Fibered View of Geometric Morphisms

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Geometric Morphisms

between toposes are usually motivated by example and analogy.

Every continuous $f : Y \to X$ induces a functor $f^* : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ by pullback. This f^* preserves finite limits and has a right adjoint f_*. The restriction of f^* to subterminal objects is (isomorphic to) the monotone map $f^{-1} : \mathcal{O}(X) \to \mathcal{O}(Y)$ which preserves finite meets and all sup's (i.e. has a right adjoint).

By **analogy** a **geometric morphism** $f : \mathbf{F} \to \mathbf{E}$ between elementary toposes is defined as an adjunction $f^* \dashv f_* : \mathbf{F} \to \mathbf{E}$ where f^* preserves finite limits. One thinks of $\mathbf{E}$ (and $\mathbf{F}$) as a generalisation of $\mathcal{O}(X)$ (and $\mathcal{O}(Y)$) and of f as a generalised continuous map.

Geometric Morphisms as Fibrations

A continuous map $f : Y \to X$ is thought of as a space Y *continuously varying* over X. **Analogously**, a geometric morphism $f : \mathbf{F} \to \mathbf{E}$ is thought of as *a topos* $\mathbf{F}$ *over* $\mathbf{E}$. Another reading of this phrase is a *fibration* $P : \mathbf{X} \to \mathbf{E}$ *of toposes with* $\mathbf{F} \simeq \mathbf{X}_1$.

What is the relation between these two readings ?

With every geometric morphism $F \dashv U : \mathbf{F} \to \mathbf{E}$ one may associate the fibration $P_F = F^* P_{\mathbf{E}} = \partial_1 : \mathbf{F}/F \to \mathbf{E}$ where $P_{\mathbf{E}} = \partial_1 : \mathbf{E}/\mathbf{E} \to \mathbf{E}$ is the fundamental fibration of $\mathbf{E}$.
The fibration P_F over $\mathbf{E}$ is a **fibration of toposes** since for every map $u : J \to I$ in $\mathbf{E}$ the pullback functor $(Fu)^* : \mathbf{F}/FI \to \mathbf{F}/FJ$ is logical.
Can one recover F from P_F and
how can one characterize fibrations of the form P_F?

Fibrations of Finite Limit Categories

Let $\mathbf{B}$ be a category with finite limits. JB has shown that $P : \mathbf{X} \to \mathbf{B}$ is a **fibration of categories with finite limits** iff P is a fibration where $\mathbf{X}$ has and P preserves finite limits. Moreover, for such fibrations cartesian arrows are stable under arbitrary pullbacks.

Let $\mathbf{B}$ be a category with 1. Then $P_{\mathbf{B}} = \partial_1 : \mathbf{B}/\mathbf{B} \to \mathbf{B}$ is a fibration iff $\mathbf{B}$ has finite limits. In this case $P_{\mathbf{B}}$ is a fibration of categories with finite limits.
Moreover, for every $F : \mathbf{A} \to \mathbf{B}$ the fibration $P_F = F^* P_{\mathbf{B}} : \mathbf{B}/F \to \mathbf{A}$ is a fibration of categories with finite limits.

Fibrations of Cats with (Internal) Sums

over a category $\mathbf{B}$ with finite limits are bifibrations $P : \mathbf{X} \to \mathbf{B}$ satisfying the **Chevalley Condition** saying that

cocartesian arrows are stable under pullbacks along cartesian arrows

i.e. for every commuting square

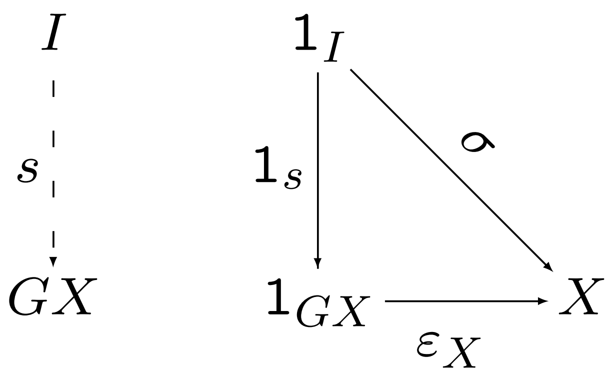
$$\begin{array}{ccc} X & \xrightarrow{\psi'} & U \\ {\scriptstyle \varphi'}\downarrow & & \downarrow{\scriptstyle \varphi} \\ Y & \xrightarrow[\psi]{} & V \end{array}$$

