Sheaf Models for CZF refuting Power Set and Full Separation

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Oberwolfach April 2008

Aim of the talk

Previously (∼2004), R. Lubarsky, B. van den Berg and I have (independently) constructed **realizability models** for CZF which refute the Power Set axiom but still validate the Full Separation schema.
(CZF + Sep has strength of 2nd Order Arith.)

The aim of this talk is to describe

sheaf models for CZF

which always **refute Full Separation** and in some natural cases also refute **Power Set**.

Our models arise as a variation of D. Scott's (pre)sheaf models for IZF but by iterating a *restricted powerset* functor $\mathcal{P}_s$ instead of the *full powerset* functor $\mathcal{P}$ of the (pre)sheaf topos.

Recap of IZF and CZF

Intuitionistic Zermelo Fraenkel set theory IZF is obtained from ZF by dropping Excluded Middle and replacing the Regularity axiom by $\in$-induction.

Constructive Zermelo Fraenkel set theory CZF is obtained from IZF by

(1) restricting the Separation scheme to **bounded** formula

(2) replacing the Powerset axiom by the following **Fullness** axiom
$(\exists c \subseteq \mathsf{mv}(a,b))(\forall r \in \mathsf{mv}(a,b))(\exists s \in c)\ s \subseteq r$
where $\mathsf{mv}(a,b)$ is the class of total relations from a to b

(3) strengthening Replacement to **Strong Collection**
$(\exists c)(\forall \vec{z})$
$\Big((\forall x \in a)(\exists y \in b)\varphi\Big) \Rightarrow (\exists d \in c)\mathbb{M}(x{:}a, y{:}d)\,\varphi$
where $\mathbb{M}(x{:}a, y{:}b)\,\varphi(x,y,\ldots)$ stands for $(\forall x \in a)(\exists y \in b)\varphi \wedge (\forall y \in b)(\exists x \in a)\varphi(x,y,\ldots)$.

Presheaf Models for IZF

As suggested by D. Scott ($\sim$1980) for arbitrary small categories $\mathcal{C}$ in $\widehat{\mathcal{C}} = \mathbf{Set}^{\mathcal{C}^{\mathsf{op}}}$ consider the **class**-valued presheaf $V(\mathcal{C})$ which is the least fixpoint of $\mathcal{P}$.
Recall that for $X \in \widehat{\mathcal{C}}$ the power object $\mathcal{P}(X)$ is given by

$$\mathcal{P}(X)(I) = \mathsf{Sub}_{\widehat{\mathcal{C}}}(\mathsf{y}(I) \times X)$$

where $\mathsf{y}(I) = \mathcal{C}(-, I)$.
For $a \in \mathcal{P}(X)(I)$ and $u : J \to I$

$$a{\cdot}u = \mathcal{P}(X)(u)(a) = \{\langle v, x\rangle \mid \langle uv, x\rangle \in a\}$$

is the **reindexing** of A along u.

Equality and membership in $V(\mathcal{C})$ are given by the following forcing clauses

$I \Vdash a \in b$ iff $\langle \mathsf{id}_I, a\rangle \in b$

$I \Vdash a = b$ iff for all $u : J \to I$ and $c \in V(\mathcal{C})(J)$
$\langle u, c\rangle \in a$ implies $J \Vdash c \in b{\cdot}u$
$\langle u, c\rangle \in b$ implies $J \Vdash c \in a{\cdot}u$

and forcing clauses for logic are as usual.

Sheaf Models for IZF

In 1980 M. Fourman has shown how to interpret IZF in **cocomplete** toposes $\mathcal{E}$ by constructing V_α in $\mathcal{E}$ for every ordinal α.
For the typical case of a Grothendieck topos $\mathcal{E} = \mathsf{Sh}(\mathcal{C}, \mathcal{J})$ a forcing definition using $V(\mathcal{C})$ can be obtained by modifying the clause for elementhood as follows

$I \Vdash a \in b$ iff there exists a $\mathcal{J}$-cover $(I_i \stackrel{u_i}{\to} I)$ and $c_i \in V(\mathcal{C})(I_i)$ s.t. for all i $I_i \Vdash a{\cdot}u_i = c_i$ and $\langle u_i, c_i \rangle \in b$

The clause for equality is as before and the clauses for $\perp$, $\vee$ and $\exists$ have to be modified as usual in Kripke-Joyal semantics.

