Forcing for IZF in Sheaf Toposes

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To Mamuka at the Occasion of his 50th Birthday

Abstract

In [Sco] D. Scott has shown how the interpretation of intuitionistic set theory IZF in presheaf toposes can be reformulated in a more concrete fashion à la forcing as known to set theorists. In this note we show how this can be adapted to the more general case of Grothendieck toposes dealt with abstractly in [Fou, Hay].

1 Introduction

Intuitionistic Zermelo Fraenkel set theory IZF was introduced by H. Friedman in the early '70s [Fri]. It is obtained from Zermelo Fraenkel set theory ZF by weakening classical to intuitionistic logic, replacing the regularity axiom by (the classically equivalent principle of) induction over $\in$ and strengthening replacement to collection.

In the '80s M. Fourman [Fou] and later but independently S. Hayashi [Hay] have shown how to interpret IZF in Grothendieck toposes, i.e. sheaf toposes $\mathcal{E} = \mathsf{Sh}(\mathbb{C}, \mathcal{J})$ where $\mathcal{J}$ is a Grothendieck topology on a small category $\mathbb{C}$. Employing the fact that $\mathcal{E}$ has all (small) colimits one may simulate the von Neumann hierarchy by defining by external transfinite recursion over the class On of ordinals a sequence $\big(\mathcal{P}^\alpha(0)\big)_{\alpha \in \mathsf{On}}$ where

$$\mathcal{P}^0(0) = 0 \quad \mathcal{P}^{\alpha+1}(0) = \mathcal{P}(\mathcal{P}^\alpha(0)) \quad \mathcal{P}^\lambda(0) = \mathrm{colim}_{\alpha<\lambda}\mathcal{P}^\alpha(0) \;\; (\lambda \text{ limit ordinal})$$

together with embeddings $\mathsf{i}^{(\mathcal{E})}_{\alpha,\beta} : \mathcal{P}^\alpha(0) \rightarrowtail \mathcal{P}^\beta(0)$ where $\mathsf{i}^{(\mathcal{E})}_{\alpha+1,\alpha+2} = \mathcal{P}(\mathsf{i}^{(\mathcal{E})}_{\alpha,\alpha+1})$ (here $\mathcal{P}$ is the *covariant* powerset functor of $\mathcal{E}$) and $\Big(\mathsf{i}^{(\mathcal{E})}_{\alpha,\lambda}\Big)_{\alpha<\lambda}$ is the colimiting cocone for limit ordinals λ.

Alas, for interpreting the language of set theory one is forced to consider an extended language containing constants V_α for the $\mathcal{P}^\alpha(0)$ and constants

$\in_\alpha$ for the subobjects of $\mathcal{P}^\alpha(0)\times\mathcal{P}^{\alpha+1}(0) = \mathcal{P}^\alpha(0)\times\mathcal{P}(\mathcal{P}^\alpha(0))$ classified by the respective evaluation maps $\mathcal{P}^\alpha(0)\times\mathcal{P}(\mathcal{P}^\alpha(0)) \to \Omega$ in $\mathcal{E}$.

Moreover, one needs auxiliary interpretations where free variables are restricted to V_α with α varying over all ordinals. Quantification over the universe is then interpreted as

$$[\![\exists x.\varphi]\!] = \bigvee_{\alpha\in\mathsf{On}} [\![\exists x \in V_\alpha.\varphi]\!] \qquad \text{and} \qquad [\![\forall x.\varphi]\!] = \bigwedge_{\alpha\in\mathsf{On}} [\![\forall x \in V_\alpha.\varphi]\!]$$

exploiting the fact that subobjects lattices in $\mathcal{E}$ are small and complete. Notice, however, that this latter problem can be avoided by considering the *class valued* sheaf $V = \mathrm{colim}_{\alpha\in\mathsf{On}}\mathcal{P}^\alpha(0)$. An even more attractive[1] solution would be to assume a Grothendieck universe U in our metatheory and to assume that $\mathbb{C}$ and $\mathcal{J}$ are elements of U.

Even under the latter ammendment the interpretation of [Fou, Hay] is fairly complicated and much more difficult to work with than the interpretations of IZF in Heyting-valued and realizability models which can be formulated in a way very similar to forcing as known from set theory (see e.g. [Gra] and [McC]). The aim of this note is to present a similarly convenient formulation of the interpretation of IZF in sheaf toposes $\mathsf{Sh}(\mathbb{C},\mathcal{J})$.

