LQD is Definable

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Abstract

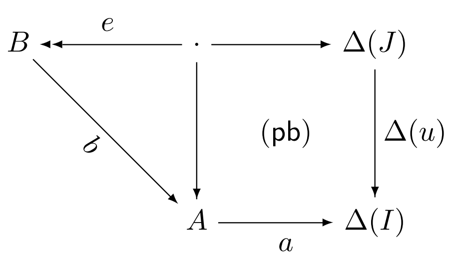
The aim of these notes is to show that for every bounded geometric morphism $\Delta \dashv \Gamma : \mathcal{E} \to \mathcal{S}$ between toposes the collection of maps in $\mathcal{E}$ which are lqd, i.e. locally are quotients of discrete maps, do form a definable subfibration of the fundamental fibration $P_{\mathcal{E}} = \partial_1 : \mathcal{E}^2 \to \mathcal{E}$ of $\mathcal{E}$.

This result will have in particular the consequence that for every object X in $\mathcal{E}$ there exists a greatest subobject $m_X : \mathcal{P}_{\mathcal{S}}(X) \rightarrowtail \mathcal{P}(X)$ such that $\pi \circ (m_X{\times}X)^*\ni_X$ is lqd.

1 Basic Facts and Definitions

Let $\Delta \dashv \Gamma : \mathcal{E} \to \mathcal{S}$ be a geometric morphism between toposes.

Definition 1.1 *For $a : A \to \Delta(I)$ and $b : B \to A$ we say that $b \in \mathsf{QD}(a)$ iff*



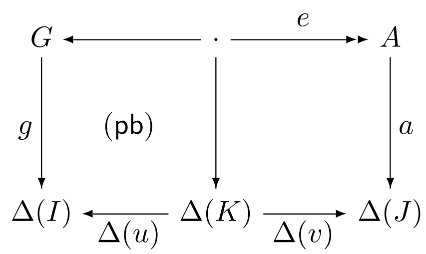
for some epi e in $\mathcal{E}$ and $u : J \to I$ in $\mathcal{S}$. A map $b : B \to A$ is in QD iff $b \in \mathsf{QD}(A)$ for some $a : A \to \Delta(I)$.

A map $b : B \to A$ is in LQD iff there exists an epi $e : C \twoheadrightarrow A$ with $e^*b \in \mathsf{QD}$. $\Diamond$

The idea behind this terminology is that D stands for "discrete", Q stands for "quotient" and L stands for "locally". We say that a map in

$\mathcal{E}$ is"a family of discrete objects" iff it appears as pullback of some $\Delta(u)$, i.e. the notion of discreteness depends on Δ which is thought of as sending $I \in \mathcal{S}$ to the I–fold coproduct of the terminal object in $\mathcal{E}$ (see *e.g.* [St]). Accordingly, a map in $\mathcal{E}$ is a "family of quotients of discrete objects" iff it is a quotient of a family of discrete objects in its fibre and a map is "locally a family of quotients of discrete objects" iff some reindexing of it along an epi in $\mathcal{E}$ is a family of quotients of discrete objects.

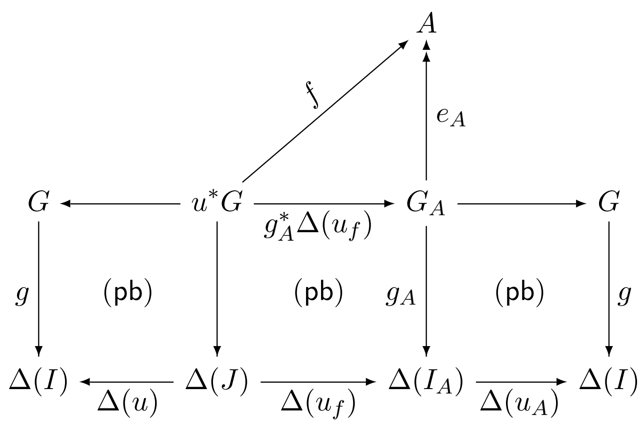
From seminal work of Bénabou and Moens (as exposed in [St]) we know that a geometric morphism $\Delta \dashv \Gamma : \mathcal{E} \to \mathcal{S}$ is *bounded* iff it admits a *generating family*, i.e. a map $g : G \to \Delta(I)$ such that for every $a : A \to \Delta(J)$ there exist maps $u : K \to I$ and $v : K \to J$ in $\mathcal{S}$ such that



for some epi e.

