Can one split dictoses?

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October 2015 – May 2017

A most natural notion of model for the Calculus of Constructions (CC) is T. Ehrhard's notion of a *dictos*, i.e. a locally cartesian closed category $\mathbb{C}$ together with a map $p_{Prop} : Prf \to Prop$ such that the class $\mathcal{P}$ of maps in $\mathbb{C}$ which can be obtained as pullback of p_{Prop} are closed under dependent products in $\mathbb{C}$, i.e. $\Pi_u a \in \mathcal{P}$ whenever $a : A \to J$ in $\mathcal{P}$ and $u : J \to I$ in $\mathbb{C}$. However, in order to interpret CC in a dictos we have to "split" it for the sake of interpreting the syntax of CC.

As well known the Yoneda functor $\mathsf{Y}_{\mathbb{C}} : \mathbb{C} \to \mathbf{Set}^{\mathbb{C}^{\mathsf{op}}}$ preserves finite limits and dependent products. Moreover, if $\mathcal{U}$ is a Grothendieck universe such that $\mathbb{C}$ lives in $\mathcal{U}$, i.e. $\mathbb{C}$ is internal to the category $\mathcal{U}$, then the Yoneda functor $\mathsf{Y}_{\mathbb{C}} : \mathbb{C} \to \mathbf{Set}^{\mathbb{C}^{\mathsf{op}}}$ factors through the inclusion $\mathcal{U}^{\mathbb{C}^{\mathsf{op}}} \hookrightarrow \mathbf{Set}^{\mathbb{C}^{\mathsf{op}}}$. We also write $\mathsf{Y}_{\mathbb{C}} : \mathbb{C} \to \mathcal{U}^{\mathbb{C}^{\mathsf{op}}}$ for the corresponding corestriction of $\mathsf{Y}_{\mathbb{C}} : \mathbb{C} \to \mathbf{Set}^{\mathbb{C}^{\mathsf{op}}}$ and notice that it also preserves finite limits and dependent products. Most of the time, however, we will simply write Y for $\mathsf{Y}_{\mathbb{C}}$.

If we choose $\mathcal{U}$ big enough for $\mathbb{C}$ being internal to $\mathcal{U}$ we may consider $\mathbb{C}$ as a small full subcategory of $\widehat{\mathbb{C}} = \mathbf{Set}^{\mathbb{C}^{\mathsf{op}}}$ as induced by a certain representable morphism $p_U : E_U \to U$ in $\widehat{\mathbb{C}}$ which can be described as follows. For $I \in \mathbb{C}$ let $U(I) = \big\{A \in \mathcal{U}^{(\mathbb{C}/I)^{\mathsf{op}}} \mid A \text{ representable}\big\}$ and for $u : J \to I$ in $\mathbb{C}$ let $U(u) = \mathcal{U}^{\Sigma_u^{\mathsf{op}}}$. We define p_U via its corresponding presheaf E_U over $\mathsf{Elts}(U)$ as follows: $\mathsf{E}_U(I, A) = A(\mathrm{id}_I)$ and $\mathsf{E}_U\big(u : u^*A \to A\big) = A\big(u : u \to \mathrm{id}_I\big)$. Notice that p_U is universal among the class $\mathcal{S}$ of representable morphisms in $\widehat{\mathbb{C}}$. This class $\mathcal{S}$ is stable under pullbacks along arbitrary morphisms in $\widehat{\mathbb{C}}$ and is it stable under dependent products in $\widehat{\mathbb{C}}$ (since $\mathsf{Y}_{\mathbb{C}}$ preserves Π).

We always can choose $\mathcal{U}$ so big that there is a Grothendieck universe $\mathcal{U}_0 \in \mathcal{U}$ with $\mathbb{C}$ internal to $\mathcal{U}_0$. Let U_P be the subpresheaf of U where $U_P(I)$ consists of all presheaves $A : (\mathbb{C}/I)^{\mathsf{op}} \to \mathcal{U}_0$ representable by a map in $\mathcal{P}$ with codomain I. Then the map p_P in

$$\begin{array}{ccc} E_P & \hookrightarrow & E_U \\ {\scriptstyle p_P}\downarrow & \lrcorner & \downarrow{\scriptstyle p_U} \\ U_P & \hookrightarrow & U \end{array}$$

is universal for the class $\mathcal{S}_P$ of $\mathcal{P}$-representable morphisms, i.e. morphisms f :

$Y \to X$ which for all $x : \mathsf{Y}(I) \to X$ fit into a pullback diagram

$$\begin{array}{ccc} \mathsf{Y}(A) & \longrightarrow & Y \\ {\scriptstyle \mathsf{Y}(a)}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ \mathsf{Y}(I) & \xrightarrow[x]{} & X \end{array}$$

for some $a \in \mathcal{P}$.