2 Forcing for IZF in Presheaf Toposes

For the case of presheaf toposes $\widehat{\mathbb{C}} = \mathbf{Set}^{\mathbb{C}^{\mathsf{op}}}$ such a simplification was found by D. Scott already back in the '70s as mentioned at the end of Fourman's paper [Fou]. Scott presented his result in talks (see [Sco]) but never published it. The most accessible source describing Scott's account is N. Gambino's paper [Gam] where he shows that Scott's result works also for the predicative set theory CZF when working in a predicative metatheory. We recall here Scott's interpretation since we need it as a basis for our generalisation to sheaf toposes in the next section.

Let $\mathbb{C}$ be a small category and $\widehat{\mathbb{C}} = \mathbf{Set}^{\mathbb{C}^{\mathsf{op}}}$ the category of presheaves over $\mathbb{C}$. We write $\mathsf{y} : \mathbb{C} \to \widehat{\mathbb{C}}$ for the Yoneda embedding. Scott's forcing formulation of the interpretation exploits the fact that in $\widehat{\mathbb{C}}$ the colimit of a sequence of inclusions[2] is given by componentwise union. For this reason the maps $\mathsf{i}_{\alpha,\beta} : \mathcal{P}^\alpha(0) \to \mathcal{P}^\beta(0)$ from the previous section are all inclusions and, therefore, we may construct $\mathrm{colim}_{\alpha\in\mathsf{On}}\mathcal{P}^\alpha(0)$ as the (componentwise) union

$$V^{(\widehat{\mathbb{C}})} = \bigcup_{\alpha\in\mathsf{On}} V_\alpha^{(\widehat{\mathbb{C}})}$$

where $V_0^{(\widehat{\mathbb{C}})} = 0$, $V_{\alpha+1}^{(\widehat{\mathbb{C}})} = \mathcal{P}(V_\alpha^{(\widehat{\mathbb{C}})})$ and $V_\lambda^{(\widehat{\mathbb{C}})} = \bigcup_{\alpha<\lambda} V_\alpha^{(\widehat{\mathbb{C}})}$ for limit ordinals

[1] since a Grothendieck universe has much better closure properties than the collection of all classes

[2] A morphism $i : A \to B$ in $\widehat{\mathbb{C}}$ is an inclusion iff for all objects I in $\mathbb{C}$ the map $i_I : A(I) \to B(I)$ is an inclusion, i.e. $i_I(x) = x$ for all $x \in A(I)$.

λ. Using the fact that in $\widehat{\mathbb{C}}$ power objects are constructed as $\mathcal{P}(A)(I) = \mathrm{Sub}_{\widehat{\mathbb{C}}}(\mathsf{y}(I)\times A)$ one may consider $V^{(\widehat{\mathbb{C}})}$ as inductively defined by the rules

$$a \in V^{(\widehat{\mathbb{C}})}(I) \quad \text{iff} \quad a \text{ is a set-valued subpresheaf of } \mathsf{y}(I)\times V^{(\widehat{\mathbb{C}})}$$

for objects I of $\mathbb{C}$. If $a \in V^{(\widehat{\mathbb{C}})}(I)$ and $u : J \to I$ is a morphism in $\mathbb{C}$ then we write $a{\cdot}u$ for the subobject of $\mathsf{y}(J)\times V^{(\widehat{\mathbb{C}})}$ where $\langle v, c\rangle \in a{\cdot}u$ iff $\langle uv, c\rangle \in a$. Notice that due to the inductive construction of $V^{(\widehat{\mathbb{C}})}$ within class-valued presheaves we have $V^{(\widehat{\mathbb{C}})} = \mathcal{P}(V^{(\widehat{\mathbb{C}})})$ where for a class-valued presheaf A the powerclass $\mathcal{P}(A)$ at stage I consist of all set-valued subpresheaves of $\mathsf{y}(I)\times A$.[3] This allows one to interpret sethood as

$$\in \rightarrowtail V^{(\widehat{\mathbb{C}})} \times \mathcal{P}(V^{(\widehat{\mathbb{C}})}) = V^{(\widehat{\mathbb{C}})} \times V^{(\widehat{\mathbb{C}})}$$

