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1 Solution to Exercise 2.3.7 (p.41) in S. Bosch's *Algebra*

Exercise

Let $R = \mathbb{Z}[X, Y]/(XY^2)$. Obviously, R is a commutative ring with 1 which, however, is not an integral domain. The elements X and $X + XY$ of R generate the same principal ideals in R but there is no unit element E of R such that $X + XY = EX$.

Solution :

We have $(1 + Y)X = X + XY$ and $(1 - Y)(X + XY) = (1 - Y)(1 + Y)X = (1 - Y^2)X = X - XY^2 = X$ from which it follows that $(X) = (X + XY)$, i.e., that X and $X + XY$ generate the same principal ideal in R .

Recall that for P in R the annihilator of P is defined as the ideal $\text{Anh}(P) = \{Q \in R \mid QP = 0\}$. If E is a unit in R then $\text{Anh}(P) = \text{Anh}(EP)$ because $QEP = 0$ iff $QP = 0$.

Suppose E is a unit in R with $X + XY = EX$. Then by the above observation we have $\text{Anh}(X) = \text{Anh}(X + XY)$. Obviously $\text{Anh}(X) = \{PY^2 \mid P \in \mathbb{Z}[Y]\}$. As $EX = X + XY = (1 + Y)X$ we have

$$E \in (1 + Y) + \text{Anh}(X)$$

and as $E^{-1}(X + XY) = X = (1 - Y)(X + XY)$ we have

$$E^{-1} \in (1 - Y) + \text{Anh}(X + XY) = (1 - Y) + \text{Anh}(X)$$

from which it follows that both E and E^{-1} are elements of $\mathbb{Z}[Y] \subseteq R$ because $1 + Y$ and $1 - Y$ are elements of $\mathbb{Z}[Y]$ and $\text{Anh}(X)$ and $\text{Anh}(X + XY)$ are subsets of $\mathbb{Z}[Y]$. But in $\mathbb{Z}[Y]$ the only units are 1 and -1 which both are not in $(1 + Y) + \text{Anh}(X)$. Thus E cannot be a unit in contradiction to our assumption. Thus, we have shown that there is no unit E in R with $EX = X + XY$.

2 Some remarks around Algebraic K -theory

2.1 The "Eilenberg Swindle"

Let R be a commutative ring with unit 1. A **countably generated projective** R -module is a module P such that $P \oplus Q \cong R^\infty := \bigoplus_{n \in \mathbb{N}} R$ for some R -module Q . If P and Q are R -modules with $P \oplus Q \cong R^\infty$ then

$$R^\infty = (P \oplus Q) \oplus \dots (P \oplus Q) \oplus \dots \cong P \oplus (Q \oplus P) \oplus \dots (Q \oplus P) \oplus \dots \cong P \oplus R^\infty$$

from which it follows that the Grothendieck group of the semigroup of isomorphism classes of countably generated R -modules vanishes (as for all countably generated projective R -modules P and Q we have $P \oplus R^\infty \cong R^\infty \cong Q \oplus R^\infty$).

2.2 Grothendieck ring and Picard group

Let R be a commutative ring with 1. Let \mathcal{M}_R be the category of R -modules and \mathbb{P}_R be the category of finitely generated projective R -modules. The Grothendieck ring $K_0(R)$ is obtained by freely adding negatives to the rig of isomorphism classes of \mathbb{P}_R w.r.t. \oplus and \otimes . The Picard group $\text{Pic}(R)$ is $K_0(R)^\times$, i.e. the abelian group of multiplicative units of $K_0(R)$. Thus $\text{Pic}(R)$ consists of isomorphism classes of modules $X \in \mathbb{P}_R$ such that $X \otimes Y \cong R$ for some $Y \in \mathbb{P}_R$. Notice that such a Y is necessarily isomorphic to $\text{Hom}_R(X, R)$, the module of R -linear maps from X to R . Alternatively, the elements of the Picard group of R can be characterized as those R -modules M where for every prime ideal \mathfrak{p} in R the module $R_{\mathfrak{p}} \otimes_R M$ over $R_{\mathfrak{p}}$ has dimension 1 (the so-called “algebraic line bundles”).

For more details see Ch. Weibel’s *Introduction to Algebraic K-theory* (book in preparation).

3 Zariski Schemes

Let R be a commutative ring with 1. The Zariski *spectrum* $\text{Spec}(R)$ consists of all prime ideals in R with the topology generated by the basic opens

$$O_f = \{\mathfrak{p} \in \text{Spec}(R) \mid f \notin \mathfrak{p}\}$$

with $f \in R$. We consider the sheaf \mathcal{O}_R over $\text{Spec}(R)$ with

$$\mathcal{O}_R(O_f) = R_f$$

where R_f is obtained from R by freely inverting all elements of the form f^n . The corresponding étale space is $\coprod_{\mathfrak{p} \in \text{Spec}(R)} R_{\mathfrak{p}}$, where $R_{\mathfrak{p}}$ is obtained by freely inverting all elements in $R \setminus \mathfrak{p}$, endowed with the least topology making all maps

$$\sigma_g^f : O_g \rightarrow \coprod_{\mathfrak{p} \in \text{Spec}(R)} R_{\mathfrak{p}} : \mathfrak{p} \mapsto \frac{f}{g} \in R_{\mathfrak{p}} \quad (f, g \in R)$$

continuous.

One thinks of $R_{\mathfrak{p}}$ as the local ring of germs of rational functions at point \mathfrak{p} . Its residue field is isomorphic to the one obtained from the integral domain R/\mathfrak{p} by freely inverting all elements different from 0.