Lemma 0.1 *If $f : Y \to X$ is in $\mathcal{S}$ and $g : Z \to Y$ is in $\mathcal{S}_P$ then $\Pi_f g$ is in $\mathcal{S}_P$.*

Proof: By assumption on f and g for all $x : \mathsf{Y}_I(I) \to X$ we have

$$\begin{array}{ccc} \mathsf{Y}(A) & \longrightarrow & Z \\ {\scriptstyle \mathsf{Y}(a)}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ \mathsf{Y}(J) & \longrightarrow & Y \\ {\scriptstyle \mathsf{Y}(u)}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ \mathsf{Y}(I) & \xrightarrow[x]{} & X \end{array}$$

for some $u : J \to I$ in $\mathbb{C}$ and $a : A \to J$ in $\mathcal{P}$.

Since Yoneda preserves Π we have $x^*\Pi_f g \cong \Pi_{\mathsf{Y}(u)}\mathsf{Y}(a) \cong \mathsf{Y}(\Pi_u a)$ from which the claim follows since $\Pi_u a$ is in $\mathcal{P}$. □

Now, if U_P were in U we could apply Voevodsky's "method of universes" for splitting the original dictos within $\mathbf{Set}^{\mathbb{C}^{\mathsf{op}}}$. Dependent products for representable morphisms are dealt with as in *loc.cit.* For impredicative universal quantification we proceed as follows. Consider the generic context

$$\Gamma_G \equiv A : U, p : U_P^{E_U(A)}$$

and the families

$$a_G \equiv \Gamma_G \vdash E_U(A) \quad \text{and} \quad p_G \equiv \Gamma_G, a : E_U(A) \vdash E_P(P(a))$$

in $\mathcal{S}$ and $\mathcal{S}_P$, respectively. By Lemma 0.1 conclude that $\Pi_{a_G} p_G$ is in $\mathcal{S}_P$. Thus, since p_P is generic for $\mathcal{S}_P$ there is a morphism $\forall : \Gamma_G \to U_P$ with

$$\begin{array}{ccc} & \longrightarrow & E_P \\ {\scriptstyle \Pi_{a_G} p_G}\downarrow & \lrcorner & \downarrow{\scriptstyle p_P} \\ \Gamma_G & \xrightarrow[\forall]{} & U_P \end{array}$$

which allows us to interpret impredicative quantification.

Unfortunately, the presheaf U_P is not representable as opposed to $\mathsf{Y}(\mathit{Prop})$. But one can define a morphism $|S| : \mathsf{Y}(\mathit{Prop}) \to U_P$ in $\widehat{\mathbb{C}}$ sending $a : I \to \mathit{Prop}$ to the presheaf $|S|_I(a)$ over $\mathbb{C}/I$ with $|S|_I(a)(u) = \{f : I \to \mathit{Prf} \mid p_{\mathit{Prop}} \circ f = a \circ u\}$ and $|S|_I(a)(v : uv \to u)(f) = f \circ v$. Obviously, every $A \in U_P(I)$ is isomorphic to $|S|_I(a)$ for some $a : I \to \mathit{Prop}$ in the category $\mathcal{U}_0^{(\mathbb{C}/I)^{\mathsf{op}}}$. Let us write S_P for the split fibration sending $I \in \mathbb{C}$ to the full subcategory of $\mathcal{U}_0^{(\mathbb{C}/I)^{\mathsf{op}}}$ on representable presheaves. Obviously, we have $U_P = |S_P|$. Let sS_P be the split fibration where $sS_P(I)$ is the category whose objects are morphisms from I to Prop and where $sS_P(I)(a,b) = \widehat{\mathbb{C}}(|S|_I(a), |S|_I(b))$. Obviously, we have $\mathsf{Y}(\mathit{Prop}) = |sS_P|$. We write S for the split cartesian functor from sS_P to S_P whose object part is given by $|S|$ and which is the identity on morphisms. Then in the category $\mathbf{Sp}(\mathbb{C})$ of split fibrations over $\mathbb{C}$ we have

