbb{B}$ has a right adjoint $(F1)^*$, iff*

1. *F preserves pullbacks*
2. *F reflects isomorphisms*
3. *F has a right adjoint U with $F \dashv U$ satisfying Funk's Frobenius reciprocity, i.e. $\varepsilon_B \circ F(Ub)^*\eta_1$ is an isomorphism for all $b : B \to F1$*
4. *the product $X \times F1$ exists for all objects X in $\mathbb{B}$.*

11 Strongly Local Geometric Morphisms and Unity and Identity of Adjoint Opposites

Recall that a *local* geometric morphism is a geometric morphism $F \dashv U : \mathbb{C} \to \mathbb{B}$ where U has a right adjoint R.

From a fibrational point of view, however, it is more natural to require that the fibred geometric morphism $\Delta \dashv \Gamma : P_F \to P_{\mathbb{B}}$ (associated with $F \dashv U$) is a *fibred local geometric morphism*, i.e. that Γ has a further fibred right adjoint ∇. In this case $F \dashv U$ is called a *strongly local geometric morphism*.

If U has an ordinary right adjoint R then for every $I \in \mathbb{B}$ the functor $\Gamma_I = \eta_I^* \circ U_{/FI}$ has a right adjoint

$$\nabla_I \equiv \eta_{FI}^* \circ R_{/UFI} \circ \Pi_{\eta_I}$$

where $\eta_{FI} : FI \to RUFI$ is the unit of $U \dashv R$ at FI and $\eta_I : I \to UFI$ is the unit of $F \dashv U$ at I.

Unfortunately, in general this does not entail that Γ has a fibred right adjoint ∇ because in this case the cartesian functor Γ has to be cocartesian, i.e. preserve internal sums, as $(Ff)^* \circ \nabla_I \cong \nabla_J \circ f^*$ iff $\Gamma_I \circ \Sigma_{Ff} \cong \Sigma_f \circ \Gamma_J$ for all $f : J \to I$ in $\mathbb{B}$.

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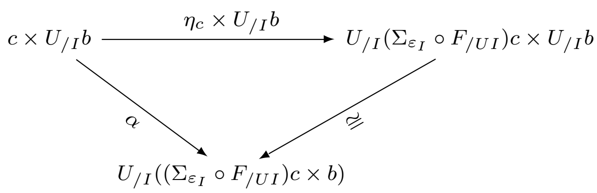
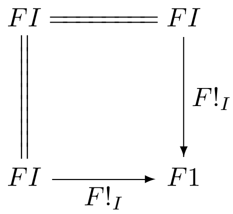
satisfies Lawvere's original Frobenius condition, i.e. $\hat{\alpha}$ is an isomorphism where

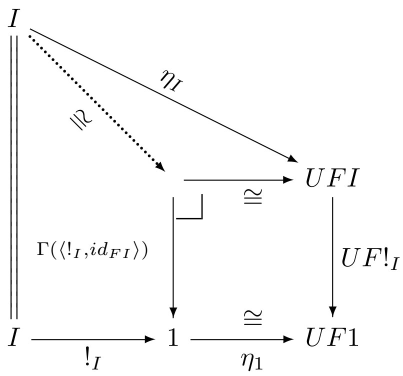


as $\hat{c} = \varepsilon_I \circ Fc = (\Sigma_{\varepsilon_I} \circ F_{/UI})c$.

But if Γ is cocartesian then $\eta_I : I \to UFI$ is an isomorphism for all $I \in \mathbb{B}$ which can be seen as follows. Applying Γ to the cocartesian arrow $\langle !_I, id_{FI} \rangle$



we get



from which it follows that η_I is an isomorphism.

On the other hand if $\eta : Id_{\mathbb{B}} \to UF$ is a natural isomorphism then Γ is cocartesian. Thus, Γ is cocartesian iff $\eta : Id_{\mathbb{B}} \to UF$ is a natural isomorphism.

Thus, $F \dashv U$ is a strongly local geometric morphism iff U has a right adjoint R and F is full and faithful, i.e. $F \dashv U \dashv R$ is a UIAO (Unity and Identiy of Adjoint Opposites) in the sense of Lawvere.