Theorem The (colimit of the) cumulative hierachy in $\mathsf{Sh}(\mathcal{C}, \mathcal{J})$ is isomorphic to $V(\mathcal{C})$ modulo the equivalence relation $=$ on $V(\mathcal{C})$ as given by forcing.

Proof (Idea)
Since $\langle u_i, c_i \rangle \in b$ is equivalent to $\langle \mathrm{id}_{I_i}, c_i \rangle \in b{\cdot}u_i$ the definition of $\in$ (implicitly) perfoms $\mathcal{J}$-closure.

“Getting Smaller” (1)

Instead of iterating the powerset functor $\mathcal{P}$ on $\widehat{\mathcal{C}}$ we consider a subfunctor $\mathcal{P}_{cg}$ which is defined as follows.

A presheaf $X \in \widehat{\mathcal{C}}$ is **countably generated** iff there exists a countable (including the empty one) family $\big(x_n \in X(I_n)\big)$ s.t. every $x \in X(I)$ is of the form $x = x_n{\cdot}u$ for some $u : I \to I_n$. We write $\mathsf{Sub}_{cg}(X)$ for the **collection of countably generated subpresheaves of** X and define $\mathcal{P}_{cg}$ as

$$\mathcal{P}_{cg}(X)(I) = \mathsf{Sub}_{cg}(\mathsf{y}(I){\times}X)$$

Now we may define the **countably generated hierachy** $U(\mathcal{C})$ in $\widehat{\mathcal{C}}$ as

$$U(\mathcal{C})_\alpha = \bigcup_{\beta<\alpha} \mathcal{P}_{cg}(V_\beta)$$

which stabilises at ω_1, i.e. $U(\mathcal{C})_{\omega_1}$ is the least fixpoint of $\mathcal{P}_{cg}$.

“Getting Smaller” (2)

We have in mind the **countable cover** topology on $\mathcal{C}$ when defining (by transfinite recursion) the forcing clauses for $=$ and $\in$ as

$I \Vdash a \in b$	iff	there exists a countable family (u_n, c_n) in b such that (u_n) covers I and for all n $\mathsf{dom}(u_n) \Vdash c_n = a{\cdot}u_n$
$I \Vdash a = b$	iff	for all $u : J {\to} I$ and $c \in U(J)$ it holds that $\langle u, c\rangle \in a$ implies $J \Vdash c \in b{\cdot}u$ and $\langle u, c\rangle \in b$ implies $J \Vdash c \in a{\cdot}u$.

for $a, b \in U(\mathcal{E})(I)$.

For **suitably chosen** $\mathcal{C}$ the structure $U(\mathcal{C})$ with $=$ and $\in$ defined as above give rise to models of CZF **refuting both the Power Set axiom and the Full Separation schema**.

Next we explain what “suitably chosen” means:

Constructive Toposes

are categories $\mathcal{E}$ such that

(1) $\mathcal{E}$ is a Heyting category, i.e. is regular and for all $f : A \to B$ in $\mathcal{E}$ the map f^{-1} : $\mathsf{Sub}_{\mathcal{E}}(B) \to \mathsf{Sub}_{\mathcal{E}}(A)$ has a right adjoint $\forall_f$

(2) $\mathcal{E}$ has stable and disjoint finite sums

(3) every equivalence relation $r = \langle r_1, r_2 \rangle$: $R \rightarrowtail A \times A$ appears as kernel pair of the coequalizer $q : A \twoheadrightarrow A_{/R}$ of r_1 and r_2

(4) $\mathcal{E}$ is locally cartesian closed (lccc), i.e. for every $f : A \to B$ the pullback functor f^* : $\mathcal{E}/B \to \mathcal{E}/A$ has a right adjoint Π_f.

Regular categories satisfying conditions (2) and (3) are called **pretoposes** provided coequalizers of equivalence relations are stable under pullbacks. This latter condition follows from (4) because f^* has a right adjoint and thus preserves colimits.

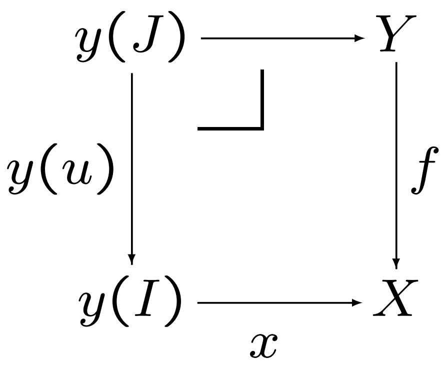
We need **constructive ∞-toposes**, i.e. constructive toposes with stable and disjoint **countable sums**. They have a **natural numbers object** (nno) $N = \coprod_\omega 1$.