$$\begin{array}{ccc}
|sS_P| & \xrightarrow{\;|S|\;} & |S_P| \\
\Big\downarrow & & \Big\downarrow \\
sS_P & \xrightarrow[\;S\;]{\;\simeq\;} & S_P
\end{array}$$

where the vertical arrows are the canonical[1] maps $|sS_P| \to sS_P$ and $|S_P| \to S_P$, respectively, and S is a weak equivalence, i.e. all S_I are weak equivalences in the ordinary sense.

The reason why we can't work with $\mathsf{Y}(\mathit{Prop})$ instead of U_P is that the latter is closed under the respective type forming operations up to equality and not just up to isomorphism as is the former.

If we start from a situation where $\mathcal{E}$ is some finite limit category, $\mathcal{S}$ is a pullback stable class of maps closed under composition and Π, contains all regular monos and a generic family $p_U : E_U \to U$ then we can use Voevodsky's "method of universes" for obtaining a split model (using global choice). If, moreover, we have a pullback stabe subclass $\mathcal{S}_P$ of $\mathcal{S}$ such that

(P1) $\Pi_f a \in \mathcal{S}_P$ whenever $a \in \mathcal{S}_P$ and $f \in \mathcal{S}$

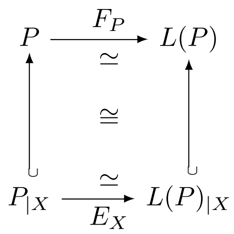
(P2) there is a subobject $m_P : U_P \rightarrowtail U$ with $m_P^* p_U$ generic for $\mathcal{S}_P$ and the terminal projection $U_P \to 1_{\mathcal{E}}$ in $\mathcal{S}$

then the above splitting of $\mathcal{S}$ restricts to one of $\mathcal{S}_P$. The attempt described in this note was motivated by establishing such a situation for $\mathcal{E} = \widehat{\mathbb{C}}$. It "only" failed in the respect that we couldn't achieve that $U_P \to 1_{\mathcal{E}}$ is in $\mathcal{S}$.

Quite generally, there arises the question whether for a finite limit category $\mathbb{C}$ together with a map $p_U : E \to U$ in $\mathbb{C}$ there does exist a splitting $S_{\mathbb{C}}$ of the fundamental fibration $P_{\mathbb{C}} = \partial_1 : \mathbb{C}^2 \to \mathbb{C}$ such that the cartesian equivalence $F : P_{\mathbb{C}} \xrightarrow{\simeq} S_{\mathbb{C}}$ restricts to an equivalence between the full subfibrations generated by p_U and $F_U(p_U)$, respectively.

[1] including its presheaf of objects into a split fibration

Well, we may achieve something quite close to this even for general Grothendieck fibrations $P : \mathbb{X} \to \mathbb{B}$ using the left adjoint splitting $L(P)$ of P. We consider the variant of $L(P)$ used by Lumsdaine and Warren making use of a normalized cleavage Cart_P of P. For $X \in \mathbb{X}$ let $P_{|X}$ be the full subfibration of P on those objects from which there exists a cartesian arrow to X and $L(P)_{|X}$ the full split subfibration of $L(P)$ on objects of the form (u, X) where the codomain of u is $P(X)$. Notice that for (u, X) there is a unique v with $(u, X) = v^*(\mathrm{id}_{PX}, X)$, namely u. One may find a non-split equivalence $E_X : P_{|X} \to L(P)_{|X}$ such that the diagram

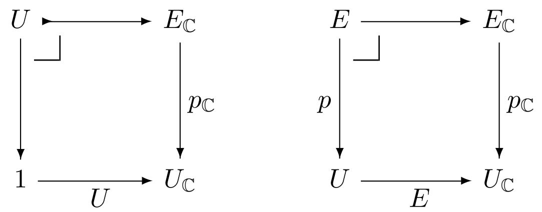


commutes up to isomorphism in $\mathbf{Fib}(\mathbb{B})$.

