Realizability Models Refuting Ishihara's Boundedness Principle

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Abstract

Ishihara's *Boundedness Principle* BD-$\mathbb{N}$ was introduced in [Ish92] and has turned out to be most useful for constructive analysis, see e.g. [Ish01]. It is equivalent to the statement that every sequentially continuous function from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$ is continuous w.r.t. the usual metric topology on $\mathbb{N}^{\mathbb{N}}$. We construct models for higher order arithmetic and intuitionistic set theory in which both every function from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$ is sequentially continuous and in which the axiom of choice from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$ holds. Since the latter is known to be inconsistent with the statement that all functions from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$ are continuous these models refute BD-$\mathbb{N}$.

Keywords: Constructive Analysis, Realizability, Independence Proofs

1. Introduction

In [Ish92] H. Ishihara introduced the so-called *boundedness principle* BD-$\mathbb{N}$ which claims that every countable pseudobounded subset of $\mathbb{N}$ is bounded. Here $S \subseteq \mathbb{N}$ is called pseudobounded iff for every sequence $a \in S^{\mathbb{N}}$ there exists an $n \in \mathbb{N}$ such that $a_k < k$ for all $k \geq n$.[1] Obviously, the principle BD-$\mathbb{N}$ is classicaly valid. Moreover, it is a most useful amendment to Bishop style constructive mathematics in the sense that it is equivalent to a lot of useful mathematical theorems over a basic theory BISH of (predicative) constructive mathematics.[2] In [Ish01] it is shown that BD-$\mathbb{N}$ is equivalent (over BISH) to each of the following prominent mathematical principles

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[1]In [Ish92] a subset S of $\mathbb{N}$ was called pseudobounded iff for every sequence (a_n) in S it holds that $\lim\limits_{n\to\infty} \frac{a_n}{n} = 0$. But both notions of pseudoboundedness give rise to equivalent boundedness principles as shown in [Ish01].

[2]As a codification of BISH one may take some variant of HA^ω or even P. Aczel's constructive set theory CZF, a predicative version of intuitionistic set theory IZF, together with number choice.

1) Every sequentially continuous mapping from a complete separable metric (csm) space to a metric space is continuous.
2) Banach's inverse mapping theorem.
3) The open mapping theorem.
4) The closed graph theorem.
5) The Banach-Steinhaus theorem.
6) The sequential completeness of the space $\mathcal{D}$ of test functions (in the sense of L. Schwartz's theory of distributions).

In [Bee85, TvD88] it is shown that both Constructive Recursive Mathematics CRM and Brouwerian Intuitionism INT allow one to prove that all functions between complete separable metric spaces are continuous and thus, in particular, also $\mathsf{BD}\text{-}\mathbb{N}$. Both CRM and INT are extensions of BISH postulating a classically inacceptable principle together with a classically valid principle stronger than BISH. From this point of view it "fits into the pattern" that BISH is in need of a further classically valid principle which exceeds basic constructivism (as represented e.g. by HA^ω or CZF) but which is still sufficiently constructive in nature. Ishihara's $\mathsf{BD}\text{-}\mathbb{N}$ is a natural candidate for such a principle since it is equivalent to each of the most desirable principles 1)-6) above and, moreover, constructively plausible since it holds both in

- number realizability combined with truth
- function realizability combined with truth

The reason is that number realizability validates CRM, function realizability validates INT and $\mathsf{BD}\text{-}\mathbb{N}$ is classically valid and thus preserved when combining these realizability interpretations with truth (see e.g. [Tro99]).

The aim of this note is to present in detail some very natural realizability models refuting $\mathsf{BD}\text{-}\mathbb{N}$ but validating even intuitionistic Zermelo Fraenkel set theory IZF. These models have been sketched in section 2.3 of the first author's PhD Thesis [Lie04]. The presentation there has been found moderately accessible by constructive mathematicians with little background in categorical logic. The current note is intended to make the result more widely accessible by reducing categorical logic to the bare minimum.