4 Lebesgue Integral

Let $\overline{\mathbb{R}}_+$ be $\{x \in \mathbb{R} \mid x \geq 0\} \cup \{\infty\}$ and $\mathcal{B}(\overline{\mathbb{R}}_+)$ the least σ -algebra containing all elements of the form $[x, \infty]$ with $x \geq 0$.

Let (X, \mathcal{A}, μ) be a measure space (i.e. \mathcal{A} is a σ -algebra on X and $\mu : \mathcal{A} \rightarrow \overline{\mathbb{R}}_+$ is a measure). A function $f : X \rightarrow \overline{\mathbb{R}}_+$ is called measurable if f^{-1} restricts to a map from $\mathcal{B}(\overline{\mathbb{R}}_+)$ to \mathcal{A} . Let \mathcal{E} be the set of all functions of the form $e = \lambda_1 \chi_{A_1} + \dots + \lambda_n \chi_{A_n}$ where $\lambda_1, \dots, \lambda_n > 0$ and $A_i \in \mathcal{A}$. We define $\int e \, d\mu$ as $\lambda_1 \mu(A_1) + \dots + \lambda_n \mu(A_n)$. One can show that every measurable function $f : X \rightarrow \overline{\mathbb{R}}_+$ is the pointwise supremum of a monotonic sequence of functions in \mathcal{E} . Moreover, whenever (e_n) is a monotonic sequence in \mathcal{E} and $e' \in \mathcal{E}$ with $e' \leq \sup e_n$ then $\int e' \, d\mu \leq \sup \int e_n \, d\mu$. Thus, when defining for $f = \sup e_n$ its integral $\int f \, d\mu$ as $\sup \int e_n \, d\mu$ this definition is independent of the choice of (e_n) . Thus, *a posteriori* we have

$$\int f \, d\mu = \sup \left\{ \int e \, d\mu \mid e \in \mathcal{E}, e \leq f \right\}$$

for all measurable $f : X \rightarrow \overline{\mathbb{R}}_+$.

For $f : X \rightarrow \overline{\mathbb{R}}$ (where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$) we define $f^+ : X \rightarrow \overline{\mathbb{R}}_+$ as $f^+(x) = \max(f(x), 0)$ and $f^- = (-f)^+$. We say that f is *integrable* (w.r.t. μ) iff $\int f^+ \, d\mu, \int f^- \, d\mu < \infty$ and define $\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu$. This makes also sense if $\int f^+ \, d\mu < \infty$ or $\int f^- \, d\mu < \infty$ in which case we say that f is *improperly integrable* (w.r.t. μ).

For details and more information see H. Bauer *Wahrscheinlichkeitstheorie und Grundzüge der Maßtheorie*.

5 Convenient Vector Spaces

(of A. Frölicher and A. Kriegl)

Let (X, \mathcal{B}) be a bornological vector space, i.e. X is a vector space, \mathcal{B} is a bornology¹ on X and vector addition and scalar multiplication are bornological maps, i.e. preserve \mathcal{B} under direct image.²

A filter \mathcal{F} in X is *Mackey convergent* to $x \in X$ (notation $\mathcal{F} \xrightarrow{M} x$) iff there exists $B \in \mathcal{B}$ such that for every $A \in \mathcal{F}$ there exists an open neighbourhood U of 0 in \mathbb{R} such that $U \cdot B \subseteq A - x$, i.e. there exists $\varepsilon > 0$ such that for all $\lambda \in \mathbb{R}$ with $|\lambda| < \varepsilon$ the set $x + \lambda \cdot B \subseteq A$.

A *Cauchy filter* in X is a filter \mathcal{F} in X such that the filter generated by $\{A - A \mid A \in \mathcal{F}\}$ is Mackey convergent to 0 (where $A - A = \{x - y \mid x, y \in A\}$).

A bornological vector space (X, \mathcal{B}) is *(Cauchy) complete* iff every Cauchy filter in X is Mackey convergent.

If X is a vector space and X' is a subspace of the algebraic dual X^* of X then a subset B of X is bounded in the *induced bornology* iff $\phi[B]$ is a bounded subset of \mathbb{R} for every $\phi \in X'$.

A *preconvenient vector space* is a vector space (over \mathbb{R}) together with a subspace X' of the algebraic dual X^* of X such that

1. X' separates points of X , i.e. $x = x'$ whenever $\phi(x) = \phi(x')$ for all $\phi \in X'$
2. every linear map $X \rightarrow \mathbb{R}$ bounded w.r.t. the induced bornology belongs to X' .

A *convenient vector space* is a preconvenient vector space (X, X') such that X is (Cauchy) complete w.r.t. the induced bornology.

Let $X = (X, X')$ be a (pre)convenient vector space. A map $c : \mathbb{R}^n \rightarrow X$ is *smooth* iff $\phi \circ c : \mathbb{R}^n \rightarrow \mathbb{R}$ is smooth for all $\phi \in X'$. For (pre)convenient vector spaces X and Y a map $f : X \rightarrow Y$ is *smooth* iff $f \circ c : \mathbb{R}^n \rightarrow Y$ is smooth (C^∞) for all smooth $c : \mathbb{R}^n \rightarrow X$.

The real numbers \mathbb{R} together with all linear functions $\mathbb{R} \rightarrow \mathbb{R}$ form a convenient vector space. From now on we understand \mathbb{R} as endowed with this structure. Obviously, a map $f : \mathbb{R} \rightarrow \mathbb{R}$ is smooth in this sense iff it is smooth in the ordinary sense. Further one can show that the smooth linear maps from X to \mathbb{R} are precisely the elements of X' (see Frölicher & Kriegl *Linear Spaces and Differentiation Theory* (1988)).

¹i.e. \mathcal{B} is a set of subsets of X s.t.