12 Locally Connected Geometric Morphisms

Recall that a geometric morphism $F \dashv U : \mathbb{C} \to \mathbb{B}$ is called *locally connected* iff Δ has a fibred left adjoint π where $\Delta \dashv \Gamma : P_F \to P_{\mathbb{B}}$ is the fibred geometric morphism associated with $F \dashv U$.

More generally we may call a left exact functor $F : \mathbb{B} \to \mathbb{C}$ *locally connected* iff the cartesian functor $\Delta : P_{\mathbb{B}} \to P_F$ has a fibred left adjoint $\pi \dashv \Delta$.

In particular this means that F has an ordinary left adjoint $L \dashv F$. Then $\Sigma_{\varepsilon_I} \circ L_{/FI} \dashv F_{/I} \equiv \Delta_I$ and Δ has an ordinary left adjoint π with $P_{\mathbb{B}} \circ \pi = P_F$.

Now it can be seen easily that this functor π is a cartesian iff for any pullback

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ {\scriptstyle a}\downarrow & \lrcorner & \downarrow{\scriptstyle b} \\ FI & \xrightarrow[Ff]{} & FJ \end{array}$$

the square

$$\begin{array}{ccc} LA & \xrightarrow{Lg} & LB \\ {\scriptstyle \hat{a}}\downarrow & \lrcorner & \downarrow{\scriptstyle \hat{b}} \\ I & \xrightarrow[f]{} & J \end{array}$$

is a pullback, too.

Let $L \dashv F \dashv U : \mathbb{E} \to \mathbb{S}$ be a geometric morphism between toposes. A morphism g in $\mathbb{E}$ is called $\mathbb{S}$–*definable* or simply *definable* iff g appears as pullback of $F(f)$ for some morphism f in $\mathbb{S}$. Accordingly, a monomorphism is definable iff it appears as pullback of $F(\top_{\mathbb{S}})$. One easily checks that a definable monomorphism $m : P \to A$ is canonically isomorphic to $\eta_A^* Lm$ and every subobject $n : Q \to LA$ is canonically isomorphic to $L\eta_A^* n$ (where in both cases η_A is the unit of $L \dashv F$ at A). This way the definable subobjects of A are in natural 1–1–correspondence with the subobjects of LA.

Let $F \dashv U : \mathbb{E} \to \mathbb{S}$ be a bounded geometric morphism beween toposes. Then $F \dashv U$ is locally connected iff [5] F preserves the locally cartesian closed structure.

Recall from [BD] that a geometric morphism $F \dashv U : \mathbb{E} \to \mathbb{S}$ is called *atomic* iff $F : \mathbb{S} \to \mathbb{E}$ is logical. It is known that in this case F has a left adjoint L (sending X to the atoms of $U(\mathcal{P}(X))$ as described in [BD]) and, therefore, the geometric morphism $F \dashv U$ is locally cartesian closed as F being logical it preserves in particular the locally cartesian closed structure. Thus, a geometric morphism $F \dashv U$ is atomic iff it is locally connected and F preserves the subobject classifier. One easily sees that for a locally connected geometric morphism $F \dashv U : \mathbb{E} \to \mathbb{S}$ its inverse image part F preserves the subobject

[5] As by the fibred adjoint functor theorem $\Delta_F : P_{\mathbb{S}} \to P_F$ has a left adjoint iff Δ_F is a limit preserving cartesian functor, i.e. F preserves the locally cartesian closed structure (F preserves finite limits anyway and F preserves Π iff Δ_F preserves internal products).

classifier if and only if all monomorphisms in $\mathbb{E}$ are ($\mathbb{S}$–)definable, i.e. iff

$$\begin{array}{ccc} P & \xrightarrow{\eta_P} & FLP \\ {\scriptstyle m}\downarrow & \lrcorner & \downarrow{\scriptstyle FLm} \\ A & \xrightarrow[\eta_A]{} & FLA \end{array}$$

is a pullback for all monomorphisms $m : P \to A$ in $\mathbb{E}$. As $\mathbb{S}$–definable monomorphisms are stable under pullbacks it suffices to require that $\top_{\mathbb{E}}$ is $\mathbb{S}$–definable.

