Computability for Quantum Theory

Thomas Streicher (TU Darmstadt)

February 2013

The Scenario

Following von Neumann Quantum Theory is part of Functional Analysis (Hilbert space theory). There is a satisfying computability theory for this, namely Weihrauch's **TTE** (Type Two Effectivity) where all objects are coded by sequences of numbers (Baire space).

This is an **implementation** of a much more abstract view based on the Hilbert lattice which suggests that the involved types of objects are kind of **domains** as known from **denotational semantics**.
But we have to ensure that we stay within **TTE** when using them.

For this purpose we have to ensure that we stay within Schröder and Simpson's **Topological Domain Theory** providing an abstract account of **TTE**.

Outline of Quantum Theory (1)

following e.g. Pták and Pulmannová's book (1991)

Orthomodular Structures as Quantum Logics

The Hilbert lattice $\mathbb{L}$ of closed linear subspaces of $\mathbb{H}$ serves as type of "quantum propositions". For $P, Q \in \mathbb{L}$ we write $P \leq Q$ for $P \subseteq Q$, $P \wedge Q$ for $P \cap Q$, $P \vee Q$ for the least closed subspace of $\mathbb{H}$ containing P and Q as subsets.
The set $P^{\perp} = \{x \in \mathbb{H} \mid \forall y \in P.\ \langle x \mid y \rangle = 0\}$ is in $\mathbb{L}$.

Though $(-)^{\perp}$ is **involutory** ($P^{\perp\perp} = P$) and **antitonic** ($Q^{\perp} \leq P^{\perp}$ whenever $P \leq Q$) the lattice $\mathbb{L}$ is **not boolean**. It is not even a Heyting algebra since it is **not even distributive**.

Outline of Quantum Theory (2)

A **state** is a function $s : \mathbb{L} \to \mathbb{I}$ such that

(s1) $s(1_{\mathbb{L}}) = 1$

(s2) $s(P^{\perp}) = 1 - s(P)$

(s3) $s(\bigvee P_n) = \sum s(P_n)$ provided $P_n \perp P_m$ for $n \neq m$

i.e. a **probability measure on quantum propositions**.

An **observable** is a function $o : \mathcal{B}(\mathbb{R}) \to \mathbb{L}$ such that

(o1) $o(\mathbb{R}) = 1$

(o2) $o(\complement A) = 1 - o(A)$

(o3) $o(\bigcup A_n) = \bigvee o(A_n)$ provided $A_n \cap A_m = \emptyset$ for $n \neq m$

i.e. an $\mathbb{L}$-**valued probability measure on** $\mathbb{R}$ (where $\mathcal{B}(\mathbb{R})$ is the set of Borel subsets of $\mathbb{R}$).

NB The composite $s \circ o : \mathcal{B}(\mathbb{R}) \to \mathbb{I}$ is a probability measure on $\mathbb{R}$.

Outline of Quantum Theory (3)

By the *Spectral Theorem* observables correspond to self adjoint operators on $\mathbb{H}$ (useful for computations in physics!).

If A is a self adjoint operator on $\mathbb{H}$ then for a unit vector x we have

$$\langle x \mid Ax \rangle = \int_{-\infty}^{\infty} \lambda \, d\mu_x(\lambda)$$

where $\mu_x = s_x \circ o$ with o corresponding to A and s_x is the state induced by x, i.e. $s_x(P) = \langle x \mid Px \rangle$ where P is identified with its projector.

Domain-Theoretic View of QT (1)

All structures involved are cpo's or opposites of cpo's. Notice that states and observables preserve suprema of ω-chains.

This suggests a **domain-theoretic view of quantum theory**.

But as observed by K. Keimel there is the following

Problem $\mathbb{L}$ is not a continuous lattice.

Proof. Suppose $\mathbb{L}$ were continuous. Then for atoms a we have $a \preceq a$. But there is an atom a and an ω-chain (b_n) with $\bigsqcup b_n = 1$ and $a \not\sqsubseteq b_n$ for all $n \in \mathbb{N}$.

One could be happy to work with cpo's. But then there is no notion of computability because this requires a basis!

Domain-Theoretic View of QT (2)

However, one may work in the

Topological Domain Theory

of M. Schröder and A. Simpson which lives within the **function realizability topos** $\mathbf{RT}(\mathcal{K}_2)$ and thus gives rise to a notion of computability à la Weihrauch's **TTE** (Type Two Effectivity).