In the first section we observe that in presence of number choice $\mathsf{AC}_{0,0}$ a fairly weak continuity principle $\mathsf{CP}_0(\mathbb{N}^+)$ suffices to show (in BISH) that all functions from $\mathbb{N}^\mathbb{N}$ to $\mathbb{N}$ are sequentially continuous. In section 2 we construct various realizability models which validate AC for finite types over $\mathbb{N}$ and $\mathsf{CP}_0(\mathbb{N}^+)$ but nevertheless refute Brouwer's continuity principle claiming that all functions from $\mathbb{N}^\mathbb{N}$ to $\mathbb{N}$ are continuous. Thus, these models will refute $\mathsf{BD}\text{-}\mathbb{N}$ since it entails that all sequentially continuous functions from $\mathbb{N}^\mathbb{N}$ to $\mathbb{N}$ are continuous. We conclude in section 3 with a discussion of related work.

2. Some Theorems in Constructive Mathematics

Let $\mathbb{N}^+$ be the one point compactification of $\mathbb{N}$ consisting of all $\alpha \in 2^\mathbb{N}$ such that $n = m$ whenever $\alpha(n) = \alpha(m) = 1$. Obviously $\mathbb{N}^+$ is a retract of $2^\mathbb{N}$ and thus also of $\mathbb{N}^\mathbb{N}$. Let $\mathsf{CP}_0(\mathbb{N}^+)$ be the principle

$$\forall F\colon \mathbb{N}^{\mathbb{N}^+}.\exists n\colon \mathbb{N}.\forall \alpha\colon \mathbb{N}^+.(\forall k < n.\alpha(k) = 0) \to F(\alpha) = F(0^\infty)$$

where 0^∞ stands for the constant function with value 0. The following theorem is inspired by Prop. 4.4 of [BS03].

Theorem 2.1. *From $\mathsf{CP}_0(\mathbb{N}^+)$ it follows (in* BISH*) using number choice $\mathsf{AC}_{0,0}$ that all functions from $\mathbb{N}^\mathbb{N}$ to $\mathbb{N}$ are sequentially continuous.*

Proof: Suppose $F : \mathbb{N}^\mathbb{N} \to \mathbb{N}$. In order to show that F is sequentially continuous suppose (α_n) is a sequence in $\mathbb{N}^\mathbb{N}$ converging to β. We will show that $\lim\limits_{n\to\infty} F(\alpha_n) = F(\beta)$.

First observe that by $\mathsf{AC}_{0,0}$ there exists $f \in \mathbb{N}^\mathbb{N}$ such that $\alpha_k(n) = \beta(n)$ whenever $k \geq f(n)$. Next we define a functional $H : \mathbb{N}^+ \times \mathbb{N} \to \mathbb{N}^\mathbb{N}$ as follows

$$H(\gamma, k)(n) = \begin{cases} \beta(n) & \text{if } \forall \ell < k.\gamma(\ell) = 0 \\ \alpha_m(n) & \text{if } m < k \text{ and } \gamma(m) = 1 \end{cases}$$

We will show now that there exists a functional $G : \mathbb{N}^+ \to \mathbb{N}^\mathbb{N}$ with $G(\gamma)(n) = \lim\limits_{k\to\infty} H(\gamma, k)(n)$. Let $n \in \mathbb{N}$ and $k_0 = f(n)$. If there exists an $m < k_0$ with $\gamma(m) = 1$ then for all $k \geq k_0$ we have $H(\gamma, k)(n) = \alpha_m(n)$ and thus $\lim\limits_{k\to\infty} H(\gamma, k)(n) = \alpha_m(n)$. On the other hand if $\forall \ell < k_0.\gamma(\ell) = 0$ then for all $k \geq k_0$ we have $H(\gamma, k)(n) = \beta(n)$ or $H(\gamma, k)(n) = \alpha_m(n)$ for some $m \geq k_0$ and thus also $H(\gamma, k)(n) = \alpha_m(n) = \beta(n)$ and accordingly $\lim\limits_{k\to\infty} H(\gamma, k)(n) = \beta(n)$. Thus, we have shown that $\lim\limits_{k\to\infty} H(\gamma, k) = G(\gamma)$ where

$$G(\gamma)(n) = \begin{cases} \beta(n) & \text{if } \forall \ell < f(n).\gamma(\ell) = 0 \\ \alpha_m(n) & \text{if } m < f(n) \text{ and } \gamma(m) = 1 \end{cases}$$

Obviously, we have $G(0^\infty) = \beta$ and $G(0^m10^\infty) = \alpha_m$.