As $\mathsf{gl}(\Delta) = \partial_1 : \mathcal{E}\downarrow\Delta \to \mathcal{S}$ is *locally small* we have the following lemma that turns out as useful for the proof of our main Theorem 2.4.

Lemma 1.1 *For every object A in $\mathcal{E}$ there is a map $u_A : I_A \to I$ and an epi $e_A : G_A \twoheadrightarrow A$ such that for every $u : J \to I$ and $f : u^*G \to A$ there is a* unique *$u_f : u \to u_A$ with $f = e_A \circ g_A^*\Delta(u_f)$ as shown in the following diagram*



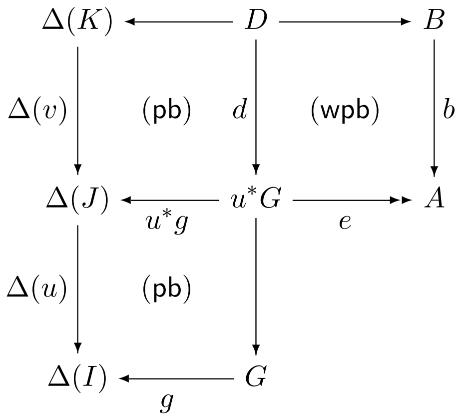
Proof. See [St] the key idea being that $I_A = \coprod_{i \in I} \mathsf{hom}(G_i, A)$. $\square$

We notice for further use that for every definable full subfibration P of $P_{\mathcal{E}}$ the full subfibration $\Delta^* P$ of $\mathsf{gl}(\Delta) = \Delta^* P_{\mathcal{E}}$ is definable, too.

2 Definability of LQD as a full subfibration of $P_{\mathcal{E}}$

In order to prove our main Theorem 2.4 we first have to establish some lemmas.

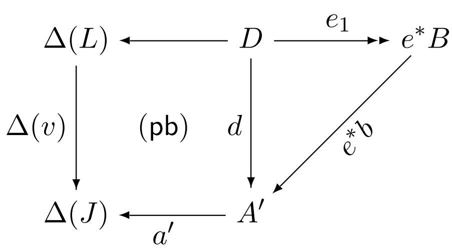
Lemma 2.1 *A map $b : B \to A$ of $\mathcal{E}$ is in LQD if and only if there exist $u : J \to I$ in $\mathcal{S}$ and an epi $e : u^*G \twoheadrightarrow A$ with $e^*b \in \mathsf{QD}(u^*g)$ as exhibited in the diagram*



where the label (wpb) indicates that the square is a weak pullback, i.e. that the mediating arrow from d to e^*b is epic.

Proof. The implication from right to left is evident.

For the reverse direction assume that $b \in \mathsf{LQD}$. That means that there is an epi $e : A' \twoheadrightarrow A$ with $e^*b \in \mathsf{QD}(a')$ for some $a' : A' \to \Delta(J)$ as exhibited in the following diagram.



As g is a generating family we have

$$\begin{array}{ccccc} G & \longleftarrow & u^*G & \overset{e_2}{\twoheadrightarrow} & A' \\ {\scriptstyle g}\downarrow & (\mathsf{pb}) & \downarrow{\scriptstyle u^*g} & & \downarrow{\scriptstyle a'} \\ \Delta(I) & \underset{\Delta(u)}{\longleftarrow} & \Delta(K) & \underset{\Delta(w)}{\longrightarrow} & \Delta(J) \end{array}$$

for some maps $u : K \to J$ and $w : K \to J$ in $\mathcal{S}$ and some epi $e_2 : u^*G \twoheadrightarrow A'$.