The aim of this talk is to show that Hilbert lattice, states and observables all live in this world and one can reason about them in the logic of $\mathbf{RT}(\mathcal{K}_2)$ which constitutes a *canonical model* for Brouwerian Intuitionism (*c.f.* Kleene & Vesley (1965)).

Recap of Topological Domain Theory (1)

The pca $\mathcal{K}_2$ is Baire space $\mathbb{B} = \mathbb{N}^\omega$ endowed with a binary partial operation $\bullet$ such that maps of the form $\alpha \bullet (-)$ are precisely the partial continuous endo maps on Baire space (whose domain of definition is a G_δ-set, i.e. a countable intersection of opens).

For α, β one defines

$$\alpha(\beta) = n \quad \text{iff} \quad \exists m. \Big(\alpha(\overline{\beta}(m)) = n+1 \wedge \forall k < m. \, \alpha(\overline{\beta}(k)) = 0\Big)$$

and

$$\alpha \bullet \beta \simeq \gamma \quad \text{iff} \quad \forall n. \, \alpha(\langle n \rangle * \beta) = \gamma(n)$$

as suggested by Brouwer in the 1920s.

Recap of Topological Domain Theory (2)

As usual we have

$$\mathbf{Mod}(\mathcal{K}_2) \subseteq \mathbf{Asm}(\mathcal{K}_2) \subseteq \mathbf{RT}(\mathcal{K}_2)$$

where $\mathbf{Mod}(\mathcal{K}_2)$ corresponds to **TTE** since its objects are representations and its morphisms are realizable maps

$$\begin{array}{ccc} R_X & \xrightarrow{\varphi} & R_Y \\ {\scriptstyle \varrho_X}\downarrow & & \downarrow{\scriptstyle \varrho_Y} \\ |X| & \xrightarrow[f]{} & |Y| \end{array}$$

where φ is a continuous map between subspaces of $\mathbb{B}$.

Recap of Topological Domain Theory (3)

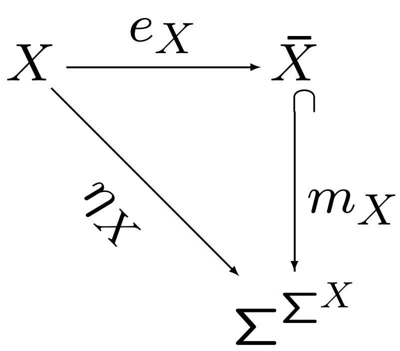
There is an obvious functor $\mathsf{Q} : \mathbf{Mod}(\mathcal{K}_2) \to \mathbf{Sp}$ sending $X = (|X|, \varrho_X)$ to the topological space $\mathsf{Q}(X)$ with underlying set $|X|$ endowed with the quotient topology induced by $\varrho_X : R_X \to |X|$.

As shown by M. Schröder the maximal full subcategory of $\mathbf{Mod}(\mathcal{K}_2)$ on which the functor Q is full and faithful are the **extensional Σ-spaces**, i.e. modest sets X such that $\eta_X : X \to \Sigma^{\Sigma^X} : x \mapsto \lambda p{:}\Sigma^X.p(x)$ is a $\neg\neg$-mono.
We write $\mathbf{Ext}_\Sigma(\mathcal{K}_2)$ for the ensuing full subcategory of $\mathbf{Mod}(\mathcal{K}_2)$.

Alternatively, extensional Σ-spaces can be characterized as **admissible representations**, i.e. for any subspace S of $\mathbb{B}$ and any continuous $f : S \to \mathsf{Q}(X)$ there is a continuous $\varphi : S \to R_X$ with $f = \varrho_X \circ \varphi$.

Recap of Topological Domain Theory (4)

For every modest set X consider



where e_X is epic and m_X a $\neg\neg$-mono. One can show that Σ^{e_X} is an isomorphism. The map e_X provides a **reflection** of X to $\mathbf{Ext}_\Sigma(\mathcal{K}_2)$. Thus $\bar{X}$ provides an admissible representation of $\mathsf{Q}(X)$.

The image of Q can be characterized as the full subcategory $\mathbf{QCB}_0$ of $\mathbf{Sp}$ consisting of T_0-quotients of countable based spaces. It contains all **complete separable metric spaces** and all ω-continuous domains.

Recap of Topological Domain Theory (5)

All spaces in $\mathbf{QCB}_0$ are **sequential** and we have

$$\mathbf{QCB}_0 \hookrightarrow \mathbf{Seq} \hookrightarrow \mathbf{Lim}$$

where $\mathbf{Lim}$ is the category of **limit spaces**. As opposed to the embedding into $\mathbf{Sp}$ the embedding into $\mathbf{Lim}$ preserves and reflects most of the relevant structure.