Equality will be interpreted as usual in presheaves. Writing out in detail the Kripke-Joyal semantics (see [MLM]) of this interpretation in presheaves over $\mathbb{C}$ gives rise to the following *forcing* clauses as on finds them in [Sco, Gam]

$I \Vdash a \in b$	iff	$\langle id_I, a\rangle \in b$
$I \Vdash a = b$	iff	$a = b$
$I \Vdash \bot$	never holds	
$I \Vdash (\phi\wedge\psi)(\vec{c})$	iff	$I \Vdash \phi(\vec{c})$ and $I \Vdash \psi(\vec{c})$
$I \Vdash (\phi\to\psi)(\vec{c})$	iff	for all $u : J\to I$ from $J \Vdash \phi(\vec{c}{\cdot}u)$ it follows that $J \Vdash \psi(\vec{c}{\cdot}u)$
$I \Vdash (\phi\vee\psi)(\vec{c})$	iff	$I \Vdash \phi(\vec{c})$ or $I \Vdash \psi(\vec{c})$
$I \Vdash \forall x.\phi(x, \vec{c})$	iff	$J \Vdash \phi(a, \vec{c}{\cdot}u)$ for all $u : J\to I$ and $a \in V^{(\widehat{\mathbb{C}})}(J)$
$I \Vdash \exists x.\phi(x, \vec{c})$	iff	$I \Vdash \phi(a, \vec{c})$ for some $a \in V^{(\widehat{\mathbb{C}})}(I)$.

3 Forcing for IZF in Sheaf Toposes

Now we will extend Scott's forcing formulation to the interpretation of set theory in sheaf toposes $\mathcal{E} = \mathsf{Sh}(\mathbb{C}, \mathcal{J})$ where $\mathcal{J}$ is a Grothendieck topology on the small category $\mathbb{C}$. The main obstacle is that colimits of transfinite chains of inclusions[4] are not simply given by (componentwise) union but rather by such unions followed by sheafification $\mathsf{a} : \widehat{\mathbb{C}} \to \mathsf{Sh}(\mathbb{C}, \mathcal{J})$ left adjoint to the inclusion $\mathsf{i} : \mathsf{Sh}(\mathbb{C}, \mathcal{J}) \hookrightarrow \widehat{\mathbb{C}}$ and that the reflection map $\eta_X : X \to \mathsf{a}(X)$ in general cannot be understood as an inclusion (see e.g. [MLM] for construction of sheafification by the "double plus construction").

[3] In analogy with set theory based on classes where for a class A the class $\mathcal{P}(A)$ consists of all sub*sets* a of A.

[4] Again a map between sheaves is an inclusion iff it is an inclusion in $\widehat{\mathbb{C}}$, i.e. iff all its components are inclusions in the set-theoretic sense.

In order to overcome this problem we will show how to obtain the model in $\mathcal{E}$ as a quotient of the model in $\widehat{\mathbb{C}}$. For this purpose we will construct by transfinite recursion a family of morphisms $\mathbf{e}_\alpha : V_\alpha^{(\widehat{\mathbb{C}})} \to V_\alpha^{(\mathcal{E})}$ where $V^{(\widehat{\mathbb{C}})}$ and $V^{(\mathcal{E})}$ refer to the cumulative hierachies in the Grothendieck toposes $\widehat{\mathbb{C}}$ and $\mathcal{E}$, respectively. This family $(\mathbf{e}_\alpha)_{\alpha \in \mathsf{On}}$ will satisfy the following conditions

(1) for all $\alpha \leq \beta$ the diagram

$$\begin{array}{ccc} V_\alpha^{(\widehat{\mathbb{C}})} & \xrightarrow{\mathbf{e}_\alpha} & V_\alpha^{(\mathcal{E})} \\ {\scriptstyle \mathsf{i}_{\alpha,\beta}^{(\widehat{\mathbb{C}})}}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{i}_{\alpha,\beta}^{(\mathcal{E})}} \\ V_\beta^{(\widehat{\mathbb{C}})} & \xrightarrow[\mathbf{e}_\beta]{} & V_\alpha^{(\mathcal{E})} \end{array}$$

commutes

(2) all $\mathbf{e}_\alpha$ are all *dense* w.r.t. the topology $\mathcal{J}$, i.e. all $\mathsf{a}(\mathbf{e}_\alpha)$ are epic in $\mathcal{E}$

(3) for successor ordinals α the map $\mathbf{e}_\alpha : V_\alpha^{(\widehat{\mathbb{C}})} \to V_\alpha^{(\mathcal{E})}$ is epic in $\widehat{\mathbb{C}}$.