- (1) every $x \in X$ is element of some $B \in \mathcal{B}$
- (2) if $B' \subseteq B \in \mathcal{B}$ then $B' \in \mathcal{B}$
- (3) \mathcal{B} is closed under finite unions.

²If (X_1, \mathcal{B}_1) and (X_2, \mathcal{B}_2) are bornological spaces then their product has underlying set $X_1 \times X_2$ and $B \subseteq X_1 \times X_2$ is bounded iff $\pi_1[B]$ and $\pi_2[B]$ are elements of \mathcal{B}_1 and \mathcal{B}_2 , respectively.

6 A non-discrete Hausdorff space all whose compact subsets are finite

Let \mathcal{U} a non-principal ultrafilter on \mathbb{N} . For every $n \in \mathbb{N}$ let $\eta(n) = \{a \in \mathcal{P}(\mathbb{N}) \mid n \in a\}$. Recall that the topology on $\beta\mathbb{N}$ is generated by the basic open sets $U_a = \{F \in \beta\mathbb{N} \mid a \in F\}$ with $a \in \mathcal{P}(\mathbb{N})$.

Then the subspace Y of $\beta\mathbb{N}$ on $\eta[\mathbb{N}] \cup \{\mathcal{U}\}$ is a Hausdorff space all whose compact subsets are finite.

Proof: Suppose K is an infinite compact subset of Y . Then $a = \eta^{-1}[K]$ is infinite.

Case $a \in \mathcal{U}$:

Let $a = b \cup c$ such that $b \cap c = \emptyset$ and b and c are both infinite. W.l.o.g. suppose that $b \in \mathcal{U}$. Then we have $K \subseteq U_b \cup \bigcup_{n \in a} U_{\{n\}}$. Since K is compact there exist $n_1, \dots, n_k \in a$ such that $K \subseteq U := U_b \cup U_{\{n_1\}} \cup \dots \cup U_{\{n_k\}}$. Let $m \in c$. Then $\eta(m) \in K$ but $\eta(m) \notin U$ contradicting $K \subseteq U$.

Case $a \notin \mathcal{U}$:

Let $b = \mathbb{N} \setminus a$. We have $K \subseteq U_b \cup \bigcup_{n \in a} U_{\{n\}}$. Since K is compact there exist $n_1, \dots, n_k \in a$ such that $K \subseteq U := U_b \cup U_{\{n_1\}} \cup \dots \cup U_{\{n_k\}}$. Let $m \in a \setminus \{n_1, \dots, n_k\}$. Then we have $\eta(m) \in K$ but $\eta(m) \notin U$ contradicting $K \subseteq U$. \square

7 Inductive dimension of spaces

(originally due to Menger and Urysohn) is defined as follows (see e.g. Engelking's book on topology).

A space X has dimension -1 iff $X = \emptyset$ and for $n \geq 0$ a space X has dimension n iff it does not have a dimension $< n$ and for all $x \in X$ and open neighbourhood U of x there exists an open neighbourhood V of x such that $V \subseteq U$ and $\partial V = \overline{V} \setminus V$ has a dimension $< n$. A space X is infinite dimensional (has dimension ∞) iff X is non-empty and does not have dimension n for any $n \in \mathbb{N}$.

Notice that for regular³ spaces for every open neighbourhood U of x there exists an open neighbourhood V of x with $\overline{V} \subseteq U$ and ∂V has a dimension $< n$. The reason is that by regularity there exist disjoint open sets U_1 and U_2 with $x \in U_1$ and $X \setminus U \subseteq U_2$ and thus one can find an open neighbourhood V of x such that $V \subseteq U_1$ and ∂V has a dimension $< n$ and for such a V it holds that $\overline{V} \cap U_2 = \emptyset$ and thus $\overline{V} \cap (X \setminus U) = \emptyset$, i.e. $\overline{V} \subseteq U$ as desired.

An open set U has empty border ∂U iff the complement of U is open, i.e. iff U is *clopen*. Thus, a space X is 0-dimensional iff its clopen subsets form a basis.

³meaning that for closed sets C and points $x \in X \setminus C$ there exist disjoint open sets U_1 and U_2 with $x \in U_1$ and $C \subseteq U_2$

8 Vietoris and Hit-and-Miss Topology

For a compact Hausdorff space X the *Vietoris topology* on $\mathcal{K}(X)$, the set of compact subsets of X , is generated by the subbasis consisting of all subsets

$$t(U) = \{K \in \mathcal{K}(X) \mid K \subseteq U\} \quad \text{and} \quad m(U) = \{K \in \mathcal{K}(X) \mid K \cap U \neq \emptyset\}$$

where $U \in \mathcal{O}(X)$.

For $U \in \mathcal{O}(X)$ let $s(U) = t(\neg U) = \{K \in \mathcal{K}(X) \mid K \cap U = \emptyset\}$. Obviously, we have $t(U) = \bigcup \{s(V) \mid V \cup U = X\}$ since $K \in \bigcup \{s(V) \mid V \cup U = X\}$ iff there exists a $V \in \mathcal{O}(X)$ with $K \subseteq \mathcal{C}V \subseteq X$ iff $K \subseteq U$ since in any Hausdorff space one may take $V = \mathcal{C}K$.

Thus, for an arbitrary Hausdorff space X one may define subsets of the set $\mathcal{A}(X)$ of all closed subsets of X

$$t(U) = \{K \in \mathcal{A}(X) \mid K \subseteq U\} \quad \text{and} \quad m(U) = \{K \in \mathcal{A}(X) \mid K \cap U \neq \emptyset\}$$

and still it holds that $t(U) = \bigcup \{s(V) \mid V \cup U = X\}$ where $s(U) = t(\neg U)$. The topology generated by the $t(U)$ and $m(U)$ (or equivalently by the $s(U)$ and $m(U)$) is called the Hit-and-Miss topology.