In particular, for Y^X we have

$$\lim f_n = f \quad \text{iff} \quad \lim f_n(x_n) = f(x) \text{ whenever } \lim x_n = x$$

Moreover $\neg\neg$-monos in $\mathbf{QCB}_0$ are those 1-1 maps which preserve and reflect convergence of sequences.

Recap of Topological Domain Theory (6)

On every object X of $\mathbf{RT}(\mathcal{K}_2)$ one has the **observational** Σ-order

$$x \sqsubseteq_X y \quad \text{iff} \quad \forall p : \Sigma^X.\, p(x) = \top \Rightarrow p(y) = \top$$

Up to iso extensional Σ-spaces are $\neg\neg$-subobjects of powers Σ^I of Σ.
If $X \subseteq_{\neg\neg} \Sigma^I$ then

$$x \sqsubseteq_X y \quad \text{iff} \quad \forall i : I.\, x(i) = \top \Rightarrow y(i) = \top$$

Such an X is called a **topological predomain** iff it is closed under **unions** of ω-chains.
It is a **topological domain** iff it has also a least element.

Let $\overline{\omega} = \{p \in \Sigma^{\mathbb{N}} \mid \forall m \leq n.\, p(n) = \top \Rightarrow p(m) = \top\}$ and ι the inclusion of $\omega = \{p \in \overline{\omega} \mid \neg\neg\exists n{:}\mathbb{N}.p(n) = \bot\}$ into $\overline{\omega}$. Then

$$X \text{ topological predomain} \iff X^\iota \text{ iso}$$

Quantum Theory in $\mathrm{RT}(\mathcal{K}_2)$ (1)

Since $\mathbb{H}$ is a csm it lives in $\mathbf{Ext}_\Sigma(\mathcal{K}_2)$ as does Σ itself. Thus $\Sigma^\mathbb{H}$ is a topological domain whose topology is the Scott topology. Notice that $A \subseteq \mathbb{H}$ is closed iff $A = p^{-1}(\bot)$ for some $p \in \Sigma^\mathbb{H}$. Under this identification we have $A \sqsubseteq B$ iff $B \supseteq A$.

Then $\mathbb{L}$ is the $\neg\neg$-closed subobject of $\Sigma^\mathbb{H}$ consisting of the $p \in \Sigma^\mathbb{H}$ s.t.

$$\forall x, y \in \mathbb{H}.\, p(x) = \bot = p(y) \Rightarrow p(x+y) = \bot$$

$$\forall x \in \mathbb{H}, \lambda \in \mathbb{C}.\, p(x) = \bot \Rightarrow p(\lambda x) = \bot$$

and thus $\mathbb{L} \in \mathbf{Ext}_\Sigma(\mathcal{K}_2)$.

The $\mathbf{QCB}_0$ topology on $\Sigma^\mathbb{H}$ is the Scott topology. Thus, the $\mathbf{QCB}_0$ topology on $\mathbb{L}$ is the **sequentialization** of the subspace topology induced by $\mathbb{L} \subseteq_{\neg\neg} \Sigma^\mathbb{H}$.

NB This is much coarser than the Scott topology on $\mathbb{L}$.

An alternative view of the Hilbert lattice

From the viewpoint of functional analysis one would identify the Hilbert lattice with the $\neg\neg$-closed subset $\mathbb{L}'$ of $\mathrm{Lin}(\mathbb{H},\mathbb{H})$ consisting of projectors $P = P \circ P$. As opposed to $\mathbb{L}$ the space $\mathbb{L}'$ is Hausdorff. Thus $\mathbb{L}$ and $\mathbb{L}'$ are not isomorphic.

Thus $\mathbf{RT}(\mathcal{K}_2)$ does **not** validate the Spectral Theorem which, in particular, entails a 1-1-correspondence between projectors and closed linear subspaces of $\mathbb{H}$.
However, it holds if we consider **located** closed linear subspaces of $\mathbb{H}$. But then we loose the domain theoretic aspect!

What is the right topology on $\mathbb{L}$?