A More Abstract View (1)

In order to prove **soundness of our model** we need a more **abstract view** of it.

Let $\mathcal{E}$ be some fixed constructive ∞-topos endowed with the **countable cover** topology: a sieve S on $I \in \mathcal{E}$ **cover**s I iff there exists a countable family $(u_n : I_n \to I)$ in S s.t. for every $u : J \to I$ in S there exists a $v : J \to I_n$ with $u = u_n v$. This topology is *subcanonical*, i.e. all representable presheaves are sheaves. We write $\mathsf{Sh}_\infty(\mathcal{E})$ for the category of sheaves over $\mathcal{E}$ w.r.t. countable cover topology.

Following a suggestion of Jean Bénabou we identify **"small"** with **representable** and call a map $f : Y \to X$ **"small"**, i.e. a **family of small objects**, iff for all $x : \mathsf{y}(I) \to X$



for some $u : J \to I$ in $\mathcal{E}$.

A More Abstract View (2)

For interpreting bounded formulas as sets, i.e. small objects, we require the equality predicates to be small maps. We call an object X **separated** iff δ_X is small and denote the ensuing full subcategory of $\mathsf{Sh}_\infty(\mathcal{E})$ by $\mathsf{Idl}_\infty(\mathcal{E})$. We write $\mathcal{S}_\mathcal{E}$ for the collection of small maps in $\mathsf{Idl}_\infty(\mathcal{E})$.

$\mathsf{Idl}_\infty(\mathcal{E})$ is thought of as the **category of classes** and $\mathcal{S}_\mathcal{E}$ as the collection of all **families of sets** indexed by classes. $\mathsf{Idl}_\infty(\mathcal{E})$ is a Heyting category and $\mathcal{S}_\mathcal{E}$ satisfies the axioms

(S1) $\mathcal{S}_\mathcal{E}$ is closed under composition and contains all isos

(S2) $\mathcal{S}_\mathcal{E}$ is stable under pullbacks along arbitrary morphisms of $\mathcal{E}$

(S3) all diagonals $\delta_A : A \rightarrowtail A \times A$ are in $\mathcal{S}_\mathcal{E}$

(S4) if e is a cover, i.e. regular epi, and $f \circ e \in \mathcal{S}_\mathcal{E}$ then $f \in \mathcal{S}_\mathcal{E}$

(S5) if $f : C \to A$ and $g : D \to A$ are in $\mathcal{S}_\mathcal{E}$ then $[f, g] : C + D \to A$ is also in $\mathcal{S}_\mathcal{E}$

Remark Under (S1) and (S2) condition (S3) is equivalent to the requirement that $g \in \mathcal{S}_\mathcal{E}$ whenever f and fg are in $\mathcal{S}_\mathcal{E}$.

A More Abstract View (3)

The following characterisation of $\mathsf{Idl}_\infty(\mathcal{E})$ within $\mathsf{Sh}_\infty(\mathcal{E})$ is useful later when constructing a universe corresponding to $U(\mathcal{E})$.

Th 1 For objects A of $\mathsf{Sh}(\mathcal{E})$ t.f.a.e.

(1) A is in $\mathsf{Idl}_\infty(\mathcal{E})$

(2) A arises as colimit in $\widehat{\mathcal{E}}$ of an ∞-**ideal diagram**, i.e. an ω_1-directed diagram of monos between representable objects

(3) for every $f : y(I) \to A$ its image in $\mathsf{Sh}_\infty(\mathcal{E})$ is representable.

NB An analogous theorem was suggested by A. Joyal and proved by S. Awodey et.al. for $\mathsf{Sh}(\mathcal{E})$, the category of sheaves w.r.t. the **finite** cover topology on $\mathcal{E}$, the full subcategory $\mathsf{Idl}(\mathcal{E})$ of separated objects and ideal diagram, i.e. ω_0-directed diagrams of monos between representable objects.
We can't take this because in the ensuing model of set theory the ordinal ω is not a set though sets host a nno!

A More Abstract View (4)

Next we consider a **small powerset** functor $\mathcal{P}_s$ on $\mathrm{Idl}_\infty(\mathcal{E})$ whose initial fixpoint will be the universe corresponding to $U(\mathcal{E})$.