Therefore, $e_2^* e^* b$ appears as quotient of $e_2^* d$ via $e_2^* e_1$. But as

$$e_2^* d = e_2^* a'^* \Delta(v) \cong (u^*g)^* \Delta(w)^* \Delta(v) \cong (u^*g)^* \Delta(w^*v)$$

it follows that $e_2^* e^* b \in \mathsf{QD}(u^*g)$ as desired. $\square$

Lemma 2.2 *Let $a : A \to \Delta(J)$. Then $b \in \mathsf{QD}(a)$ iff*

$$\mathsf{ev} \circ a^* \varepsilon^{(J)}_{\Pi_a(b)} : a^* \Delta_J \Gamma_J \Pi_a(b) \to b$$

is epic where $\varepsilon^{(J)}$ is the counit of the adjunction $\Delta_J \dashv \Gamma_J$.

Proof. First recall that Δ_J is given by $\Delta_{/J} : \mathcal{S}/J \to \mathcal{E}/\Delta(J)$. Therefore, the map b is in $\mathsf{QD}(a)$ iff there is an epic map $e : a^*\Delta_J(v) \to b$ for some $v : K \to J$ in $\mathcal{S}$ (as a map e in the slice over A is epic iff $\partial_0(e)$ is epic, *i.e.* iff it is epic as a map in $\mathcal{E}$). Thus, the map b is in $\mathsf{QD}(a)$ iff there is a map $e' : \Delta_J(v) \to \Pi_a(b)$ such that $\mathsf{ev} \circ \Delta_J(e')$ is epic. Using the adjunction $\Delta_J \dashv \Gamma_J$ this is further equivalent to the existence of a map $e'' : v \to \Gamma_J \Pi_a(b)$ with $\mathsf{ev} \circ a^*(\varepsilon^{(J)}_{\Pi_a(b)} \circ \Delta_J(e'')) : a^*\Delta_J(v) \to b$ epic where $\varepsilon^{(J)}$ is the counit of the adjunction $\Delta_J \dashv \Gamma_J$. But as the morphism $\mathsf{ev} \circ a^* \varepsilon^{(J)}_{\Pi_a(b)}$ is epic iff $\mathsf{ev} \circ a^* \varepsilon^{(J)}_{\Pi_a(b)} \circ a^* \Delta_J(e'')$ is epic for some $e'' : v \to \Gamma_J \Pi_a(b)$ we finally get that $b \in \mathsf{QD}(a)$ iff $\mathsf{ev} \circ a^* \varepsilon^{(J)}_{\Pi_a(b)} : a^* \Delta_J \Gamma_J \Pi_a(b) \to b$ is epic. $\square$

Lemma 2.3 *Let $a : A \to \Delta(J)$ and $b : B \to A$. Then there exists a (necessarily unique) subobject $m : K \rightarrowtail J$ such that*

(i) $\Delta(m)^* b \in \mathsf{QD}(\Delta(m)^* a)$ *and*

(ii) *f factors through m whenever $\Delta(f)^* b \in \mathsf{QD}(\Delta(f)^* a)$.*

Proof. As the fibration $\mathsf{gl}(\Delta)$ has internal products and $\Delta \dashv \Gamma$ extends to a fibred adjunction between $P_{\mathcal{S}}$ and $\mathsf{gl}(\Delta)$ it follows by Lemma 2.2 that the condition $\Delta(f)^*b \in \mathsf{QD}(\Delta(f)^*a)$ is equivalent to $\Delta(f)^*(\mathsf{ev} \circ a^*\varepsilon^{(J)}_{\Pi_a(b)})$ being epic. But as being epic is a definable property w.r.t. the fundamental fibration $P_{\mathcal{E}}$ it is also a definable property w.r.t. the fibration $\Delta^*P_{\mathcal{E}} = \mathsf{gl}(\Delta)$ (see the remark immediately following Lemma 1.1). Thus, the subobject m of J satisfying the required conditions (i) and (ii) does exist. $\square$

Now we are ready to prove our Main Theorem.