Quantum Theory in $\mathrm{RT}(\mathcal{K}_2)$ (2)

Externally for every $A \subseteq \mathbb{H}$ the set

$$A^\perp = \{x \in \mathbb{H} \mid \forall y \in A.\, \langle x \mid y \rangle = 0\}$$

is in $\mathbb{L}$. Since the map

$$(-)^\perp : \mathbb{L} \to \mathbb{L} : A \mapsto A^\perp$$

reverses $\subseteq$ it is not a morphism in $\mathbf{RT}(\mathcal{K}_2)$.

But we still have

$$\forall A \in \mathbb{L}. \neg\neg\exists! B \in \mathbb{L}. \forall x \in \mathbb{H}.\, x \in B \Leftrightarrow \forall y \in A.\, \langle x \mid y \rangle = 0$$

which suffices for most purposes.

Quantum Theory in $\mathrm{RT}(\mathcal{K}_2)$ (3)

As opposed to $(-)^{\perp}$ the map $(-)^{\perp\perp} : \mathbb{L} \to \mathbb{L} : A \mapsto A^{\perp\perp}$ lives in $\mathbf{RT}(\mathcal{K}_2)$ since it commutes with arbitrary intersections.

For this reason the map

$$\vee : \mathbb{L} \times \mathbb{L} \to \mathbb{L} : (A, B) \mapsto (A \cup B)^{\perp\perp}$$

lives in $\mathbf{RT}(\mathcal{K}_2)$.

Moreover, the relation

$$A \perp B \qquad \Longleftrightarrow \qquad \forall x \in A. \forall x \in B.\, \langle x \mid y \rangle = 0$$

on $\mathbb{L}$ is $\neg\neg$-closed.

States in $\mathrm{RT}(\mathcal{K}_2)$

First observe that **externally** a **state** is a function $s : \mathbb{L} \to \mathbb{I}$ such that

(S1) $s(0_{\mathbb{L}}) = 0$ and $s(1_{\mathbb{L}}) = 1$

(S2) $s(P \vee Q) = s(P) + s(Q)$ whenever $P \perp Q$

(S3) s preserves infima of decreasing ω-chains.

Recall that $\mathbb{L}$ is a topological domain with $P \sqsubseteq Q$ iff $Q \leq P$. Moreover, the interval $\mathbb{I}$ ordered by $\geq$ is a continuous lattice and thus also a topological domain with $x \sqsubseteq y$ iff $x \geq y$.

Notice that $s \in \mathbb{I}^{\mathbb{L}}$ automatically validate (S3). Let St be the $\neg\neg$-closed subobject of $\mathbb{I}^{\mathbb{L}}$ of all s validating (S1) and (S2). It is easily seen to be a topological predomain and closed under countable convex combinations. Thus, since all pure states are in St it follows by Gleason's Theorem that St contains all states.

Observables in $\mathrm{RT}(\mathcal{K}_2)$

First observe that an **observable** is a function $o : \mathcal{B}(\mathbb{R}) \to \mathbb{L}$ such that

(O1) $o(\emptyset) = 0_{\mathbb{L}}$ and $o(\mathbb{R}) = 1_{\mathbb{L}}$

(O2) $o(A) \perp o(B)$ and $o(A \cup B) = o(A) \vee o(B)$ whenever $A \cap B = \emptyset$

(O3) o preserves infima of descending ω-chains.

It suffices to consider the restriction of o to closed sets. Thus, we consider maps $o : \Sigma^{\mathbb{R}} \to \mathbb{L}$ thought of as sending a closed set $p^{-1}(\perp)$ to $o(p)^{-1}(\perp)$.

Recall that $\Sigma^{\mathbb{H}}$ and $\Sigma^{\mathbb{R}}$ are $\mathbf{QCB}_0$ spaces carrying the Scott topology.

Since sequentialisation is a coreflection and $\mathbb{L}$ is the sequentialisation of a subspace of $\Sigma^{\mathbb{H}}$ the $\mathbf{QCB}_0$ maps from $\Sigma^{\mathbb{R}}$ to $\mathbb{L}$ are precisely the Scott continuous ones.

Thus, an **observable** is an $o \in \mathbb{L}^{\Sigma^{\mathbb{R}}}$ satisfying (O1) and (O2).

The collection Obs of observables forms a topological predomain.

Conclusion and Further Work

- essentially all ingredients of quantum theory live within topological predomains

- ω-continuity properties are automatic, one only needs equations to specify states and observables

- explicitate constructions in $\mathbf{QCB}_0$ (i.e. externally)
(*e.g.* the $\mathbf{QCB}_0$ topology on St)

- can one axiomatize properties of $\mathbb{L}$ or even develop a

  "Synthetic Quantum Theory" ?