in $\mathbf{X}$ above a pullback square in $\mathbf{B}$ with φ and φ' cartesian

ψ *cocartesian implies* ψ' *cocartesian*

Fibrations with (Lawvere) Comprehension

$P : \mathbf{X} \to \mathbf{B}$ is a fibration of cats with terminal objects iff P has a right adjoint right inverse 1 (picking a terminal object in each fibre).
Such a P has *(Lawvere) Comprehension* iff 1 has a right adjoint G, i.e. for every $\sigma : 1_I \to X$ there is a unique $s : I \to GX$ with



Thus GX can be thought of as $\mathrm{hom}(1_{PX}, X)$ in the sense of Bénabou's notion of *local smallness*.

Fibrational Motivation of GM's

In his 1974 Montreal lectures JB has proven that for a functor $F : \mathbf{A} \to \mathbf{B}$ between finite limit categories with $F1$ terminal it holds that

(1) P_F has internal sums iff F preserves pullbacks

(2) P_F has comprehension iff F has a right adjoint U.

One obtains F from P_F since $FI \cong \coprod_I 1_I = \Delta(I)$.
Moreover, we have $UX = \mathrm{hom}(1, X) = \Gamma(X)$.

Thus, a functor $F : \mathbf{S} \to \mathbf{E}$ between toposes is the inverse image part of a geometric morphism $\mathbf{E} \to \mathbf{S}$ iff F preserves 1 and P_F is a fibration of locally small toposes with internal sums.

Thus all geometric morphisms are of the form $\Delta \dashv \Gamma$.

Moens' Lemma (1)

Characterize those fibrations which are of the form P_F for some finite limit preserving functor $F : \mathbf{A} \to \mathbf{B}$ beween finite limit cats $\mathbf{A}$ and $\mathbf{B}$. They are fibrations $P : \mathbf{X} \to \mathbf{B}$ of finite limit cats with internal sums having certain properties. These have been identified by *J.-L. Moens* in his 1982 Thése as the following ones

(1) internal sums are **stable**, i.e. cocartesian arrows are stable under pullbacks along arbitrary vertical morphism

(2) internal sums are **disjoint**, i.e. δ_φ is cocartesian whenever φ is cocartesian.

Moens' Lemma (2)

In presence of (1) condition (2) is equivalent to the requirement that every commuting square

$$\begin{array}{ccc} Y & \xrightarrow[\text{cocart.}]{\psi} & V \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\beta} \\ X & \xrightarrow[\varphi]{\text{cocart.}} & U \end{array}$$

in $\mathbf{X}$ with α, β vertical is a pullback square.

Thus one can combine conditions (1) and (2) into a single one.

Moens' Lemma in Terms of Extensivity (1)

A fibration $P : \mathbf{X} \to \mathbf{B}$ is equivalent to one of the form P_F for a finite limit preserving functor F between categories with finite limits iff P is a fibration of categories with finite limits and internal sums which are **extensive** in the sense that a commuting diagram

$$\begin{array}{ccc} Y & \xrightarrow{\psi} & V \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ X & \xrightarrow[\varphi]{\text{cocart.}} & U \end{array}$$

in $\mathbf{X}$ with α, β vertical

is a pullback square iff ψ is cocartesian.