Let $V^{(\widehat{\mathbb{C}})}$ and $V^{(\mathcal{E})}$ be the colimits of the $V_\alpha^{(\widehat{\mathbb{C}})}$ and $V_\alpha^{(\mathcal{E})}$ in $\widehat{\mathbb{C}}$ and $\mathcal{E}$, respectively, and $\mathbf{e} : V^{(\widehat{\mathbb{C}})} \to V^{(\mathcal{E})}$ the unique mediating arrow between them, i.e.

$$\begin{array}{ccc} V_\alpha^{(\widehat{\mathbb{C}})} & \xrightarrow{\mathbf{e}_\alpha} & V_\alpha^{(\mathcal{E})} \\ {\scriptstyle \mathsf{i}_{\alpha}^{(\widehat{\mathbb{C}})}}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{i}_{\alpha}^{(\mathcal{E})}} \\ V^{(\widehat{\mathbb{C}})} & \xrightarrow[\mathbf{e}]{} & V^{(\mathcal{E})} \end{array}$$

commutes in $\widehat{\mathbb{C}}$ for all ordinals α (where the i_α denote the components of the respective colimiting cones). The following proposition is the key to our extension of Scott's forcing formulation to sheaf toposes.

Proposition 3.1 *The map* $\mathbf{e} : V^{(\widehat{\mathbb{C}})} \to V^{(\mathcal{E})}$ *is an epimorphism in* $\widehat{\mathbb{C}}$.

Proof: Since all $\mathsf{i}_{\alpha,\beta}^{(\mathcal{E})} : V_\alpha^{(\mathcal{E})} \to V_\beta^{(\mathcal{E})}$ are monomorphisms between $\mathcal{J}$-separated objects (actually $\mathcal{J}$-sheaves) the colimit of the $V_\alpha^{(\mathcal{E})}$ in $\widehat{\mathbb{C}}$ gives rise to a $\mathcal{J}$-separated object $\widetilde{V}$. Thus (see e.g. [MLM]) its sheafification is obtained as $\widetilde{V}^+$. For every $\mathcal{J}$-cover $S \hookrightarrow \mathsf{y}(I)$ every generalized element $f : S \to \widetilde{V}$ factors[5]

[5]For every $u \in S$ the element $f(u)$ appears already in some $V_{\alpha_u}^{(\mathcal{E})}$. Since S is a set by the axiom of replacement the collection $\{\alpha_u \mid u \in S\}$ is also a set and thus bounded by some ordinal α. Therefore, the map $f : S \to \widetilde{V}$ factors through $V_\alpha^{(\mathcal{E})}$.

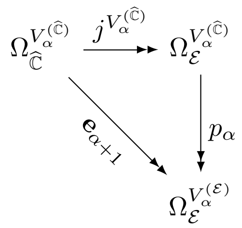
through some $V_\alpha^{(\mathcal{E})}$ which is a sheaf and thus $f : S \to \widetilde{V}$ appears as restriction of some $\overline{f} : \mathsf{y}(I) \to \widetilde{V}$. From this it follows that $\widetilde{V}^+$ is isomorphic to $\widetilde{V}$ via the reflection map $\eta_{\widetilde{V}} : \widetilde{V} \to \widetilde{V}^+$. Thus, we have shown that the colimit of the $V_\alpha^{(\mathcal{E})}$ in $\mathcal{E}$ coincides with their colimit in $\widehat{\mathbb{C}}$, i.e. $\left(\mathsf{i}_\alpha^{(\mathcal{E})}\right)_{\alpha \in \mathsf{On}}$ is a colimiting cone in $\widehat{\mathbb{C}}$.