Theorem 2.4
The full subfibration LQD *of the fundamental fibration* $P_{\mathcal{E}} : \mathcal{E}^2 \to \mathcal{E}$ *is definable, i.e. for every map* $b : B \to A$ *in* $\mathcal{E}$ *there is a (uniquely determined) subobject* $m : P \rightarrowtail A$ *such that*

(i) $m^*b \in \mathsf{LQD}$ *and*

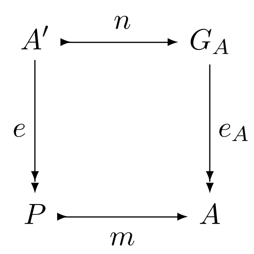
(ii) f *factors through* m *whenever* $f^*b \in \mathsf{LQD}$.

Proof. Let $g_A : G_A \to \Delta(I_A)$ and $e_A : G_A \twoheadrightarrow A$ as in Lemma 1.1. Instantiating Lemma 2.3 by g_A for a and e_A^*b for b there exists a subobject $\widetilde{m} : J \rightarrowtail I_A$ such that

(a) $\Delta(\widetilde{m})^*e_A^*b \in \mathsf{QD}(\Delta(\widetilde{m})^*g_A)$ and

(b) f factors through $\widetilde{m}$ whenever $\Delta(f)^*e_A^*b \in \mathsf{QD}(\Delta(f)^*g_A)$.

Let $n : A' \rightarrowtail G_A$ be the subobject of G_A obtained by pulling back $\Delta(\widetilde{m})$ along g_A and consider the epi–mono–factorisation of $e_A \circ n$

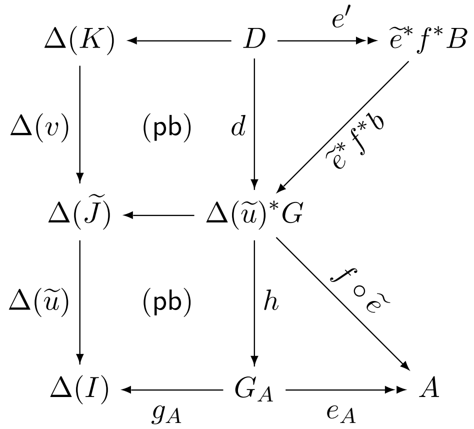


with e epic and m monic.

We now show that m satisfies the requirements (i) and (ii).

The mono m satisfies requirement (i) as e is an epi and $e^*m^*b \cong n^*e_A^*b \cong \Delta(\widetilde{m})^*e_A^*b \in \mathsf{QD}(\Delta(\widetilde{m})^*g_A)$ by (a).

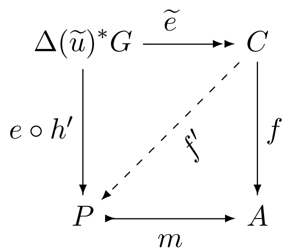
For showing requirement (ii) assume that $f^*b \in \mathsf{LQD}$ for some $f : C \to A$. Then by Lemma 1.1 and Lemma 2.1 there exist a map $\widetilde{u} : \widetilde{J} \to I_A$ in $\mathcal{S}$ and an epi $\widetilde{e} : \Delta(\widetilde{u})^*G \twoheadrightarrow C$ and such that



for some appropriate morphism $v : K \to \widetilde{J}$ in $\mathcal{S}$ and epi e' in $\mathcal{E}$. Thus, we have that $\Delta(\widetilde{u})^* e_A^* b \cong \widetilde{e}^* f^* b \in \mathsf{QD}(\Delta(\widetilde{u})^* g_A)$ from which it follows by condition (b) that $\widetilde{u}$ factors through $\widetilde{m}$ via some $w : \widetilde{J} \to J$. Accordingly, the map $h = g_A^* \Delta(\widetilde{u})$ factors through $n = g_A^* \Delta(\widetilde{m})$ via $h' := g_A^* \Delta(w)$. Now it holds that

$$f \circ \widetilde{e} = e_A \circ h = e_A \circ n \circ h' = m \circ e \circ h'$$

and, therefore, we have



for a necesarily unique f'. Thus, the map f factors through m as required by (ii). $\square$