A relation $r : R \rightarrowtail A \times I$ is called **small** iff $\pi_2 \circ r : R \to I$ is in $\mathcal{S}_\mathcal{E}$, i.e. r is small iff it is a(n I-indexed) family of small subobjects of A. Obviously, for $u : J \to I$ in $\mathcal{E}$ the relation $(A{\times}u)^* r$ is also small.

Th 2 For every A in $\mathrm{Idl}_\infty(\mathcal{E})$ there is a small relation $\in_A \rightarrowtail A \times \mathcal{P}_s(A)$ such that for every small relation $r : R \rightarrowtail A \times I$ there exists a unique map $\varrho : I \to \mathcal{P}_s(A)$ in $\mathrm{Idl}_\infty(\mathcal{E})$ s.t.

$$\begin{array}{ccc} R & \longrightarrow & \in_A \\ {\scriptstyle r}\downarrow & \lrcorner & \downarrow \\ A \times I & \xrightarrow{A \times \varrho} & A \times \mathcal{P}_s(A) \end{array}$$

The object $\mathcal{P}_s(A)$ is called the *power class* of A. Its elements are subsets of A. But in general $\mathcal{P}_s(A)$ is not small even if A is small.

A More Abstract View (5)

Yoneda tells us that

$$\mathcal{P}_s(A)(I) \cong \mathsf{Idl}_\infty(\mathcal{E})(y(I), \mathcal{P}_s(A)) \cong$$

$$\cong \{r : R \rightarrowtail A \times y(I) \mid r \text{ small relation}\}$$

Since the powerset functor $\mathcal{P}$ for $\mathsf{Sh}_\infty(\mathcal{E})$ is given by $\mathcal{P}(A)(I) = \mathsf{Sub}_{\mathsf{Sh}_\infty(\mathcal{E})}(A \times y(I))$ it follows that

$$\mathcal{P}_s(A) \subseteq \mathcal{P}(A)$$

for $A \in \mathsf{Sh}_\infty(\mathcal{E})$.

Remark Notice that $\mathcal{P}(A)$ is the class of *subclasses* of A (an "impredicative" notion of class is available in $\mathsf{Sh}_\infty(\mathcal{E})$!) whereas $\mathcal{P}_s(A)$ is the subclass of $\mathcal{P}(A)$ consisting of (families of) *subsets* of A.

Th 3 $\mathcal{P}_s(1)$ is small iff $\mathcal{E}$ is a topos.

Proof. We have
$\mathcal{P}_s(1)(I) \cong \{\text{repr. subobj. of } \mathsf{y}(I)\} \cong \mathsf{Sub}_{\mathcal{E}}(I)$.
Thus $\mathcal{P}_s(1)$ is small iff $\mathsf{Sub}_{\mathcal{E}}$ is representable iff $\mathcal{E}$ is a topos.

A More Abstract View (6)

Th 4 The functor $\mathcal{P}_s : \mathsf{Idl}_\infty(\mathcal{E}) \to \mathsf{Idl}_\infty(\mathcal{E})$ preserves ∞-ideal colimits.

Proof (Sketch)
If $y(J) \cong R \overset{r}{\rightarrowtail} A \times y(I)$ then by Th.1(3) the map $r_1 = \pi_1 \circ r : y(J) \to A$ factors through $m : y(K) \rightarrowtail A$ via some regular epimorphism $e : y(J) \twoheadrightarrow y(K)$ and we have

$$\begin{array}{ccc} y(J) & \xrightarrow{\langle e, r_2 \rangle} & y(K) \times y(I) \\ & \searrow^{r} & \downarrow{\scriptstyle m \times y(I)} \\ & & A \times y(I) \end{array}$$

and $y(K) \times y(I) \cong y(K \times I)$.