Now applying assumption $\mathsf{CP}(\mathbb{N}^+)$ to the functional $F \circ G : \mathbb{N}^+ \to \mathbb{N}$ we obtain an $n \in \mathbb{N}$ such that $F(G(\gamma)) = F(G(0^\infty)) = F(\beta)$ whenever $\gamma(k) = 0$ for all $k < n$. Thus for all $m \geq n$ we have $F(\alpha_m) = F(G(0^m10^\infty)) = F(\beta)$, i.e. $\lim\limits_{n\to\infty} F(\alpha_n) = F(\beta)$ as desired. □

Let $\mathsf{CP}(\mathbb{N}^\mathbb{N})$ be Brouwer's Continuity principle claiming

$$\forall F\colon \mathbb{N}^\mathbb{N} \to \mathbb{N}.\forall \alpha\colon \mathbb{N}^\mathbb{N}.\exists n\colon \mathbb{N}.\forall \beta \in \bar{\alpha}(n).F(\alpha) = F(\beta)$$

i.e. that all functionals from $\mathbb{N}^\mathbb{N}$ to $\mathbb{N}$ are continuous. The following theorem is a direct consequence of Cor. 9.6.11 in [TvD88].

Theorem 2.2. *In HA^ω with extensionality and axiom of choice for all finite types one can prove $\neg\mathsf{CP}(\mathbb{N}^\mathbb{N})$.*

The following corollary will be crucial subsequently.

Corollary 2.1. *In* HA^ω *with extensionality and axiom of choice the principles* $\mathsf{CP}_0(\mathbb{N}^+)$ *and* $\mathsf{BD}\text{-}\mathbb{N}$ *are inconsistent.*

Proof: Using $\mathsf{BD}\text{-}\mathbb{N}$ from $\mathsf{CP}_0(\mathbb{N}^+)$ one can derive $\mathsf{CP}(\mathbb{N}^\mathbb{N})$. But by Th. 2.2 one can prove $\neg\mathsf{CP}(\mathbb{N}^\mathbb{N})$. □

In the next section we will construct strong and natural realizability models which will validate HA^ω with extensionality and axiom of choice for finite types together with $\mathsf{CP}_0(\mathbb{N}^+)$ and thus refute $\mathsf{BD}\text{-}\mathbb{N}$.

3. Natural Realizability Models Refuting BD-$\mathbb{N}$

A comprehensive account of realizability models can be found in J. van Oosten's book [vOo08]. There it is explained how every partial combinatory algebra (pca) $\mathcal{A}$ – thought of as an untyped model of computation[3] – gives rise to the realizability topos $\mathbf{RT}(\mathcal{A})$ and the *extensional realizability topos* $\mathbf{Ext}(\mathcal{A})$.

The full subcategory of $\mathbf{RT}(\mathcal{A})$ on $\neg\neg$-separated objects is much easier to work with since it is equivalent to the category $\mathbf{Asm}(\mathcal{A})$ of "assemblies" which can be described very briefly as follows. An *assembly* (over $\mathcal{A}$) is a pair $X = \big(|X|, ||\cdot||_X\big)$ where $|X|$ is a set and $||\cdot||_X$ is a function sending elements of $|X|$ to nonempty subsets of $\mathcal{A}$ and a morphism from X to Y is a function $f : |X| \to |Y|$ which is realized by some $e \in \mathcal{A}$, i.e. whenever $a \in ||x||_X$ then $ea{\downarrow}$ and $ea \in ||f(x)||_Y$. For assemblies X and Y we may construct their *exponential* Y^X whose underlying set is $\mathbf{Asm}(\mathcal{A})(X,Y)$ and where $||f||_{Y^X}$ consists of all realizers of f. As explained in [vOo08] in every pca $\mathcal{A}$ one can do arithmetic by associating with every $n \in \mathbb{N}$ an element $\underline{n} \in \mathcal{A}$ in such a way that every partial recursive function gets tracked by some element of $\mathcal{A}$. In particular, the category $\mathbf{Asm}(\mathcal{A})$ hosts a natural numbers object also denoted by $\mathbb{N}$ whose underlying set is $\mathbb{N}$ and where $||n||_\mathbb{N} = \{\underline{n}\}$. The underlying set of $\mathbb{N}^\mathbb{N}$ consists of functions from $\mathbb{N}$ to $\mathbb{N}$ and contains at least all total recursive functions.[4] The one point compactification $\mathbb{N}^+$ of $\mathbb{N}$ will be the regular[5] subobject of $\mathbb{N}^\mathbb{N}$ whose underlying set consists of all functions from $\mathbb{N}$ to 2 which assume the value 1 at most once. The category $\mathbf{Asm}(\mathcal{A})$ has a lot of other useful properties. In particular, it gives rise to a model of (even impredicative) Martin-Löf type theory as described in e.g. [Str91, Jac99].