Moreover, for any Hausdorff space X the set $\{\{x\} \mid x \in X\}$ is a closed subset of $\mathcal{A}(X)$ which can be seen as follows. The open set $t(\emptyset) = \{\emptyset\}$ is a neighbourhood of \emptyset containing no singleton sets as elements. If $A \in \mathcal{A}(X)$ containing two distinct elements x and y then for disjoint open neighbourhoods U and V of x and y , respectively, we have $A \in m(U) \cap m(V)$ but $m(U) \cap m(V)$ contains no singleton set (since otherwise $U \cap V$ were nonempty).

9 Vietoris vs. Hausdorff

The motivation for the Vietoris topology is that for compact metric spaces (X, d) it coincides with the topology induced by the Hausdorff metric d_H on $\mathcal{K}(X)$ which is defined as follows: first for $x \in X$ and $A \in \mathcal{K}(X)$ put $d(x, A) = \inf_{y \in A} d(x, y)$ and then

$$d_H(A, B) = \max(\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A))$$

for $A, B \in \mathcal{K}(X)$.

We observe that for $\varepsilon > 0$ we have $d(x, B) < \varepsilon$ if $x \in \mathcal{U}_\varepsilon(B) = \bigcup_{b \in B} \mathcal{U}_\varepsilon(b)$ and thus

$$d_H(A, B) < \varepsilon \quad \text{iff} \quad A \subseteq \mathcal{U}_\varepsilon(B) \text{ and } B \subseteq \mathcal{U}_\varepsilon(A)$$

for $A, B \in \mathcal{K}(X)$.

We first show that the Vietoris topology is finer than the one induced by d_H . For this purpose it suffices to show that for every $A \in \mathcal{K}(X)$ and $\varepsilon > 0$ there exists a Vietoris open V with $A \in V$ and $d_H(A, B) < \varepsilon$ for all $B \in V$. Since $A \subseteq \mathcal{U}_{\frac{\varepsilon}{2}}$ and A is compact there exists $a_1, \dots, a_n \in A$ such that $A \subseteq \mathcal{U}_{\frac{\varepsilon}{2}}(a_1) \cup \dots \cup \mathcal{U}_{\frac{\varepsilon}{2}}(a_n)$. Put $U = \mathcal{U}_\varepsilon(A)$ and $U_i = \mathcal{U}_{\frac{\varepsilon}{2}}(a_i)$ for $i = 1, \dots, n$. We define $V = t(U) \cap m(U_1) \cap \dots \cap m(U_n)$. Obviously $A \in V$. Suppose $B \in V$. Then $B \subseteq U = \mathcal{U}_\varepsilon(A)$. Suppose $x \in A$. Then for some i we have $x \in U_i$, i.e. $d(x, a_i) < \frac{\varepsilon}{2}$. Since $B \in m(U_i)$ there exists a $y \in B$ with $y \in U_i$, i.e. $d(a_i, y) < \frac{\varepsilon}{2}$. Thus $d(x, y) < d(x, a_i) + d(a_i, y) < \varepsilon$, i.e. $x \in \mathcal{U}_\varepsilon(y)$. Thus we have shown $A \subseteq \mathcal{U}_\varepsilon(B)$. This finishes the argument that $d_H(A, B) < \varepsilon$.

Next we show that the topology induced by d_H is finer than the Vietoris topology. For this purpose it suffices to show that for every subbasic open V in the Vietoris topology and $A \in V$ there exists a $\varepsilon > 0$ such that $B \in V$ whenever $d_H(A, B) < \varepsilon$.

Let $U \in \mathcal{O}(X)$ and $A \in t(U)$. i.e. $A \subseteq U$. Then $K = X \setminus U \in \mathcal{K}(X)$. Since $d(-, K) : X \rightarrow \mathbb{R}$ is continuous and A is compact $\varepsilon = \inf_{x \in A} d(x, K) > 0$. Thus $\mathcal{U}_\varepsilon(A) \subseteq U$. Suppose $d_H(A, B) < \varepsilon$. Then $B \subseteq \mathcal{U}_\varepsilon(A) \subseteq U$, i.e. $B \in t(U)$ as desired.

Suppose $A \in m(U)$ where $U \in \mathcal{O}(X)$. Then $A \cap U \neq \emptyset$, i.e. there exists a $x \in A \cap U$. Since $x \in U$ there exists an $\varepsilon > 0$ with $\mathcal{U}_\varepsilon(x) \subseteq U$. Suppose $d_H(A, B) < \varepsilon$, i.e. $A \subseteq \mathcal{U}_\varepsilon(B)$ and $B \subseteq \mathcal{U}_\varepsilon(A)$. Thus there exists a $y \in B$ with $x \in \mathcal{U}_\varepsilon(y)$, i.e. $d(x, y) < \varepsilon$. Thus $y \in \mathcal{U}_\varepsilon(x) \subseteq U$ from which it follows that $B \cap U \neq \emptyset$, i.e. $B \in m(U)$ as desired.

10 Sp is not l.f.p.

An object K of a category \mathcal{C} is *finitely presentable* iff the functor $\mathcal{C}(K, -) : \mathcal{C} \rightarrow \mathbf{Set}$ preserves directed colimits, i.e., more explicitly, for every directed diagram $D : (\cdot, \leq) \rightarrow \mathcal{C}$ and colimiting cone $(c_i : D_i \rightarrow D)_{i \in I}$ and morphism $f : K \rightarrow D$ it holds that

- (i) there is an $i \in I$ and a morphism $g : K \rightarrow D_i$ with $f = c_i g$ and this factorisation is essentially unique in the sense that
- (ii) whenever $c_i g = f = c_i g'$ then $D(i \rightarrow j) \cdot g = D(i \rightarrow j) \cdot g'$ for some $j \geq i$.