Thus, if A is the ∞-ideal colimit of $(y(A_i))_{i \in I}$ then

$$\mathcal{P}_s(\mathsf{colim}_{i \in I}) \cong \mathsf{colim}_{i \in I} \mathcal{P}_s(y(A_i))$$

A More Abstract View (7)

Since $\mathcal{P}_s$ preserves ∞-ideal colimits (Th 4) it has an initial fixpoint $U_\mathcal{E}$ which is attained after ω_1 iterations, i.e.

$$U_\mathcal{E} = \mathsf{colim}_{\alpha<\omega_1}\mathcal{P}_s^\alpha(0)$$

Interpreting $\in$ as

$$\in_{U_\mathcal{E}} \rightarrowtail U_\mathcal{E} \times \mathcal{P}_s(U_\mathcal{E}) \cong U_\mathcal{E} \times U_\mathcal{E}$$

and equality as $\delta_{U_\mathcal{E}}$ gives rise to a first order structure (in $\mathsf{Idl}_\infty(\mathcal{E})$ thus in $\widehat{\mathcal{E}}$) which can be shown to be isomorphic to the forcing model $U(\mathcal{E})$ modulo $=$ as defined by forcing.
More precisely, $U_\mathcal{E}$ is isomorphic to $U(\mathcal{E})_{/\sim}$ – where $a \sim b$ means $I \Vdash a = b$ for $a, b \in U(\mathcal{E})$ – and this isomorphism respects $\in$.

From work of Awodey, Simpson et.al. it follows that

Th 5
$U_\mathcal{E}$ is a model of $\mathsf{CZF}^-_{\mathsf{Exp}}$, i.e. CZF without Infinity and Fullness but with Exponentiation.

Countable Ordinals in $U(\mathcal{E})$

We associate with every countable ordinal α a global element $\widehat{\alpha}$ of $U(\mathcal{E})$ as follows: if $\alpha = \sup_{n\in\omega} \alpha_n$ then let $\widehat{\alpha}$ be the subpresheaf of $\mathsf{y}(1) \times U(\mathcal{E})$ generated by the countable set $\{\langle n, \widehat{\alpha_n}\rangle \mid n \in \omega\}$ where $n : 1 \to N$ is the n-th numeral of nno N in $\mathcal{E}$.

One easily checks that $\widehat{\omega}$ witnesses the set-theoretic Infinity axiom. Thus we have that

Th 6 $U(\mathcal{E})$ is a model of $\mathsf{CZF}_{\mathsf{Exp}}$.

One easily checks that

Lemma 1
$\widehat{\alpha} \in U(\mathcal{E})_\alpha$ fails in $U(\mathcal{E})$ for $\alpha < \omega_1$. Thus, there is **no** set a in $U(\mathcal{E})$ such that $\widehat{\alpha} \in a$ holds in $U(\mathcal{E})$ for all $\alpha < \omega_1$.

Fullness

For maps $a : A \to I$ and $b : B \to I$ in $\mathcal{E}$ we write $M_I(a,b)$ for the collection of all subobjects $r : R \rightarrowtail A \times B$ such that $\pi_1 \circ r : R \twoheadrightarrow A$.

Type-Theoretic Fullness Axiom (TTFA)
For all $a : A \to I$ and $b : B \to I$ in $\mathcal{E}$ there exist a cover $\widetilde{e} : \widetilde{I} \twoheadrightarrow I$, a map $c : C \to \widetilde{I}$ and $R \in M_C(c^*\widetilde{e}^*A, c^*\widetilde{e}^*B)$ such that for every $f : D \to \widetilde{I}$ and $S \in M_D(f^*\widetilde{e}^*A, f^*\widetilde{e}^*B)$ there exists a cover $e : E \twoheadrightarrow D$ and a map $g : E \to C$ with $fe = cg$ and $g^*R \subseteq e^*S$.

Th 7
If $\mathcal{E}$ validates the type-theoretic fullness axiom TTFA then $U(\mathcal{E})$ validates the Fullness axiom of CZF.

“Good” Examples of $\mathcal{E}$

Th 8 If $\mathcal{T}$ is the **typed pca** $\mathbf{Mod}(K_2)$, $\mathbf{Mod}(\mathcal{P}\omega)$ or QCB_0 (i.e. T_0 quotients of subspaces of $\mathcal{P}\omega$) then for the realizability model $\mathbf{RC}(\mathcal{T})$, i.e. the ex/reg completion of $\mathbf{Asm}(\mathcal{T})$, we have

(1) $\mathbf{RC}(\mathcal{T})$ is a constructive ∞-topos

(2) $\mathbf{RC}(\mathcal{T})$ validates TTCA

(3) $\mathbf{RC}(\mathcal{T})$ has no subobject classifier.