A Natural Notion of (Impredicative) Universe

In Awodey's article on 'natural semantics' of type theory he did not consider universes. We now suggest a "natural notion of universe" within a locally cartesian closed category $\mathbb{C}$. First, for sake of exposition, we rebaptize $p_U : E_U \to U$ as $p_{\mathbb{C}} : E_{\mathbb{C}} \to U_{\mathbb{C}}$ because it is a representable family within $\widehat{\mathbb{C}}$ from which all representable morphisms can be obtained as pullback.

Now a universe in $\mathbb{C}$ is given by a global element $U : 1 \to U_{\mathbb{C}}$ together with a map (typically an inclusion!) $E : U \to U_{\mathbb{C}}$ as depicted in



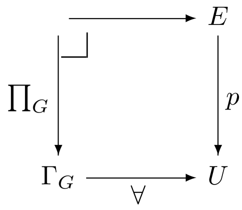
where U and E are identified with their weak classifiers. Now consider the generic context

$$\Gamma_G \equiv A : U_{\mathbb{C}}, P : E_{\mathbb{C}}(A) \to U$$

and the families

$$a_G \equiv \Gamma_G \vdash E_{\mathbb{C}}(A) \quad \text{and} \quad p_G \equiv \Gamma_G, a : E_{\mathbb{C}}(A) \vdash E(P(a))$$

where both a_G and p_G are representable. Of course, the dependent product $\prod_G \equiv \prod_{a_G} p_G$ is a representable morphism with codomain Γ_G. We say that the universe (U,E) is **impredicative** iff $\prod_G$ can be obtained from p via pullback along some $\forall : \Gamma_G \to U$



This may be seen as a rational reconstruction of the notion of **split dictos** as considered in M. Hofmann's 1994 CSL paper.

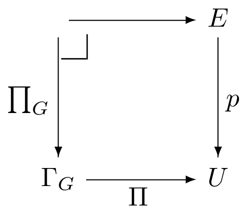
Of course, we can define a **predicative notion of universe closed under dependent products** by considering the generic context

$$\Gamma_G \equiv A : U, B : E(A) \to U$$

and the families

$$a_G \equiv \Gamma_G \vdash E(A) \quad \text{and} \quad b_G \equiv \Gamma_G, a : E(A) \vdash E(B(a))$$

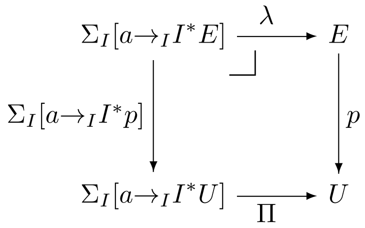
and requiring that $\prod_G \equiv \prod_{a_G} b_G$ fits into a pullback square



for some map $\Pi : \Gamma_G \to U$.

Voevodsky's Version of Universes in LCCC's

Let $\mathcal{E}$ be a locally cartesian closed category or even topos which typically will later be instantiated by $\widehat{\mathbb{C}}$. A map $p : E \to U$ in $\mathcal{E}$ may be understood as a universe in $\mathcal{E}$. For $a : A \to I$ in $\mathcal{E}$ we say that *p is closed under a-products* iff $\Sigma_I[a{\to}_I I^*p]$ appears as pullback of p, i.e.



for some maps Π and λ.

Of course, when instantiating a by p this amounts to the requirement that p is closed under dependent products. But it is a stronger condition than requiring that the class of pullbacks of p be closed under dependent products.