[3]Typical examples of pca's are the first and second Kleene algebra corresponding to Kleene's number and function realizability, respectively. The first Kleene algebra $\mathcal{K}_1$ is given by the set $\mathbb{N}$ on which application is defined as $nm \simeq \{n\}(m)$ where $\{n\}$ is the partial recursive function with Gödel number n. The second Kleene algebra $\mathcal{K}_2$ is given by the set $\mathbb{N}^\mathbb{N}$ on which application $\alpha\beta$ is defined as $\{\alpha\}(\beta)$ where $\{\alpha\} : \mathbb{N}^\mathbb{N} \rightharpoonup \mathbb{N}^\mathbb{N}$ is the partial continuous functional as represented by the *neighbourhood function* α as conceived by L. Brouwer (see [TvD88] and [vOo08] for background information).

[4]In case of $\mathcal{K}_1$ it contains precisely the total recursive functions and in case of $\mathcal{K}_2$ is contains all functions from $\mathbb{N}$ to $\mathbb{N}$.

[5]meaning that it inherits realizability from $\mathbb{N}^\mathbb{N}$

The topos $\mathbf{RT}(\mathcal{A})$ is obtained from $\mathbf{Asm}(\mathcal{A})$ by "adding quotients" and $\mathbf{Ext}(\mathcal{A})$ is obtained from $\mathbf{Asm}(\mathcal{A})$ via the so-called "setoid construction", i.e. adding quotients of "proof-relevant" equivalence relations. For the purposes of this paper it is enough to consider the full subcategory of $\mathbf{Ext}(\mathcal{A})$ on $\neg\neg$-separated objects which is equivalent to the much simpler category $\mathbf{ExtAsm}(\mathcal{A})$. The latter differs from $\mathbf{Asm}(\mathcal{A})$ only in the following respects: $||\cdot||_X$ sends elements of $|X|$ to nonempty *partial equivalence relations* on $\mathcal{A}$ which have to be respected by realizers e of morphims $f : X \to Y$, i.e. whenever $a\,||x||_X\,b$ then ea and eb are both defined and $ea\,||f(x)||_Y\,eb$.

Notice that all realizability and extensional realizability toposes also interpret IZF (see e.g. [vOo08]).

For readers who can't make any sense of the previous paragraph and don't want to get too deeply into [vOo08] we can offer the following alternative view. The $\mathbf{Ext}(\mathcal{A})$ model of type theory can be seen as a generalization of Beeson's realizability model for type theory (see section 20 of Ch. XI of [Bee85]) which is based on the first Kleene algebra $\mathcal{K}_1$ to the arbitrary pca $\mathcal{A}$. The $\mathbf{RT}(\mathcal{P}\omega)$ model of HA^ω coincides with the one defined and studied in sections 4.9-4.14 in Chapter 9 of [TvD88] where it is also shown that it validates extensionality and choice for all finite types.[6] The same applies to $\mathbf{RT}(\mathbb{N}{\rightharpoonup}\mathbb{N})$ since $\mathbb{N}{\rightharpoonup}\mathbb{N}$ is a coherently complete countably algebraic domain containing all other such domains as retracts (see [Plo78]).

Thus all extensional realizability toposes $\mathbf{Ext}(\mathcal{A})$ and the domain realizability toposes $\mathbf{RT}(\mathcal{P}\omega)$ and $\mathbf{RT}(\mathbb{N}{\rightharpoonup}\mathbb{N})$ gives rise to models of extensional HA^ω satisfying AC at all finite types.