A category \mathcal{C} is *locally finitely presentable* (l.f.p.) iff \mathcal{C} is cocomplete and there is a set \mathcal{A} of finitely presentable objects such that every object of \mathcal{C} arises as a directed colimit of objects in \mathcal{A} . Typical example l.f.p. categories are categories of algebraic and/or relational structures.

Non-examples are the category **Cpo** of complete partial orders and Scott continuous maps and the category **Sp** of topological spaces and continuous maps. These facts can be seen as follows.

No non-empty cpo A is finitely presentable. Let $\bar{\omega}$ be the directed colimit of the finite ordinals n , i.e. the natural numbers \mathbb{N} with an additional top element ∞ . Then the constant map $f : A \rightarrow \bar{\omega}$ with value ∞ factors through no finite ordinal n . Thus in **Cpo** there are not enough finitely presentable objects.

In **Sp** non-discrete spaces are not finitely presentable. Suppose A is a space with non-open subset M . Let $D_n = A + \mathbb{N}$ whose proper open subsets are those proper subsets U whose intersection with A is M and whose intersection with \mathbb{N} is cofinite and disjoint from $n = \{0, \dots, n-1\}$. Obviously D_n is a subspace of D_{n+1} and the colimit D of $(D_n)_{n \in \mathbb{N}}$ is the set $A + \mathbb{N}$ with the indiscrete topology. Then the inclusion map $f : A \hookrightarrow D$ is continuous but does not factor through any of the D_n as a continuous map since $M = f^{-1}[M \cup \{k \in \mathbb{N} \mid k \geq n\}]$ is not open in A . Since directed colimits of indiscrete spaces are indiscrete the category **Sp** does not have enough finitely presentable objects.

11 Semantics of Nondeterministic/Probabilistic Computation

The interval $[0, 1]$ is a continuous lattice w.r.t. \leq . For a topological space X let $\mathcal{V}(X)$ be the set of all Scott continuous maps $\mu \in [0, 1]^{\mathcal{O}(X)}$ such that

$$\mu(\emptyset) = 0 \quad \mu(X) = 1 \quad \mu(U \cup V) + \mu(U \cap V) = \mu(U) + \mu(V)$$

for all $U, V \in \mathcal{O}(X)$. Obviously $\mathcal{V}(X)$ is a dcpo w.r.t. the pointwise order (although in general it will not have a least element).

Let S be a countable set (of states). Then $\mathcal{V}(S_{\perp})$ is isomorphic to the pointwise ordered set of maps $\mu : S \rightarrow [0, 1]$ satisfying $\sum_{s \in S} \mu(s) \leq 1$. We write $\mu(\perp)$ for $1 - \sum_{s \in S} \mu(s)$, the probability of divergence.

The direct semantics of a pGCL program P is a function $\llbracket P \rrbracket : S \rightarrow \mathcal{P}(\mathcal{V}(S_{\perp}))$ where \mathcal{P} stands for the Plotkin powerdomain. If P is probabilistic but deterministic(ally chooses the output sub-probability measure on S) then $\llbracket P \rrbracket : S \rightarrow \mathcal{V}(S_{\perp})$. The *predicate transformer* associated with an $f : S \rightarrow \mathcal{P}(\mathcal{V}(S_{\perp}))$ is the continuous function $\mathbf{wp}(f) : [0, 1]^S \rightarrow [0, 1]^S$ with

$$\mathbf{wp}(f)(B)(s) = \bigwedge_{\mu \in f(s)} \int B d\mu$$

for $B : S \rightarrow [0, 1]$ and $s \in S$ (where $\int B d\mu$ stands for $\sum_{s \in S} B(s) \cdot \mu(s)$).

One can show that for a while program **while** b **do** P we have

$$\mathbf{wp}(\llbracket \mathbf{while} \ b \ \mathbf{do} \ P \rrbracket)(B) = \mu(\Phi) = \mu X. (b \wedge \mathbf{wp}(\llbracket P \rrbracket)(X)) \vee (\neg b \wedge B)$$

for all $B \in [0, 1]^S$. However, for verifying while loops by the well established method of loop invariants one would need that the right hand side is a greatest fixpoint. Suppose $I \in [0, 1]^S$ is a postfixpoint of Φ , i.e. $I \leq (b \wedge \mathbf{wp}(\llbracket P \rrbracket)(I)) \vee (\neg b \wedge B)$ which is equivalent to the conjunction of $b \wedge I \leq \mathbf{wp}(\llbracket P \rrbracket)(I)$ and $\neg b \wedge I \leq B$, then $I \leq \nu(\Phi)$ (where $\nu(\Phi)$ is the greatest fixpoint of Φ , i.e. $\bigwedge \{I \mid I \leq \Phi(I)\}$).

For this reason A. McIver and C. Morgan have suggested to consider a *partial correctness* variant of their predicate transformer semantics for pGCL where while loops are interpreted as greatest fixpoints. They suggested to associate with every pGCL program P a predicate transformer $\mathbf{wlp}(P) : [0, 1]^S \rightarrow [0, 1]^S$ which for loops is defined as

$$\mathbf{wlp}(\mathbf{while} \ b \ \mathbf{do} \ P)(B) = \mu(\Phi) = \nu X. (b \wedge \mathbf{wlp}(P)(X)) \vee (\neg b \wedge B)$$

and for all other cases like for \mathbf{wp} . *Formally*, such a procedure is certainly possible but there arises the question whether the formally defined \mathbf{wlp} factors along the direct semantics of programs, i.e. whether we have $\mathbf{wlp}(P) = \mathbf{wlp}(\llbracket P \rrbracket)$ for some appropriately defined map

$$\mathbf{wlp} : [S \rightarrow \mathcal{P}(\mathcal{V}(S_{\perp}))] \rightarrow [[0, 1]^S \rightarrow [0, 1]^S]$$

