Precohesive Toposes over Arbitrary Base Toposes

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Caveat

We give a **fibered** (or indexed) account of Lawvere's notion of **precohesive topos** which, following the terminology of [LM15], might be called **stably precohesive**.

Since we find the latter the conceptually correct one we would refer to it as **precohesive** and to Lawvere's original notion as **weakly precohesive** or something alike.

At the end we will discuss the problem of separating the weak and the strong notion of precohesiveness by a counterexample.

Precohesive Toposes over Set

A **Grothendieck topos** over $\mathbf{Set}$ is a topos $\mathcal{E}$ which has **small sums**, is **locally small** and admits a **bound**, i.e. an object S whose subobjects form a generating family.

We write $\Gamma : \mathcal{E} \to \mathbf{Set}$ for the global sections functor $\mathcal{E}(1, -)$ and Δ for the left adjoint of Γ sending $I \in \mathbf{Set}$ to $\coprod_I 1_{\mathcal{E}}$.

Such a topos $\mathcal{E}$ is called **precohesive** iff

(1) $\mathcal{E}$ is **locally connected**, i.e. Δ has a left adjoint Π sending an object in $\mathcal{E}$ to its set of connected components

(2) $\mathcal{E}$ is **local**, i.e. Γ has a right adjoint ∇ sending $I \in \mathbf{Set}$ to the *codiscrete* object $\nabla(I)$ (necessarily satisfying $\Gamma\nabla I \cong I$)

(3) $\mathcal{E}$ is **hyperconnected**, i.e. $\Gamma(\Omega_{\mathcal{E}}) \cong \Omega_{\mathbf{Set}}$ (i.e. $\mathcal{E}$ is 2-valued).

Condition (3) is equivalent to $\Delta/I : \mathcal{S}/I \to \mathcal{E}/\Delta(I)$ restricting to an equivalence between $\mathsf{Sub}_{\mathbf{Set}}(I)$ and $\mathsf{Sub}_{\mathcal{E}}(\Delta(I))$ for all $I \in \mathbf{Set}$.

Lawvere's original formulation

As shown by Johnstone (TAC 2011) for such toposes the functor $\Pi : \mathcal{E} \to \mathbf{Set}$ always preserves finite products and conditions (2)+(3) can be replaced by the requirement that every connected component of an object in $\mathcal{E}$ contains a point, i.e. a global element.

Thus, a Grothendieck topos $\mathcal{E}$ is locally connected, local and hyperconnected iff $\Pi \dashv \Delta \dashv \Gamma \dashv \nabla : \mathbf{Set} \hookrightarrow \mathcal{E}$ such that Π preserves finite products and every noninitial object has a global element.

The latter condition is called **Nullstellensatz** by Lawvere and formulated as the requirement that for every $A \in \mathcal{E}$ the canonical map $\vartheta_A : \Gamma A \to \Pi A$ is epic, i.e. **"every piece has a point"**.

Geometric Morphisms as Fibrations 1

With a finite limit preserving $F : \mathcal{S} \to \mathcal{E}$ between elementary toposes one may associate the (Grothendieck) fibration

$$P_F = F^* P_{\mathcal{E}} = \partial_1 : \mathcal{E}/F \to \mathcal{S}$$

where $P_{\mathcal{E}} = \partial_1 : \mathcal{E}/\mathcal{E} \to \mathcal{E}$ is the **fundamental fibration** of $\mathcal{E}$.

The fibration P_F over $\mathcal{S}$ is a **fibration of toposes** since all $\mathcal{E}/FI$ are toposes and $(Fu)^* : \mathcal{E}/FI \to \mathcal{E}/FJ$ is logical for all $u : J \to I$.
Moreover, the fibration P_F has **internal sums** which are necessarily stable and disjoint.
By results of J.-L. Moens and M. Jibladze (in the 1980s) these are precisely the fibred toposes with internal sums, i.e. every such fibration $P : \mathcal{X} \to \mathcal{S}$ is equivalent to P_F where $F : \mathcal{S} \to \mathcal{E} = \mathcal{X}_1$ is the functor sending $I \in \mathcal{S}$ to $\coprod_I 1_I$.

Geometric Morphisms as Fibrations 2

As shown by Bénabou (in his 1974 Montreal Lectures) the fibration P_F is **locally small** iff F has a right adjoint U. Accordingly, one calls such fibrations **geometric** since $F \dashv U : \mathcal{E} \to \mathcal{S}$ is a geometric morphism between toposes.
Moreover, such fibrations admit a **small generating family** iff the geometric morphism $F \dashv U : \mathcal{E} \to \mathcal{S}$ is **bounded**, i.e. there is an $S \in \mathcal{E}$ s.t. every $A \in \mathcal{E}$ appears as subquotient of $S \times FI$ for some $I \in \mathcal{S}$.