We will now show that $\mathbf{Ext}(\mathcal{K}_1)$ and $\mathbf{Ext}(\mathcal{K}_2)$ and the domain realizability toposes $\mathbf{RT}(\mathcal{P}\omega)$ and $\mathbf{RT}(\mathbb{N}{\rightharpoonup}\mathbb{N})$ validate $\mathsf{CP}_0(\mathbb{N}^+)$ and thus by Cor. 2.1 refute the boundedness principle BD-$\mathbb{N}$.

Theorem 3.1. *Both* $\mathbf{Ext}(\mathcal{K}_1)$ *and* $\mathbf{Ext}(\mathcal{K}_2)$ *validate* $\mathsf{CP}_0(\mathbb{N}^+)$ *and thus refute* BD-$\mathbb{N}$*.*

Proof: The finite types over $\mathbb{N}$ in $\mathbf{RT}(\mathcal{K}_1)$ coincide with the hereditary effective operations (see [TvD88]). By the Kreisel-Lacombe-Shoenfield theorem there exists an $m \in \mathbb{N}$ which for every e realizing a type 2 functional F over $\mathbb{N}$ and every Gödel number n of a total recursive function f the computation men terminates and for all total recursive functions g with $\forall k < men.f(k) = g(k)$ it holds that $F(f) = F(g)$, i.e. men provides a modulus of continuity for F at f. From m one obtains a Gödel number $\widetilde{m}$ of an algorithm which for every e realizing $F : \mathbb{N}^\mathbb{N} \to \mathbb{N}$ gives rise to a number $\widetilde{m}e$ which is the least n such that $F(0^\infty) = F(\alpha)$ for all $\alpha \in \mathbb{N}^+$ with $\alpha(k) = 0$ for $k < n$.[7] Apparently $\widetilde{m}$ realizes

[6] The reason why choice holds for finite types is that for every finite type σ the realizers for an element $a \in [\![\sigma]\!]$ are closed under union (see 4.11 of Chapter 9 of [TvD88]).

[7] One uses m to compute a in general not minimal n' such that $F(0^\infty) = F(\alpha)$ for all $\alpha \in \mathbb{N}^+$ with $\alpha(k) = 0$ for $k < n'$. Since there are only finitely many $\alpha \in \mathbb{N}^+$ with $\alpha(k) = 0$ for $k < n'$ and one can decide effectively whether $F(\alpha)$ coincides with $F(0^\infty)$ one can effectively determine the minimal n from n'. This explains how one gets $\widetilde{m}$ effectively from m.

a functional $M : \mathbb{N}^{\mathbb{N}^+} \to \mathbb{N}$ which computes a minimal modulus of continuity of functionals of type $\mathbb{N}^{\mathbb{N}^+}$ at 0^∞. Using this M one easily obtains a realizer for $\mathsf{CP}_0(\mathbb{N}^+)$ in $\mathbf{Ext}(\mathcal{K}_1)$.

For $\mathbf{Ext}(\mathcal{K}_2)$ the argument is similar. One just uses instead of the Kreisel-Lacombe-Shoenfield theorem the fact that one can continuously choose a modulus of continuity at 0^∞ from realizers of type 2 functionals over $\mathbb{N}$. □

Let $\mathsf{CP}(2^\mathbb{N})$ be the principle claiming

$$\forall F \colon 2^\mathbb{N} \to \mathbb{N}. \forall \alpha \colon 2^\mathbb{N}. \exists n \colon \mathbb{N}. \forall \beta \in \bar{\alpha}(n). F(\alpha) = F(\beta)$$

i.e. that all functionals from $2^\mathbb{N}$ to $\mathbb{N}$ are continuous. Since $\mathbb{N}^+$ is a (definable) retract of $2^\mathbb{N}$ it entails continuity of all functions from $\mathbb{N}^+$ to $\mathbb{N}$ and thus in particular $\mathsf{CP}_0(\mathbb{N}^+)$.

Theorem 3.2. *The domain realizability toposes* $\mathbf{RT}(\mathcal{P}\omega)$ *and* $\mathbf{RT}(\mathbb{N} \rightharpoonup \mathbb{N})$ *validate* $\mathsf{CP}(2^\mathbb{N})$ *and thus refute* $\mathsf{BD}\text{-}\mathbb{N}$.