For showing that $\mathbf{e} : V^{(\widehat{\mathbb{C}})} \to V^{(\mathcal{E})}$ is epic in $\widehat{\mathbb{C}}$ suppose $x \in V^{(\mathcal{E})}(I)$. Then we have $x = \mathsf{i}_\alpha^{(\mathcal{E})}(x')$ for some $x' \in V_\alpha^{(\mathcal{E})}(I)$ where α is a successor ordinal. Since $\mathbf{e}_\alpha$ is epic in $\widehat{\mathbb{C}}$ there exists a $y \in V_\alpha^{(\widehat{\mathbb{C}})}$ with $\mathbf{e}_\alpha(y) = x'$. Thus, since $\mathsf{i}_\alpha^{(\mathcal{E})} \circ \mathbf{e}_\alpha = \mathbf{e} \circ \mathsf{i}_\alpha^{(\widehat{\mathbb{C}})}$ we have $x = \mathsf{i}_\alpha^{(\mathcal{E})}(\mathbf{e}_\alpha(y)) = \mathbf{e}(\mathsf{i}_\alpha^{(\widehat{\mathbb{C}})}(y))$. □

Let us now give the construction of an appropriate family $(\mathbf{e}_\alpha)_{\alpha \in \mathsf{On}}$. There is a unique map $\mathbf{e}_0 : V_0^{(\widehat{\mathbb{C}})} \to V_0^{(\mathcal{E})}$ since $V_0^{(\widehat{\mathbb{C}})}$ is initial in $\widehat{\mathbb{C}}$ and obviously $\mathbf{e}_0$ is dense w.r.t $\mathcal{J}$. For limit ordinals λ condition (1) suggests to take for $\mathbf{e}_\lambda$ the unique mediating arrow from the colimiting cocone $\left(\mathsf{i}_{\alpha,\lambda}^{(\widehat{\mathbb{C}})}\right)_{\alpha<\lambda}$ to the cone $\left(\mathsf{i}_{\alpha,\lambda}^{(\mathcal{E})} \circ \mathbf{e}_\alpha\right)_{\alpha<\lambda}$. This map $\mathbf{e}_\lambda$ is dense w.r.t. $\mathcal{J}$ since the sheafification functor a preserves colimits. Now let us consider the case of a successor ordinal $\alpha+1$. By induction hypothesis the map $\mathbf{e}_\alpha : V_\alpha^{(\widehat{\mathbb{C}})} \to V_\alpha^{(\mathcal{E})}$ is dense w.r.t. $\mathcal{J}$ and, therefore, the map $\Omega_{\mathcal{E}}^{\mathbf{e}_\alpha} : \Omega_{\mathcal{E}}^{V_\alpha^{(\mathcal{E})}} \to \Omega_{\mathcal{E}}^{V_\alpha^{(\widehat{\mathbb{C}})}}$ is monic. Actually, the map $\Omega^{\mathbf{e}_\alpha}$ is split monic as exhibited by the map $p_\alpha : \Omega_{\mathcal{E}}^{V_\alpha^{(\widehat{\mathbb{C}})}} \to \Omega_{\mathcal{E}}^{V_\alpha^{(\mathcal{E})}}$ defined as

$$p_\alpha(P)(x) \equiv \exists y{:}V_\alpha^{(\widehat{\mathbb{C}})}.P(y) \wedge \mathbf{e}_\alpha(y) = x$$

employing the internal language of $\widehat{\mathbb{C}}$. Let $j : \Omega_{\widehat{\mathbb{C}}} \to \Omega_{\widehat{\mathbb{C}}}$ be the local operator corresponding to $\mathcal{J}$ (see e.g. [MLM, Joh] for explanation of these notions and their properties). Then $\Omega_{\mathcal{E}}$ is the image of j and the epi/mono factorisation of j exhibits $\Omega_{\mathcal{E}}$ as a retract of $\Omega_{\widehat{\mathbb{C}}}$. Writing (also) $j : \Omega_{\widehat{\mathbb{C}}} \twoheadrightarrow \Omega_{\mathcal{E}}$ for the epi part of this retraction we observe that $j^{V_\alpha^{(\widehat{\mathbb{C}})}} : \Omega_{\widehat{\mathbb{C}}}^{V_\alpha^{(\widehat{\mathbb{C}})}} \to \Omega_{\mathcal{E}}^{V_\alpha^{(\widehat{\mathbb{C}})}}$ is also split epic (since all functors preserve split epis!). Now we define $\mathbf{e}_{\alpha+1}$ as



and conclude that it is a split epi since it arises as a composition of split epis.