In order to arrive at this goal we first consider the case of *deterministic nonprobabilistic* programs $f: S \rightarrow S_\perp$. For such f its *weakest liberal precondition* at $B \subseteq S$ is given by

$$\text{wlp}(f)(B) = f^{-1}(B \cup \{\perp\})$$

i.e. $s \in \text{wlp}(f)(B) \Leftrightarrow f(s) \in B \cup \{\perp\} \Leftrightarrow (f(s) \neq \perp \Rightarrow f(s) \in B)$. Obviously $\text{wlp}(f)$ commutes with arbitrary unions and nonempty intersections. Notice, however, that wlp is antitone, i.e. $\text{wlp}(f)(B) \supseteq \text{wlp}(g)(B)$ whenever $f \sqsubseteq g$. Moreover, for every ascending ω -chain $(f_n)_{n \in \mathbb{N}}$ in $[S \rightarrow S_\perp]$ we have

$$\begin{aligned} s \in \text{wlp}(\bigsqcup_{n \in \mathbb{N}} f_n)(B) & \text{ iff } (\exists n \in \mathbb{N}. f_n(s) \in B) \vee (\forall n \in \mathbb{N}. f_n(s) = \perp) \\ & \text{ iff } \forall n \in \mathbb{N}. [(\exists n \in \mathbb{N}. f_n(s) \in B) \vee f_n(s) = \perp] \\ & \text{ iff } \forall n \in \mathbb{N}. [f_n(s) \neq \perp \Rightarrow (\exists n \in \mathbb{N}. f_n(s) \in B)] \\ & \text{ iff } \forall n \in \mathbb{N}. [f_n(s) \neq \perp \Rightarrow f_n(s) \in B] \\ & \text{ iff } \forall n \in \mathbb{N}. s \in \text{wlp}(f_n)(B) \\ & \text{ iff } s \in \bigcap_{n \in \mathbb{N}} \text{wlp}(f_n)(B). \end{aligned}$$

i.e. $\text{wlp}(\bigsqcup_{n \in \mathbb{N}} f_n)$ is the pointwise infimum of the decreasing sequence $\text{wlp}(f_n)$.

Now given $b \subseteq S$ and $f: S \rightarrow S_\perp$ the direct semantics of **while** b **do** f is given by the least fixpoint of $\Psi: [S \rightarrow S_\perp] \rightarrow [S \rightarrow S_\perp]$ with

$$\Psi(g)(s) = \begin{cases} g(f(s)) & \text{if } s \in b \\ s & \text{otherwise} \end{cases}$$

We have to show that for all $B \subseteq S$ it holds that

$$\mu(\Psi)^{-1}(B \cup \{\perp\}) = \nu(\Phi_B)$$

where $\Phi_B: \mathcal{P}(S) \rightarrow \mathcal{P}(S): X \mapsto (b \cap \text{wlp}(f)(X)) \cup (\neg b \cap B)$. For this purpose we show by induction on $n \in \mathbb{N}$ that

$$(\Psi^n(\lambda s. \perp))^{-1}(B \cup \{\perp\}) = \Phi_B^n(S)$$

which suffices since Φ_B preserves infima of decreasing chains. The claim holds for $n = 0$ since $(\lambda s. \perp)^{-1}(B \cup \{\perp\}) = S = \Phi_B^0(S)$. Assume as induction hypothesis that $(\Psi^n(\lambda s. \perp))^{-1}(B \cup \{\perp\}) = \Phi_B^n(S)$. Then we have

$$\begin{aligned} s \in (\Psi^{n+1}(\lambda s. \perp))^{-1}(B \cup \{\perp\}) & \text{ iff} \\ s \in (b \cap f^{-1}((\Psi^n(\lambda s. \perp))^{-1}(B \cup \{\perp\}))) \cup (\neg b \cup B) & \text{ iff (by i.h.)} \\ s \in (b \cap f^{-1}(\Phi_B^n(S))) \cup (\neg b \cap B) & \text{ iff} \\ s \in \Phi_B^{n+1}(S) & \end{aligned}$$

i.e. $(\Psi^{n+1}(\lambda s. \perp))^{-1}(B \cup \{\perp\}) = \Phi_B^{n+1}(S)$ as desired.

Next we consider *deterministic probabilistic* programs $f: S \rightarrow \mathcal{V}(S_\perp)$ for which their *weakest liberal precondition* is given by

$$\begin{aligned} \text{wlp}(f)(B)(s) &= \text{wp}(f)(B) + f(s)(\perp) = \\ &= \sum_{s' \in S} B(s') \cdot f(s)(s') + 1 - \sum_{s' \in S} f(s)(s') = \\ &= 1 + \sum_{s' \in S} (B(s') - 1) \cdot f(s)(s') \end{aligned}$$

The first equality shows that $\text{wlp}(f)$ is continuous (since $\text{wp}(f)$ is continous). The third equality shows that wlp reverses order (because the $B(s') - 1$ are ≤ 0). Suppose $(f_n)_{n \in \mathbb{N}}$ is an increasing sequence in $[S \rightarrow \mathcal{V}(S_\perp)]$. Then for $B \subseteq S$ and $s \in S$ we have

$$\begin{aligned} \text{wlp}(\bigsqcup_{n \in \mathbb{N}} f_n)(B)(s) &= 1 + \sum_{s' \in S} (B(s') - 1) \cdot (\bigsqcup_{n \in \mathbb{N}} f_n)(s)(s') \\ &= 1 + \sum_{s' \in S} (B(s') - 1) \cdot (\sup_{n \in \mathbb{N}} f_n)(s)(s') \\ &= 1 + \sum_{s' \in S} \inf_{n \in \mathbb{N}} (B(s') - 1) \cdot f_n(s)(s') \\ &= \inf_{n \in \mathbb{N}} 1 + \sum_{s' \in S} (B(s') - 1) \cdot f_n(s)(s') \\ &= \inf_{n \in \mathbb{N}} \text{wlp}(f_n)(B)(s) \end{aligned}$$

i.e. $\text{wlp}(\bigsqcup_{n \in \mathbb{N}} f_n)$ is the pointwise infimum of the decreasing sequence $\text{wlp}(f_n)$.