Moens' Lemma in Terms of Extensivity (2)

It suffices to require this for the case where $X = 1_I$ and $\varphi = \varphi_I : 1_I \to \Delta(I)$ is cocartesian over $I \to 1$. Thus extensivity says that for $u : J \to I$ in $\mathbf{B}$ the adjunction

$$\coprod_u /1_I : \mathbf{X}_I \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \mathbf{X}_1/\Delta(I) : \varphi_I^*$$

is an equivalence (which coincides with the usual notion of extensivity for sums when $P = \mathsf{Fam}(\mathbf{C})$).

Thus, from commutation of

$$\begin{array}{ccc} 1_J & \xrightarrow{\varphi_J} & \Delta(J) \\ {\scriptstyle 1_u}\downarrow & & \downarrow{\scriptstyle \Delta(u)} \\ 1_I & \xrightarrow[\varphi_I]{} & \Delta(I) \end{array}$$

Moens' Lemma in Terms of Extensivity (3)

it follows that

$$\begin{array}{ccc} \mathbf{X}_J & \xrightarrow[\simeq]{\varphi_J^*} & \mathbf{X}_1/\Delta(J) \\ {\scriptstyle u^*}\uparrow & & \uparrow{\scriptstyle \Delta(u)^*} \\ \mathbf{X}_I & \xrightarrow[\varphi_I^*]{\simeq} & \mathbf{X}_1/\Delta(I) \end{array}$$

i.e. that

$$P \simeq P_\Delta$$

as desired.

Generalised Moens' Lemma

This argument goes through if P is just a bifibration not necessarily validating the Chevalley condition.

Thus, a fibration $P : \mathbf{X} \to \mathbf{B}$ is equivalent to one of the form P_F for a terminal object preserving functor F between categories with finite limits iff P is a bifibration, $\mathbf{X}$ has and P preserves finite limits and which is extensive in the sense that a commuting diagram

$$\begin{array}{ccc} Y & \xrightarrow{\psi} & V \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ X & \xrightarrow[\varphi]{\text{cocart.}} & U \end{array}$$

in $\mathbf{X}$ with α, β vertical is a pullback square iff ψ is cocartesian.

“Cartesian” Bifibrations

were recently introduced by M. Zawadowski (for very different purposes). They are fibrations $P : \mathbf{X} \to \mathbf{B}$ of finite limit cats over a finite limit cat $\mathbf{B}$ which are also cofibrations where for every $u : J \to I$ in $\mathbf{B}$ the functor $\coprod_u : \mathbf{X}_J \to \mathbf{X}_I$ preserves pullbacks and both unit and counit of the adjunction $\coprod_u \dashv u^*$ are “cartesian” natural transformations, i.e. all naturality squares are pullbacks.

One can show that cartesian bifibrations over $\mathbf{B}$ are up to equivalence those of the form P_F for some terminal object preserving functor $F : \mathbf{B} \to \mathbf{C}$ between finite limit cats.

Fibrations of Grothendieck Toposes (1)

A Grothendieck topos is a locally small elementary topos with small sums and a small generating family. This can be straightforwardly generalised to fibrations.

For locally small fibrations $P : \mathbf{X} \to \mathbf{B}$ over a base with finite products a small generating family is a $G \in \mathbf{X}_I$ such that every $A \in \mathbf{X}$ fits into a diagram

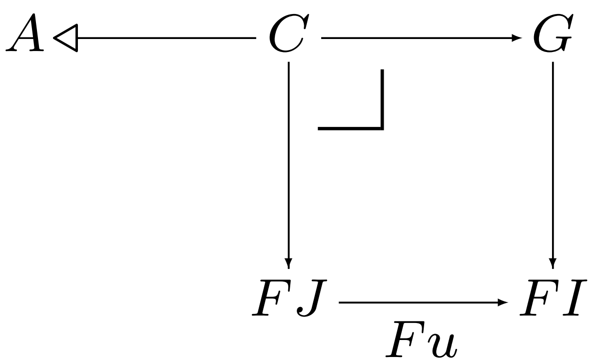
$$A \xleftarrow{\;e\;} \bullet \xrightarrow{\;\varphi\;} G$$

with φ cartesian and e **collectively epic** (i.e. for vertical α, β, $\alpha e = \beta e$ implies $\alpha = \beta$).

