Simplicial Sets within Cubical Sets

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Computational Meaning of HoTT? (1)

HoTT is **Intensional Type Theory** together with Voevodsky's **Univalence Axiom** (UA) and **Higher Inductive Types** (HITs).

Let us recall what *UA* says. This requires a few definitions

$$\mathrm{iscontr}(X : \mathrm{Set}) = (\Sigma x : X)(\Pi y : X)\ \mathrm{Id}_X(x, y)$$

$$\mathrm{hfiber}(X, Y : \mathrm{Set})(f : X \to Y)(y : Y) = \\ = (\Sigma x : X)\ \mathrm{Id}_Y(f(x), y)$$

$$\mathrm{isweq}(X, Y : \mathrm{Set})(f : X \to Y) = \\ = (\Pi y : Y)\ \mathrm{iscontr}(\mathrm{hfiber}(X, Y, f, y))$$

$$\mathrm{Weq}(X, Y : \mathrm{Set}) = (\Sigma f : X \to Y)\ \mathrm{isweq}(X, Y, f)$$

Computational Meaning of HoTT? (2)

Using the eliminator J for identity types one easily constructs a map

$$\mathrm{eqweq}(X, Y : \mathrm{Set}) : \mathrm{Id}_{\mathrm{Set}}(X, Y) \to \mathrm{Weq}(X, Y)$$

Then the **Univalence Axiom**

$$\mathrm{UA} : (\Pi X, Y : \mathrm{Set})\ \mathrm{isweq}(\mathrm{eqweq}(X, Y))$$

postulates that all maps $\mathrm{eqweq}(X, Y)$ are weak equivalences.

It has been shown that **simplicial sets** provide a model of HoTT interpreting types as Kan complexes.
But UA as it is **lacks computational meaning**:
what should be rewrite rules for the constant UA?

T. Coquand et.al. [CCHM] have developed a *Cubical Type Theory* with computational meaning and in which one can **derive** UA.

From Simplicial Sets to Cubical Sets (1)

Bezem and Coquand have shown that the theory of simplicial sets is **not constructive**, e.g. one cannot show constructively that Kan complexes are closed under exponentiation!

This limitation, however, can be overcome when working in **cubical sets**.

The crucial property is that representable objects are closed under finite products. In **sSet** the interval $\mathbb{I}$ is representable but $\mathbb{I} \times \mathbb{I}$ is not!

The site of **sSet** is Δ, the full subcategory of **Poset** on finite non-empty linear posets.
The site of **cSet** is $\square$, the full subcategory of **Poset** of finite powers of $\mathbf{2}$.
Splitting idempotents in $\square$ gives rise to **FL**, the full subcategory of **Poset** on finite lattices.

From Simplicial Sets to Cubical Sets (2)

Notice that $\square$ is op-equivalent to the category of *free finitely generated distributive lattices*. Coquand et.al. – just for convenience – used the opposite of *free finitely generated de Morgan algebras*.

Let $i : \Delta \to \mathbf{FL}$ be the inclusion functor. The restriction functor i^* from $\mathbf{cSet} = \widehat{\mathbf{FL}}$ to $\mathbf{sSet} = \widehat{\Delta}$ has left and right adjoints $i_!$ and i_*, respectively. Since the restriction of the nerve functor Nv to **FL** is given by $i^* \circ \mathrm{Y}_{\mathbf{FL}}$ we have

$$i_*(X)(L) \cong \mathbf{cSet}(\mathrm{Y}(L), i_*(X)) \cong \mathbf{sSet}(i^*\mathrm{Y}(L)), X) \cong \mathbf{sSet}(\mathrm{Nv}(L), X)$$

from which it follows that i_* is full and faithful (and thus also $i_!$). Since i^* has a left adjoint $i_!$ (given by left Kan extension of $\mathrm{Y}_{\mathbf{FL}} \circ i$ along Y_Δ) it preserves (finite) limits and thus $i^* \dashv i_*$ is an **injective geometric morphism**.

The topology on **FL** inducing **sSet**

A sieve $S \subseteq \mathsf{Y}_{\mathbf{FL}}(L)$ **covers** L iff $i^*S = i^*\mathsf{Y}_{\mathbf{FL}}(L) = \mathrm{Nv}(L)$ iff S contains all chains in L, i.e. all monotone maps $[n] \to L$.

The corresponding **closure operator** $j : \Omega \to \Omega$ sends $S \subseteq \mathsf{Y}_{\mathbf{FL}}(L)$ to all $u : K \to L$ such that $uc \in S$ for all chains $c : [n] \to K$.

Cisinski Model Structures on **cSet** and **sSet** (1)

For Cisinski model structures on $\widehat{\mathbb{C}}$ its class of **cofibrations** consist of all monos.
Its class of trivial fibrations consists of maps weakly right orthogonal to all monos.
A **naive fibration** is a map weakly right orthogonal to **all cylinders**, i.e. monos of the form

$$(\{\varepsilon\} \times X) \cup (\mathbb{I} \times Y) \hookrightarrow \mathbb{I} \times X$$

where $Y \subseteq X$, $\varepsilon \in \{0,1\} \subseteq \mathbb{I} = \mathrm{Y}(2)$.
The **trivial cofibrations** are the maps weakly left orthogonal to all naive fibrations.