Now it is a routine task to verify that the so defined family $(\mathbf{e}_\alpha)_{\alpha \in \mathsf{On}}$ satisfies the conditions (1)-(3). Moreover, we have the following

Proposition 3.2 *The map* $\mathbf{p} : \mathcal{P}(V^{(\widehat{\mathbb{C}})}) \to \mathcal{P}(V^{(\mathcal{E})})$ *defined as*

$$\mathbf{p}(P)(x) \equiv j\left(\exists y{:}V^{(\widehat{\mathbb{C}})}.P(y) \wedge \mathbf{e}(y) = x\right)$$

makes the diagram

$$\begin{array}{ccc} V^{(\widehat{\mathbb{C}})} & = & \mathcal{P}(V^{(\widehat{\mathbb{C}})}) \\ \mathbf{e}\downarrow & & \downarrow\mathbf{p} \\ V^{(\mathcal{E})} & \xrightarrow{\cong} & \mathcal{P}(V^{(\mathcal{E})}) \end{array}$$

commute.

Let $\mathbf{s} : \mathcal{P}(V^{(\mathcal{E})}) \to \mathcal{P}(V^{(\widehat{\mathbb{C}})})$ *with* $\mathbf{s}(P)(y) = i(P(\mathbf{e}(y)))$ *where* $i : \Omega_{\mathcal{E}} \hookrightarrow \Omega_{\widehat{\mathbb{C}}}$ *is the mono part of the epi/mono factorisation of* j*. Then* $\mathbf{p} \circ \mathbf{s} = id_{\mathcal{P}(V^{(\mathcal{E})})}$.

Proof: The first claim is immediate from Proposition 3.1 and the construction of the $\mathbf{e}_{\alpha+1}$ from the p_α. The second claim follows from the fact that $p_\alpha \circ \Omega_{\mathcal{E}}^{\mathbf{e}_\alpha} = id_{\Omega_{\mathcal{E}}^{V_\alpha^{(\mathcal{E})}}}$ holds for all ordinals α and $j \circ i = \Omega_{\mathcal{E}}$. $\Box$

Propositions 3.1 and 3.2 allow us to reformulate the interpretation of the language of set theory in $V^{(\mathcal{E})}$ in terms of $V^{(\widehat{\mathbb{C}})}$, namely by pulling back along $\mathbf{e} \times \mathbf{e}$. More elementarily, this may be expressed by the following forcing clauses for elementhood and equality

$I \Vdash a \in b$ iff for some $\mathcal{J}$-cover $(u_j : I_j \to I)_{j \in J}$ for all $j \in J$ there exists a $c \in V^{(\widehat{\mathbb{C}})}(I_j)$ with $\langle u_j, c\rangle \in b$ and $I_j \Vdash c = a{\cdot}u_j$

$I \Vdash a = b$ iff for all $u : J \to I$ and $c \in V^{(\widehat{\mathbb{C}})}(J)$ it holds that
$\langle u, c\rangle \in a$ implies $J \Vdash c \in b{\cdot}u$ and
$\langle u, c\rangle \in b$ implies $J \Vdash c \in a{\cdot}u$

employing an implicit transfinite recursion on the rank of a and b.

The clauses for conjunction, implication and universal quantification are as in the previous section. For the remaining connectives and the existential quantifier the forcing clauses have to be adapted as follows

$I \Vdash \bot$ iff the empty cover of I is in $\mathcal{J}$

$I \Vdash (\phi \vee \psi)(\vec{c})$ iff there exists a cover $(u_j : I_j \to I)_{j \in J}$ such that $I_j \Vdash \phi(\vec{c}{\cdot}u_j)$ or $I_j \Vdash \phi(\vec{c}{\cdot}u_j)$ for all $j \in J$

$I \Vdash \exists x.\phi(x, \vec{c})$ iff there exists a cover $(u_j : I_j \to I)_{j \in J}$ such that for all $j \in J$ there exists $a \in V^{(\widehat{\mathbb{C}})}(I_j)$ with $I_j \Vdash \phi(a, \vec{c}{\cdot}u_j)$

as follows from the Kripke-Joyal semantics for Grothendieck toposes (see [MLM]).

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