Fibrations of Grothendieck Toposes (2)

Task Characterize those geometric morphisms $F \dashv U : \mathbf{E} \to \mathbf{S}$ between toposes for which P_F admits a small generating family, i.e. the Grothendieck toposes over $\mathbf{S}$.

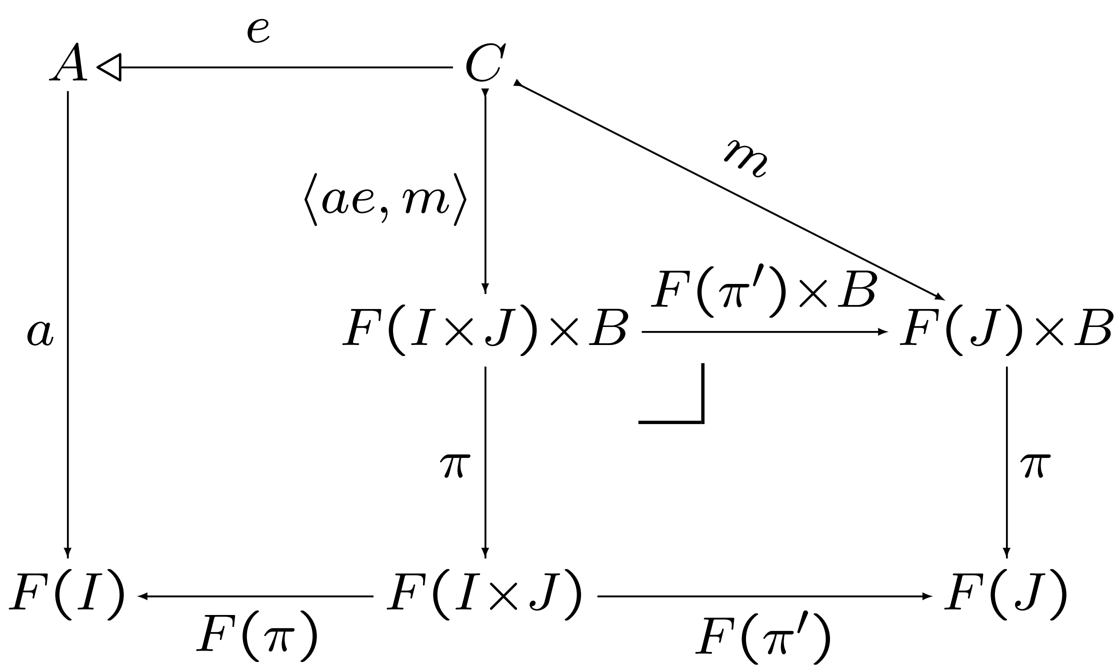
First notice that collective epis in $\mathbf{E}/F$ are those squares whose top arrow is an epi. Thus, if $g : G \to FI$ is a small generating family for P_F then G is a bound for $F \dashv U$ because



and $C \rightarrowtail FJ \times G$.

Fibrations of Grothendieck Toposes (3)

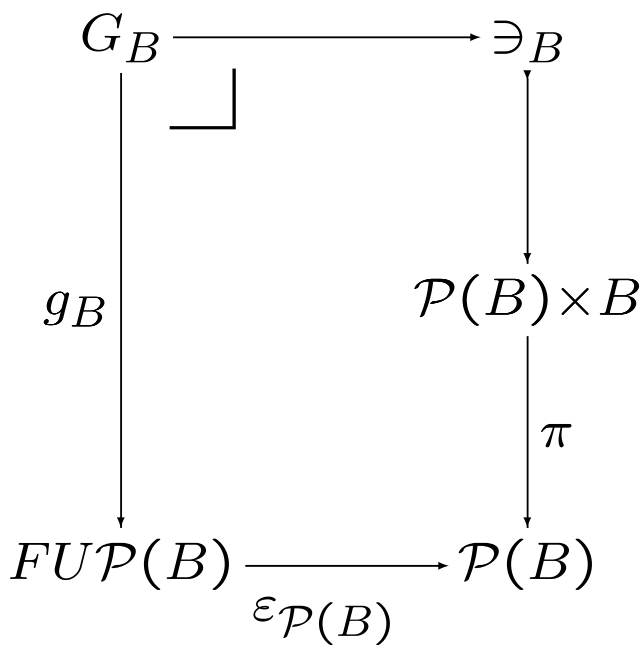
Suppose B is a bound for $F \dashv U$ and $a : A \to FI$. Consider