References

[St] T. Streicher
Fibred Categories à la Bénabou Notes (2000) electronically available as `www.mathematik.tu−darmstadt.de/~streicher/FIBR/fib.ps.gz`

Appendix Some Explicitations

We have shown that for every $b : B \to A$ there exists a greatest subobject $m : P \rightarrowtail A$ with $m^*b \in \mathsf{LQD}$ such that every $f : C \to A$ with $f^*b \in \mathsf{LQD}$ factors through m. Part of the construction of m from b can be performed inside $\mathcal{E}$ but now and then we have "to switch universe" in an essential way using the adjunction $\Delta \vdash \Gamma : \mathcal{E} \to \mathcal{S}$.

First we explicitate the construction of g_A and e_A from Lemma 1.1. This means to explicitate how $\mathsf{gl}(\Delta)$ is locally small. Let $g : G \to \Delta(I)$ be some generating family for $\mathsf{gl}(\Delta)$. Let $e : E \to \Delta(I)$ be the fibrewise exponential $[g{\to}_{\Delta(I)}\Delta(I)^*A]$ where $\Delta(I)^*A = \pi_2 : \Delta(I){\times}A \to A$. Now we may consider the pullback

$$\begin{array}{ccc} I_A & \xrightarrow{\;q\;} & \Gamma E \\ {\scriptstyle u_A}\downarrow & (\mathsf{pb}) & \downarrow{\scriptstyle \Gamma e} \\ I & \xrightarrow[\eta_I]{} & \Gamma\Delta I \end{array}$$

where $u_A = \Gamma_I(e)$. The counit $\varepsilon_e^{(I)}$ of $\Delta_I \vdash \Gamma_I$ at e is given by $\varepsilon_E \circ \Delta(q)$ where ε_E is the counit of $\Delta \vdash \Gamma$ at E. Now the map $g_A : G_A \to \Delta(I_A)$ appears as pullback of g along $\Delta(u_A)$ and finally $e_A : G_A \twoheadrightarrow A$ is given by $\pi_2 \circ \mathsf{ev} \circ (\varepsilon_e^{(I)} {\times}_{\Delta(I)} g)$ where $\pi_2 : \Delta(I){\times}A \to A$.

Now we describe more explicitly the subobject $\widetilde{m} : J \rightarrowtail I_A$ as constructed in the proof of Theorem 2.4. Let $\widetilde{b} := {e_A}^*b$ and $p := \Pi_{g_A}\widetilde{b} : P \to \Delta(I_A)$. The counit $\varepsilon_p^{(I_A)}$ of $\Delta_{I_A} \vdash \Gamma_{I_A}$ at p is constructed as $\varepsilon_P \circ \Delta(q')$ where q' is given by the following diagram.

$$\begin{array}{ccc} I_A & \xrightarrow{\;q'\;} & \Gamma P \\ {\scriptstyle \Gamma_{I_A}p}\downarrow & (\mathsf{pb}) & \downarrow{\scriptstyle \Gamma p} \\ I_A & \xrightarrow[\eta_{I_A}]{} & \Gamma\Delta I_A \end{array}$$

Let $f := \mathsf{ev} \circ {g_A}^*(\varepsilon_P \circ \Delta(q'))$. Now let $m' : P' \rightarrowtail \Delta(I_A)$ be the subobject of $\Delta(I_A)$ "consisting of all $i \in \Delta(I_A)$ such that f_i is epic". Then $\widetilde{m} : J \rightarrowtail I_A$ appears as the pullback of $\Gamma(m')$ along η_{I_A}.

Finally, we obtain the desired subobject m of A as the image of the map $e_A \circ {g_A}^*\Delta(\widetilde{m})$.