Now given $b \subseteq S$ and $f : S \rightarrow \mathcal{V}(S_\perp)$ the direct semantics of **while** b **do** f is given by the least fixpoint of $\Psi : [S \rightarrow \mathcal{V}(S_\perp)] \rightarrow [S \rightarrow \mathcal{V}(S_\perp)]$ with

$$\Psi(g)(s) = \begin{cases} (f; g)(s) & \text{if } s \in b \\ \eta(s) & \text{otherwise} \end{cases}$$

for $B \in [0, 1]^S$ and $s \in S$ where

$$(f; g)(s)(s') = \sum_{s'' \in S} g(s'')(s') \cdot f(s)(s'')$$

and

$$\eta(s)(s') = \begin{cases} 1 & \text{if } s = s' \\ 0 & \text{otherwise} \end{cases}$$

For $B \in [0, 1]^S$ we have to show that

$$\mu(\Psi)^{-1}(B \cup \{\perp\}) = \nu(\Phi_B)$$

where $\Phi_B : [0, 1]^S \rightarrow [0, 1]^S : X \mapsto (b \wedge \text{wlp}(f)(X)) \vee (-b \wedge B)$. We prove by induction on $n \in \mathbb{N}$ that

$$\text{wlp}(\Psi^n(\lambda s. \lambda s'. 0))(B) = \Phi_B^n(\lambda s. 1)$$

which suffices since wlp sends sups of ω -chains to pointwise infima and Φ_B preserves such infima. For $n = 0$ we have $\text{wlp}(\lambda s. \lambda s'. 0)(B) = \lambda s s. 1$ which establishes the induction basis. Since we have $\text{wlp}(\Psi(g))(B) = \Phi_B(\text{wlp}(g))$ for all $g : S \rightarrow \mathcal{V}(S_\perp)$ it follows from the induction hypothesis that $\text{wlp}(\Psi^{n+1}(\lambda s. \lambda s'. 0))(B) = \text{wlp}(\Psi(\Psi^n(\lambda s. \lambda s'. 0)))(B) = \Phi_B(\text{wlp}(\Psi^n(\lambda s. \lambda s'. 0))) = \Phi_B(\Phi_B^n(\lambda s. 1)) = \Phi_B^{n+1}(\lambda s. 1)$.

Next we consider *nondeterministic probabilistic* programs $f : S \rightarrow \mathcal{P}(\mathcal{V}(S_\perp))$ where \mathcal{P} is the Plotkin powerdomain. For $\mu \in \mathcal{V}(S_\perp)$ let $\widehat{\mu} : [0, 1]^S \rightarrow [0, 1]$ be defined as $\widehat{\mu}(B) = 1 + \sum_{s \in S} (1 - B(s)) \cdot \mu(s)$. Obviously, the construction $\widehat{(\cdot)}$ is antitonic, i.e. $\widehat{\mu}_1 \geq \widehat{\mu}_2$ whenever $\mu_1 \leq \mu_2$. For this reason it makes sense to define $\text{wlp} : [S \rightarrow \mathcal{P}(\mathcal{V}(S_\perp))] \rightarrow [[0, 1]^S \rightarrow [0, 1]^S]$ as

$$\text{wlp}(f)(B)(s) = \inf_{\mu \in f(s)} \widehat{\mu}(B)$$

11.1 An explicit construction of the extension of $f : A \rightarrow \mathcal{P}\mathcal{V}B$ to $\widehat{f} : \mathcal{V}A \rightarrow \mathcal{P}\mathcal{V}B$

Let $f : A \rightarrow \mathcal{P}\mathcal{V}B$ be continuous. We write $\Pi_c(f)$ for the set of continuous maps $g : A \rightarrow \mathcal{V}B$ with $g(a) \in f(a)$ for all $a \in A$. Under reasonable assumptions about A and B it holds that $\Pi_c(f)$ is a compact saturated subset of $[A \rightarrow \mathcal{V}B]$. Then, for $\mu \in \mathcal{V}A$ the set

$$\widetilde{f}(\mu) := \{\lambda V \in \mathcal{O}(B). \int g(x)(V) d\mu(x) \mid g \in \Pi_c(f)\}$$

is a compact subset of $\mathcal{V}B$ since it arises as continuous image of the compact set $\Pi_c(f)$. Since $\widetilde{f}(\mu)$ is compact its upward closure $\uparrow \widetilde{f}(\mu)$ is an element of $\mathcal{P}\mathcal{V}B$. Thus we may define $\widehat{f}(\mu)$ as $\uparrow \widetilde{f}(\mu)$ or more explicitly as $\widehat{f}(\mu) = \uparrow \{\lambda V \in \mathcal{O}(B). \int g(x)(V) d\mu(x) \mid g \in \Pi_c(f)\}$ One readily checks that the so defined \widehat{f} is continuous and linear. Moreover, for simple valuations $\mu = \sum \lambda_i \cdot \eta_{a_i}$ we have $\widehat{f}(\mu) = \uparrow \{\lambda V. \sum \lambda_i \cdot g_i(V) \mid g \in \Pi f(a_i)\}$ Thus \widehat{f} is the unique continuous extension of the (continuous) function $\sum \lambda_i \cdot \eta_{a_i} \mapsto \uparrow \{\lambda V. \sum \lambda_i \cdot g_i(V) \mid g \in \Pi f(a_i)\}$ on simple valuations which is the way how it was constructed – slightly more implicitly – by Keimel and Plotkin.