Thus, a geometric morphism $F \dashv U : \mathbb{E} \to \mathbb{S}$ is atomic iff it is locally connected and

$$\begin{array}{ccc} 1_{\mathbb{E}} & \xrightarrow{\eta_{1_{\mathbb{E}}}} & FL1_{\mathbb{E}} \\ {\scriptstyle \top_{\mathbb{S}}}\downarrow & \lrcorner & \downarrow{\scriptstyle FL\top_{\mathbb{E}}} \\ \Omega_{\mathbb{E}} & \xrightarrow[\eta_{\Omega_{\mathbb{E}}}]{} & FL\Omega_{\mathbb{E}} \end{array}$$

is a pullback.

13 Connected Geometric Fibrations

Traditionally, a geometric morphism between toposes is called *connected* iff the unit of the adjunction is an isomorphism. Accordingly, a geometric fibration $P : \mathbb{X} \to \mathbb{B}$ is called **connected** iff the unit η^P of the fibred adjunction $\Delta_P \vdash \Gamma_P$ is a natural isomorphism.

More explicitely, this means that for every morphism $u : I \to J$ in $\mathbb{B}$ the morphism $\eta^P_u : I \to GX$ is an isomorphism

$$\begin{array}{ccc} 1_I & \xrightarrow{1_{\eta^P_u}} & 1_{GX} \\ & {\scriptstyle \varphi}\searrow & \downarrow{\scriptstyle \varepsilon_X} \\ & & X \cong \coprod_u 1_I \end{array}$$

where φ is cocartesian over u. Notice that η^P_u is an isomorphism iff $G\varphi$ is an isomorphism as $1_{G\varphi} = 1_{\eta^P_u} \circ \varepsilon_{1_I}$

$$\begin{array}{ccc} 1_{G(1_I)} & \xrightarrow{1_{G\varphi}} & 1_{GX} \\ {\scriptstyle \cong}\Big\downarrow{\scriptstyle \varepsilon_{1_I}} & & \Big\downarrow{\scriptstyle \varepsilon_X} \\ 1_I & \xrightarrow[\varphi]{} & X \end{array}$$

and ε_{1_I} is an isomorphism.

Thus, P is a connected geometric fibration iff [6] G inverts the cocartesian arrows of $\mathbb{X}$ (i.e. send them to isomorphisms) which in turn is equivalent to the requirement that Γ_P is cocartesian [7].

Accordingly, we call a geometric fibration $P : \mathbb{X} \to \mathbb{B}$ is **local** iff Γ_P has a fibred right adjoint ∇_P. But, then Γ_P is cocartesian and, therefore, local geometric fibrations are in particular connected. Thus, a geometric fibration P is local iff for the corresponding geometric morphism $F \dashv U : \mathbb{E} \equiv \mathbb{X}_1 \to \mathbb{B}$ it holds that F is full and faithful and U has a right adjoint R (which has to be full and faithful, too). Thus, local geometric fibrations correspond to UIAO's in the sense of Lawvere.

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[6] If G inverts cocartesian arrows $\varphi : 1_I \to \coprod_u 1_I$ then $G\psi$ is an iso for every cocartesian $\psi : X \to Y$ over u as G preserves pullbacks and, therefore, we have

$$\begin{array}{ccc} GX & \xrightarrow[\cong]{G\psi} & GY \\ {\scriptstyle G\alpha}\Big\downarrow & \lrcorner & \Big\downarrow{\scriptstyle G\beta} \\ G1_I & \xrightarrow[G\varphi]{\cong} & G\coprod_u 1_I \end{array}$$

where α and β are the unique vertical arrows with

$$\begin{array}{ccc} X & \xrightarrow{\psi} & Y \\ {\scriptstyle \alpha}\Big\downarrow & \lrcorner & \Big\downarrow{\scriptstyle \beta} \\ 1_I & \xrightarrow[\varphi]{} & \coprod_u 1_I \end{array} \quad .$$

[7] As $\Gamma_P(\varphi : X \to Y)$ is

$$\begin{array}{ccc} GX & \xrightarrow{G\varphi} & GY \\ {\scriptstyle P\varepsilon_X}\Big\downarrow & & \Big\downarrow{\scriptstyle P\varepsilon_Y} \\ PX & \xrightarrow[P\varphi]{} & PY \end{array}$$

and, therefore, $\Gamma_P(\varphi)$ is cocartesian iff $G\varphi$ is an isomorphism

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