On **sSet** this construction gives the classical model structure and on **cSet** the one employed by Coquand et.al. in [CCHM].

Cisinski Model Structures on **cSet** and **sSet** (2)

Since i^* preserves $\mathbb{I}$ and finite limits and it retracts monos of **cSet** onto monos of **sSet** it follows that i^* retracts cylinders in **sSet** onto cylinders in **sSet**.
Thus $i_* p$ is a fibration in **cSet** iff p is a fibration in **sSet**.
Accordingly, we have $\mathcal{F}_{\mathbf{sSet}} = \mathbf{sSet} \cap \mathcal{F}_{\mathbf{cSet}}$.

Thus $i^* \dashv i_*$ is a *Quillen pair*.

We don't know whether it is a Quillen equivalence.
But presumably not!

i_* preserves and reflects "equality" of fibrant objects

Theorem If A and B are fibrant in **sSet**, i.e. Kan complexes, a map $f : A \to B$ is a weak equivalence in **sSet** iff it is a weak equivalence in **cSet**.

Proof.
One can show that $i_*(\mathrm{hfiber}(f)) \simeq \mathrm{hfiber}(i_* f)$ and thus
$\forall m \in \mathrm{Mono}(\mathbf{cSet})(m \perp \mathrm{hfiber}(i_* f))$ iff
$\forall m \in \mathrm{Mono}(\mathbf{cSet})(m \perp i_*(\mathrm{hfiber}(f)))$ iff
$\forall m \in \mathrm{Mono}(\mathbf{cSet})(i^* m \perp \mathrm{hfiber}(f))$ iff[1]
$\forall m \in \mathrm{Mono}(\mathbf{sSet})(m \perp \mathrm{hfiber}(f))$
i.e.[2] $i_* f$ is a weak equivalence in **cSet** iff f is a weak equivalence in **sSet**. □

[1] since the monos in **sSet** are precisely the sheafifications of monos in **cSet**

[2] as shown by Voevodsky for fibrant objects A and B a map $w : A \to B$ is a weak equivalence iff $\mathrm{hfiber}(w)$ is a trivial cofibrations, i.e. $m \perp \mathrm{hfiber}(w)$ for all monos m

Universes in **cSet** and **sSet**

In both **cSet** and **sSet** we may construct universes

$$\pi_c : E_c \to U_c \quad \text{and} \quad \pi_s : E_s \to U_s$$

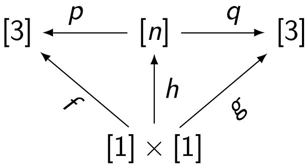
à la Yoneda where $U(I)$ consists of all small fibrations over $\mathrm{Y}(I)$.
In both cases one could show the universe to be fibrant and univalent!
However, in case of **sSet** this requires heavy choice (due to use of minimal fibrations!) but not so for **cSet**.

If i^* preserved fibrations, i.e. $i_!$ preserved cylinders, then $i^*\pi_c$ were a universe generic for small Kan fibrations.
Composing this with the map sending a small fibration over $\mathrm{Y}_\square([n])$ to its sheafification which is a small fibration over $\mathrm{Y}_\Delta([n])$ we would obtain a univalent universe equivalent to π_s.

But $i_!$: **sSet** $\hookrightarrow$ **cSet** does not preserve finite limits!

Since both $i_!$ and i^* are cocontinuous and $i_!$ is full and faithful the full embedding $i_!$ preserves and reflects colimits. Thus **sSet** is a full subcategory of **cSet** closed under colimits (taken in **cSet**).

By a well known theorem $i_!$ preserves finite limits iff $\mathbf{FL}(L, i(-))$ is **flat** for all $L \in \mathbf{FL}$ which, however, is not the case:
let $L = [1] \times [1]$ then there are (precisely two) 1-1 maps $f, g : [1] \times [1] \to [3]$ not fitting into a diagram of the form



Staying within **cSet** avoids the problem!

Cut U_c down to the subuniverse U_{cs} consisting at stage $L \in \mathbf{FL}$ of all $A \in U(L)$ such that $a : \mathsf{El}(A) \to \mathsf{Y}(L)$ is a family of sheaves, i.e.

$$\begin{array}{ccc} \mathsf{El}(A) & \xrightarrow{\eta_{\mathsf{El}(A)}} & i_* i^* \mathsf{El}(A) \\ {\scriptstyle a}\downarrow & \lrcorner & \downarrow{\scriptstyle i_* i^* a} \\ \mathsf{Y}(L) & \xrightarrow[\eta_{\mathsf{Y}(L)}]{} & i_* i^* \mathsf{Y}(L) \end{array}$$

or equivalently a is strictly right orthogonal to all subobjects of representables which are dense, i.e. inverted by i^*.

This universe U_{cs} classifies small families of sheaves which are fibrations in **cSet**. If the indexing object is a sheaf such families coincide with images of fibrations under i_*.

Conclusion and Outlook

- We have shown that simplicial sets form a(n essential) subtopos of cubical sets.
- Moreover, **sSet** is a submodel of **cSet** since the inclusion $i_* : \mathbf{sSet} \hookrightarrow \mathbf{cSet}$ preserves Σ and Π and also the interval $\mathbb{I}$ and thus also identity types.
- Cutting down the universe U_c to families of sheaves weakly classifies (small) fibrations of sheaves.