from which it follows that

Fibrations of Grothendieck Toposes (4)

the map $g_B : G_B \to FU\mathcal{P}(B)$ in



is a small generating family for P_F.

If g_B is a small generating family for P_F then B is a bound for $F \dashv U$.

Fibrations of Grothendieck Toposes (5)

Thus, we have shown that

> a geometric morphism $F \dashv U : \mathbf{E} \to \mathbf{S}$ is bounded
> iff
> P_F is a fibered Grothendieck topos over $\mathbf{S}$.

Finite limit preserving functors and inverse image parts of (bounded / localic) geometric morphisms are closed under composition.

These facts can be understood as **iteration theorems** for the respective topos extensions.

Geometric View of Triposes (1)

Triposes have been defined by Pitts et.al. to unify Heyting valued sets and realizability toposes.

A (moral) **tripos** over a base topos $\mathbf{S}$ is a **posetal** hyperdoctrine $\mathbb{P}$ over $\mathbf{S}$ (pre-Heyting algebra fibred over $\mathbf{S}$ with internal sums and products) such that for every $I \in \mathbf{S}$ there is a predicate $\in_I$ in $\mathbb{P}_{I \times P(I)}$ such that for every $R \in \mathbb{P}_{I \times J}$ it holds that

$$\forall j{:}J.\exists p{:}P(I).\forall i{:}I.\ R(i,j) \leftrightarrow i \in_I p$$

i.e. $\mathbb{P}$ is a model of higher order intuitionistic logic over $\mathbf{S}$.

Geometric View of Triposes (2)

For every posetal hyperdoctrine $\mathbb{P}$ over $\mathbf{S}$ one can "add quotients" obtaining $\Delta : \mathbf{S} \to \mathbf{S}[\mathbb{P}]$ which preserves finite limits. In his Thesis Pitts has shown that $\mathbf{S}[\mathbb{P}]$ is a topos iff every object X of $\mathbf{S}[\mathbb{P}]$ appears as subquotient of some $\Delta(I)$. Notice that $\mathbb{P} \simeq \Delta^*\mathsf{Sub}_{\mathbf{S}[\mathbb{P}]}$.

Thus, triposes over $\mathbf{S}$ correspond to cocomplete toposes over $\mathbf{S}$ where subobjects of 1 generate, i.e.

"localic toposes over $\mathbf{S}$ not necessarily locally small"

The corresponding Δ are called **"weakly localic"**.

Notice that $\mathbf{S}[\mathbb{P}]$ locally small over $\mathbf{S}$ iff $\mathbb{P}$ locally small over $\mathbf{S}$ iff $\mathbb{P} \simeq \mathsf{Fam}(\Omega)$ for some cHa Ω.

Geometric View of Triposes (3)

Is there an "Iteration Theorem for Tripos Extensions", i.e.

Are weakly localic functors closed under composition?

Pitts has shown that for weakly localic F and G their composite GF is weakly localic whenever G preserves regular epis.
Can one drop this additional assumption?

Idea: For every Grothendieck topos $\mathbf{E}$ over $\mathbf{Set}$ the functor $\Gamma : \mathbf{E} \to \mathbf{Set}$ is always weakly localic but in general does not preserve regular epis. Can one find a weakly localic $F : \mathbf{F} \to \mathbf{E}$ such that $\Gamma F : \mathbf{F} \to \mathbf{Set}$ is not weakly localic, i.e. not bounded by 1.