Proof (ideas)
(1) For any $\mathcal{T}$ one knows that $\mathbf{RC}(\mathcal{T})$ is a constructive topos. Stable and disjoint countable sums exist in $\mathbf{RC}(\mathcal{T})$ since they exists in the categories $\mathbf{Mod}(K_2)$, $\mathbf{Mod}(\mathcal{P}\omega)$ and QCB_0.
(2) Essentially as in Aczel's verification Fullness w.r.t. his interpretation in type theory.
(3) Lietz and S. have shown that $\mathbf{RC}(\mathcal{T})$ is a topos iff $\mathcal{T}$ has a universal type which is not the case for the $\mathcal{T}$'s under consideration.

Failure of Full Separation 1

In $\mathrm{CZF}_{\mathrm{Exp}}$ Brouwer's 2nd Number Class W_1 appears as an inductively defined subclass of the set $\omega^{(\omega^\omega)}$, namely as the least C s.t.

(1) $\lambda f.0 \in C$

(2) if $F \in C$ then the functional

$$\mathrm{succ}(F)(f) = \begin{cases} 1 & \text{if } f(0) = 0 \\ F(\lambda n.f(n+1)) & \text{otherwise} \end{cases}$$

is in C as well

(3) if $(F_n)_{n\in\omega}$ is a sequence in C then

$$\Big(\sup_{n\in\omega} F_n\Big)(f) = \begin{cases} 2 & \text{if } f(0) = 0 \\ F_{f(0)-1}(\lambda n.f(n+1)) & \text{otherwise} \end{cases}$$

is in C as well.

By transfinite recursion over W_1 we define a class function $E : W_1 \to \mathrm{Ord}$

$$E(t) = \begin{cases} \emptyset & \text{if } t = 0 \\ E(t') \cup \{E(t')\} & \text{if } t = \mathrm{succ}(t') \\ \bigcup_{n\in\omega} E(t_n) & \text{if } t = \sup_{n\in\omega} t_n \end{cases}$$

Failure of Full Separation 2

Since $\mathcal{E}$ has countable sums ω^ω contains all functions on ω and thus $\omega^{(\omega^\omega)}$ contains all *continuous* functionals (corresponding to countably branching wellfounded trees).

Thus $U(\mathcal{E}) \models \widehat{\alpha} \in E[W_1]$ for all $\alpha < \omega_1$.
Thus, by Lemma 1, in $U(\mathcal{E})$ it does not hold that $E[W_1]$ is a set. If $U(\mathcal{E})$ validated Full Separation then $E[W_1]$ were a set

Th 9 If $\mathcal{E}$ is a constructive ∞-topos then Full Separation fails in $U(\mathcal{E})$.

Remark
Obviously, the class $E[W_1]$ is a subclass of the class ω_1 of all countable ordinals. Thus, in $U(\mathcal{E})$ the class ω_1 cannot be a set which means that REA, the Regular Extension axiom, fails in $U(\mathcal{E})$ since REA allows one to prove that ω_1 is a set.

Main Theorems

Theorem 10
If $\mathcal{E}$ is a cocomplete topos validating TTFA then $U(\mathcal{E})$ is a model for CZF validating the Power Set axiom but refuting the Full Separation scheme.

Proof Immediate from Th 7 and Th 3 since the Powerset axiom holds whenever $\mathcal{P}(1)$ is a set.

Theorem 11
If $\mathcal{T}$ is $\mathbf{Mod}(K_2)$, $\mathbf{Mod}(\mathcal{P}\omega)$ or QCB_0 then $U(\mathbf{RC}(\mathcal{T}))$ refutes both the Power Set axiom and the Full Separation scheme.

Proof Immediate from Th 10 and Th 8.

Theorem 12
If $\mathcal{E}$ is the ex/reg-completion of $\mathbf{Mod}(K_2)$ or $\mathbf{Mod}(\mathcal{P}\omega)$ then $U(\mathcal{E})$ refutes both the Power Set axiom and the Full Separation scheme.

Proof Like for Th 11 since an analogue of Th 8 holds for the ex/reg-completions of $\mathbf{Mod}(K_2)$ and $\mathbf{Mod}(\mathcal{P}\omega)$.

Summary

1. We have extended D. Scott's presheaf model for IZF to arbitrary sheaf models (Grothendieck toposes) thus simplifying M. Fourman's treatment.

2. Iterating a "small" version $\mathcal{P}_s$ of the power functor $\mathcal{P}$ in particular sheaf models we have obtained models of CZF refuting both Powerset and Full Separation.

3. In these models the class ω_1 is not a set and thus REA fails.

4. The site for the simplest such sheaf model is the *exact completion* of the category of countably based T_0-spaces endowed with the countable cover topology.