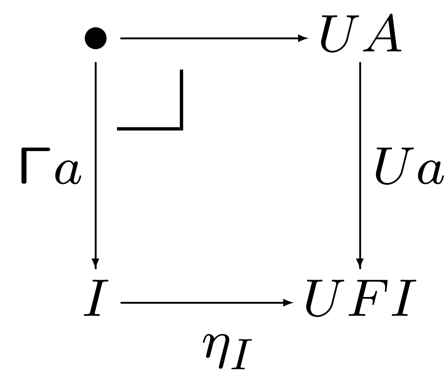
Thus, bounded geometric morphisms to $\mathcal{S}$ correspond to Grothendieck toposes over $\mathcal{S}$, i.e. toposes fibered over $\mathcal{S}$ which have internal sums, are locally small and admit a small generating family.

See my notes on *Fibered Categories à la Jean Bénabou* for details.

Geometric Morphisms as Fibrations 3

For every (bounded) geometric morphism $F \dashv U : \mathcal{E} \to \mathcal{S}$ there is a fibered functor $\Delta = \Delta_F : P_{\mathcal{S}} \to P_F$ sending $u : J \to I$ to $Fu : FJ \to FI$.

Moreover, the fibered functor Δ has a fibered right adjoint Γ sending $a : A \to FI$ in $P_F(I)$ to $\eta_I^* Ua$



where η is the unit of $F \dashv U$.

We call $\Delta \dashv \Gamma : P_F \to P_{\mathcal{S}}$ the **fibered geometric morphism** induced by the ordinary geometric morphism $F \dashv U$.

Locally Connected Geometric Morphisms 1

A geometric morphism $F \dashv U : \mathcal{E} \to \mathcal{S}$ is called **locally connected** or **molecular** iff $\Delta_F : P_{\mathcal{S}} \to P_F$ has a fibered left adjoint Π, i.e. F has an ordinary left adjoint L such that

$$\begin{array}{ccc} B & \xrightarrow{\;f\;} & A \\ {\scriptstyle b}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle a} \\ FJ & \xrightarrow[\;Fu\;]{} & FI \end{array} \qquad \text{implies} \qquad \begin{array}{ccc} LB & \xrightarrow{\;Lf\;} & LA \\ {\scriptstyle \widehat{b}}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \widehat{a}} \\ J & \xrightarrow[\;u\;]{} & I \end{array}$$

with $\widehat{a} = \varepsilon_I \circ La$ and $\widehat{b} = \varepsilon_J \circ Lb$ where ε is the counit of $L \dashv F$.

A locally connected geometric morphism $F \dashv U : \mathcal{E} \to \mathcal{S}$ is **connected**, i.e. F is full and faithful, iff L preserves terminal objects.

Locally Connected Geometric Morphisms 2

As shown in C3.3.1 of Johnstone's *Elephant* for a geometric morphism $F \dashv U : \mathcal{E} \to \mathcal{S}$ the following are equivalent

(1) $F \dashv U$ is is locally connected

(2) F preserves dependent function types (i.e. dependent products)

(3) $F/I : \mathcal{S}/I \to \mathcal{E}/FI$ preserves exponentials for all $I \in \mathcal{S}$.

Notice that condition (2) provides a characterization of locally connectedness avoiding any reference to fibrational notions.

Locally Connected Geometric Morphisms 3

Let $F \dashv U : \mathcal{E} \to \mathcal{S}$ be a geometric morphism which is connected, i.e. F is full and faithful.

Then $F \dashv U$ is locally connected iff F has a left adjoint L sending pullbacks of cospans in $\mathcal{E}$ with one side in the image of F to pullbacks in $\mathcal{S}$.

Moreover, as observed in [LM15], the geometric morphism $F \dashv U$ is locally connected with all $F/I : \mathcal{S}/I \hookrightarrow \mathcal{E}/FI$ exponential ideals iff F has a left adjoint L sending pullbacks of cospans in $\mathcal{E}$ with common codomain in the image of F to pullbacks in $\mathcal{S}$.*

*As observed in *loc.cit.* for locally connected geometric morphisms this is equivalent to L preserving finite products.