Proof: It is a well known fact that the finite type hierarchy over $\mathbb{N}$ in both toposes coincides with the Kleene-Kreisel continuous functionals (see e.g. [Nor99]). But in the continuous functionals we have the Gandy-Berger functional which computes a modulus of continuity for functionals from $2^\mathbb{N}$ to $\mathbb{N}$ (see e.g. [Nor99]) which can be used for realizing $\mathsf{CP}(2^\mathbb{N})$. Since $\mathsf{CP}(2^\mathbb{N})$ entails $\mathsf{CP}_0(\mathbb{N}^+)$ it follows by Cor. 2.1 that both toposes refute $\mathsf{BD}\text{-}\mathbb{N}$. □

From Prop. 4.4 of [BS03] it follows that $\mathsf{CP}_0(\mathbb{N})$ entails sequential continuity for all functions between complete separable metric spaces. Thus the models exhibited in the previous two theorems validate sequential continuity of all functions between complete separable metric spaces though by Th. 2.2 continuity for such functions doesn't hold in general since our models validate $\neg\mathsf{CP}(\mathbb{N}^\mathbb{N})$. Thus, we have constructed natural models for IZF where continuity is stronger than sequential continuity. Finding such models was the original motivation for our investigations and refutation of $\mathsf{BD}\text{-}\mathbb{N}$ was sort of a byproduct.

In $\mathbf{Ext}(\mathcal{K}_2)$ and the domain realizability models of Th. 3.2 the principle $\mathsf{CP}(2^\mathbb{N})$ holds[8] though they refute $\mathsf{CP}(\mathbb{N}^\mathbb{N})$. Thus continuity for functions between compact complete separable metric spaces does not entail continuity for functions between complete separable metric spaces.

4. Discussion of Related Work

In [B$^+$05] it has been observed that one can use Beeson's *fp*-realizability interpretation for showing that $\mathsf{BD}\text{-}\mathbb{N}$ is not derivable in HA and some of its extensions. In his PhD Thesis M. Beeson has come up with a model refuting

[8]For $\mathbf{Ext}(\mathcal{K}_2)$ this follows from the fact that the finite type hierarchy over $\mathbb{N}$ in $\mathbf{Ext}(\mathcal{K}_2)$ coincides with the continuous functionals.

BD-$\mathbb{N}$ based on his *fp*-realizability introduced for the purpose of showing that the Kreisel-Lacombe-Shoenfield and the Myhill-Shepherdson Theorem are not derivable in $\mathsf{HA} + \mathsf{ECT}_0$ where ECT_0 is "Extended Church's Thesis" (see e.g. [TvD88]). In [B+05] it has been observed that using a few purely mathematical results of H. Ishihara one can prove in $\mathsf{HA}+\mathsf{ECT}_0$ that BD-$\mathbb{N}$ is equivalent to the Kreisel-Lacombe-Shoenfield theorem. Thus $\mathsf{HA} + \mathsf{ECT}_0$ does not prove BD-$\mathbb{N}$.

It is well known (see e.g. [Bee75] that in $\mathsf{HA}+\mathsf{ECT}_0+\mathsf{MP}$ (where MP stands for Markov's principle) one can prove the Kreisel-Lacombe-Shoenfield theorem. Our realizability models, however, do validate MP and are thus essentially different. Moreover, Beeson's *fp*-realizability interpretation is based on a *formal provability* predicate Pr whereas our realizability models are purely semantical and make no reference to formal provability, i.e. syntax.

We can adapt our extensional realizability models with Kreisel's modified realizability (see [vOo08] for a semantical account of the latter) and thus obtain natural syntax-free models which refute both BD-$\mathbb{N}$ and MP.

In recent unpublished work [Lub10] R. Lubarsky has constructed a topological model of IZF which refutes BD-$\mathbb{N}$ in a very strong sense. He considers a topological space T whose underlying set are the bounded sequences in $\mathbb{N}^{\mathbb{N}}$. In $\mathbf{Sh}(T)$ he exhibits a subset B of $\mathbb{N}$ which is pseudobounded but not bounded in the internal logic of $\mathbf{Sh}(T)$. For models based on realizability it is not known whether such a set B exists. They rather fail to validate BD-$\mathbb{N}$ due to a *lack of uniformity*, i.e. one cannot compute in a uniform way a bound for a set from a realizer for its pseudoboundedness. However, in contrast to our realizability models Lubarsky's topological model does not validate number choice which is usually considered as an intrinsic part of Bishop style constructive mathematics.

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