If $\mathbb{C}$ is a locally cartesian closed category a map $p : E \to U$ in $\mathbb{C}$ is an *impredicative universe* iff $\mathsf{Y}(p)$ is closed under $p_{\mathbb{C}}$-products in $\widehat{\mathbb{C}}$. One may consider this as a rational reconstruction of Hofmann's notion of "split dictos".[2]

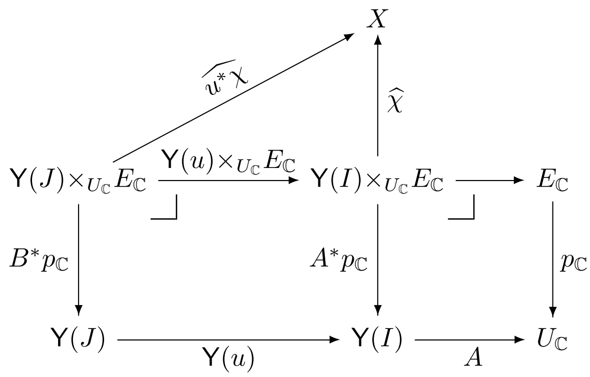
For $X \in \widehat{\mathbb{C}}$ we explicitate $\Sigma_{U_{\mathbb{C}}}[p_{\mathbb{C}} \to_{U_{\mathbb{C}}} U_{\mathbb{C}}^* X]$ as follows. A generalized element at stage $I \in \mathbb{C}$ is given by $A \in U_{\mathbb{C}}(I)$ together with a natural transformation $\chi : A \times_{U_{\mathbb{C}}} p_{\mathbb{C}} \to U_{\mathbb{C}}^* X$ in $\widehat{\mathsf{Elts}(U_{\mathbb{C}})}$. Let us write P_A for $A \times_{U_{\mathbb{C}}} p_{\mathbb{C}}$ in $\widehat{\mathsf{Elts}(U_{\mathbb{C}})}$. For $J \in \mathbb{C}$ and $B \in U_{\mathbb{C}}(J)$ we have

$$P_A(\langle J, B\rangle) = \{\langle u, b\rangle \mid u : J \to I, B = u^* A \text{ and } b \in B(\mathrm{id}_J)\}$$

and

$$v^*\langle u, b\rangle = \langle uv, (v : uv \to u)(b)\rangle$$

for $v : K \to J$.[3] Now we explicitate reindexing in $\Sigma_{U_{\mathbb{C}}}[p_{\mathbb{C}} \to_{U_{\mathbb{C}}} U_{\mathbb{C}}^* X]$ along $u : J \to I$. Given $\langle A, \chi\rangle$ in $\Sigma_{U_{\mathbb{C}}}[p_{\mathbb{C}} \to_{U_{\mathbb{C}}} U_{\mathbb{C}}^* X]$, i.e. $A \in U_{\mathbb{C}}(I)$ and $\chi : A \times_{U\mathbb{C}} p_{\mathbb{C}} \to U_{\mathbb{C}}^* X$ which we identify with its transpose $\widehat{\chi} : \mathsf{Y}(I) \times_{U_{\mathbb{C}}} E_{\mathbb{C}} \to X$, its reindexing $u^*\langle A, \chi\rangle = \langle B, u^*\chi\rangle$ where $B = u^* A$ and $u^*\chi$ is the transpose of $\widehat{u^*\chi}$ as depicted in



where $\mathsf{Y}(u) \times_{U_{\mathbb{C}}} E_{\mathcal{C}}$ sends $\langle v, b\rangle$ to $\langle uv, b\rangle$. Thus, we have $u^*\langle A, \chi\rangle = \langle u^* A, u^*\chi\rangle$ with $u^*\chi(\langle v, b\rangle) = \chi(\langle uv, b\rangle)$.

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[2]The map

$$\Sigma_{U_{\mathbb{C}}}[p_{\mathbb{C}} \to_{U_{\mathbb{C}}} \mathsf{Y}(p)]$$

is the display map for the family of types

$$A : U_{\mathbb{C}}, p : E_{\mathbb{C}}(A) \to \mathsf{Y}(U) \vdash (\Pi a : E_{\mathbb{C}}(a)) E(p(a))$$

considered in the previous paragraph.

[3]Alternatively, we may describe P_A as the corresponding morphism $p_A : \int P_A \to U_{\mathbb{C}}$ in $\widehat{\mathbb{C}}$. The fibre of $\int P_A$ over J consists of all pairs $\langle u, a\rangle$ where $u : J \to I$ and $a \in A(u)$. For $v : K \to J$ reindexing along v sends $\langle u, a\rangle$ to $\langle uv, A(v : uv \to u)(a)\rangle$. The map p_A sends $\langle u, a\rangle$ to $u^* A$.