Local Geometric Morphisms

A geometric morphism $F \dashv U : \mathcal{E} \to \mathcal{S}$ is called **local** iff $\Gamma : P_F \to P_\mathcal{S}$ has a fibered right adjoint ∇.

This is equivalent to F being full and faithful and U having a right adjoint R (which necessarily is full and faithful as well).

Hyperconnected Geometric Morphisms

A geometric morphism $F \dashv U : \mathcal{E} \to \mathcal{S}$ is called **hyperconnected** iff U preserves subobject classifiers.

This condition is equivalent to the requirement that for every $I \in \mathcal{S}$ the functor $F/I : \mathcal{S}/I \to \mathcal{E}/FI$ restricts to an equivalence between $\mathsf{Sub}_{\mathcal{S}}(I)$ and $\mathsf{Sub}_{\mathcal{E}}(FI)$.

A further equivalent characterization is that for $F \dashv U$ all its units are isos and all its counits are monic.

Precohesion over Arbitrary Base Toposes

The above considerations suggest to define a geometric morphism $F \dashv U : \mathcal{E} \to \mathcal{S}$ to be **precohesive** iff it is **locally connected**, **local** and **hyperconnected**.

In his recent (2021) JPAA paper Matias Menni has shown that a hyperconnected geometric morphism $F \dashv U : \mathcal{E} \to \mathcal{S}$ is local iff U preserves coequalizers.
The latter condition says that $\Gamma : P_F \to P_{\mathcal{S}}$ is cocontinuous. Thus, the claim follows by an appropriate fibered adjoint functor theorem when $F \dashv U$ is assumed as bounded. Menni's argument, however, avoids this assumption.

This allows one to characterize precohesiveness of geometric morphisms $F \dashv U$ purely in terms of preservation properties, namely that

(1) F preserves dependent function types and

(2) U preserves coequalizers and subobject classifiers.

Two Recent Independence Results

In their 2020 APCS paper Hemelaer and Rogers have come up with an essential geometric morphism between toposes of monoid actions validating (2) but not (1).
They have shown that the leftmost adjoint does not preserve finite products which by a result from Johnstone's 2011 TAC paper follows from the further assumption of locally connectedness.

In a TAC paper from 2021 R. Garner and I have come up with an example of an essential local geometric morphism $F \dashv U$ such that $L \dashv F$ preserves finite products (and thus F is an exponential ideal) but F does not preserve dependent function types.†

†As already observed in Barr and Paré's 1980 paper *Molecular Toposes* the leftmost adjoint L is $\mathcal{S}$-**enriched** iff F **preserves exponentials** and L is **fibered over** $\mathcal{S}$ iff F **preserves dependent products**.

An outline of our Counterexample

Let $\mathcal{B}$ be a small locally cartesian closed (lcc) category and $r \dashv i : \mathcal{A} \hookrightarrow \mathcal{B}$ a full reflective subcategory.
We consider $r^* \dashv i^* : \widehat{\mathcal{B}} \to \widehat{\mathcal{A}}$ and show that r preserves the lcc structure whenever r^* does. From a result by Garner and Lack [GL] it follows that $\mathcal{A}$ is lcc whenever $\mathcal{B}$ is.

Thus it suffices to find a small lccc $\mathcal{B}$ with a small reflective subcat $\mathcal{A}$ which is not lcc.

We exhibit the following two instances
(1) let $\mathcal{B}$ be finite reflexive graphs and $\mathcal{A}$ the full reflective subcat of finite preorders
(2) let $\mathcal{B}$ be the category of modest sets over the 1$^{\text{st}}$ Kleene algebra and $\mathcal{A}$ the full reflective subcategory of $\neg\neg$-closed subobjects of powers of $\mathbb{N}$.

An open question

Alas, our counterexample is not hyperconnected and thus leaves open Menni's question whether inverse image parts of hyperconnected and local geometric morphisms preserve dependent function spaces whenever they preserve ordinary exponentials.

In a note from 2021 Menni has shown that essential local geometric morphisms $F \dashv U : \mathcal{E} \to \mathcal{S}$ with $\mathcal{S}$ boolean are locally connected whenever the left adjoint L of F preserves finite products.

I expected that hyperconnected and local geometric morphisms to the Sierpiński topos $\widehat{\mathbb{2}}$ need not be locally connected even if their inverse image part preserves ordinary exponentials. This, however, was shown to be impossible by Jens Hemelaer in his 2022 CTGD paper.
But I still believe that there does exist a counterexample over some non-boolean base topos simply because dependent function spaces cannot be reduced to ordinary ones without using classical logic.