Lemma 11.1 *It remains to show that for a continuous domain A , a coherent continuous domain B and a continuous map $f : A \rightarrow \mathcal{P}(B)$ the set $\Pi_c(f) = \{g \in [A \rightarrow B] \mid \forall a \in A. g(a) \in f(a)\}$ is a compact subset of $[A \rightarrow B]$.*

Proof: Since B was assumed to be a coherent continuous domain it is compact w.r.t. the Lawson topology thus $\prod_{a \in |A|} B$ is also compact w.r.t. the Lawson topology.

Consider the diagram of pullbacks

$$\begin{array}{ccccc} \Pi_c(f) & \hookrightarrow & \Pi_m(f) & \hookrightarrow & \prod_{a \in |A|} f(a) \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ [A \rightarrow B] & \hookrightarrow & [A \rightarrow_m B] & \hookrightarrow & \prod_{a \in |A|} B \end{array}$$

where obviously $\Pi_m(f) = \{g \in [A \rightarrow_m B] \mid \forall a \in A. g(a) \in f(a)\}$. Notice that both $\Pi_m(f)$ and $\prod_{a \in |A|} f(a)$ are closed subspaces of $\prod_{a \in |A|} B$ w.r.t. the Lawson topology. Thus $\Pi_m(f)$ is a compact subset of $[A \rightarrow_m B]$ w.r.t. the Lawson topology and thus *a fortiori* w.r.t. the Scott topology.⁴

Next we show that there exists a continuous retraction r on the domain $[A \rightarrow_m B]$ such that

⁴We still have to check that the Lawson topology on $[A \rightarrow_m B]$ coincides with the subspace topology inherited from $\prod_{a \in |A|} B$.

- (1) $[A \rightarrow B]$ is the image of r and
- (2) r preserves $\Pi_m(f)$.

For $g \in [A \rightarrow_m B]$ and $x \in A$ we put $r(g)(x) = \bigsqcup_{y \ll x} g(y)$. Obviously $r(g)$ is monotonic. We next show that $r(g)$ is continuous. Suppose $X \subseteq A$ is directed. Then we have

$$r(g)(\bigsqcup D) = \bigsqcup_{y \ll x} r(g)(y) = \bigsqcup_{x \in X} \bigsqcup_{y \ll x} g(y) = \bigsqcup_{x \in X} r(g)(y)$$

since by definition of the way-below-relation \ll we have $y \ll \bigsqcup X$ iff $y \ll x$ for some $x \in X$. If g is already continuous then $g(x) = g(\bigsqcup_{y \ll x} y) = r(g)(y)$ for all $x \in A$, i.e. $g = r(g)$. Thus $r(g)$ is the greatest continuous function $h \sqsubseteq g$. Finally, we show that r preserves $\Pi(f)$. Suppose $g \in \Pi_m(f)$ and $x \in A$. Then

$$r(g)(x) = \bigsqcup_{y \ll x} g(y) \in \bigcap_{y \ll x} f(y) = \bigsqcup_{y \ll x} f(y) = f(x)$$

since directed suprema in $\mathcal{PV}(B)$ are given by intersection and all $g(y) \in f(y)$ due to assumption $g \in \Pi_m(f)$.

Thus $\Pi_c(f)$ is the image of the compact set $\Pi_m(f)$ under the continuous map r and, accordingly, the set $[A \rightarrow B]$ is a compact subset of $[A \rightarrow B]$ as desired. \square

Recap on Lawson topology. If X is a dcpo then a subset C of X is Scott closed iff C is downward closed and closed under suprema of directed subsets (taken in X). The Lawson topology on X is the least topology whose closed sets contain all Scott closed sets and all sets of the form $\uparrow x$ for $x \in X$. Thus, the Lawson closed subsets of X are precisely those sets which can be obtained as arbitrary intersections of sets of the form $C \cup \uparrow x_1 \cup \dots \cup \uparrow x_n$ where C is a Scott closed subset of X and the x_i are elements of X . Sets of this form are called *basic Lawson closed sets*.

The Lawson topology on $\prod_{a \in |A|} B$ is generated by the sets of the form $\prod_{a \in |A|} C_a$ with the C_a Scott closed subsets of B and the closed sets of the form $\uparrow f$ with $f \in \prod_{a \in |A|} B$. Thus, for showing that the Lawson topology on $[A \rightarrow_m B]$ coincides with the subspace topology induced by the inclusion into $\prod_{a \in |A|} B$ it suffices to show that

- (1) the Scott topology on $[A \rightarrow_m B]$ is generated by closed sets of the form $\prod_{a \in |A|} C_a$ with the C_a Scott closed subsets of B and
- (2) for every $f \in \prod_{a \in |A|} B$ the set $\uparrow f \cap [A \rightarrow_m B]$ is Lawson closed in $[A \rightarrow_m B]$.

However, if A is some countable discrete poset of states and $B = \mathcal{V}(S)$ (as is the case when modelling pGCL) then $[S \rightarrow \mathcal{V}(S)]$ carries the product topology (of $\prod_S \mathcal{V}(S)$) anyway both for the Scott and the Lawson case. Now if $f : S \rightarrow \mathcal{P}(\mathcal{V}(S))$ then $\prod_{s \in S} f(s)$ is a closed subspace of $\prod_S \mathcal{V}(S)$. Thus $\prod_{s \in S} f(s)$ is compact in the Lawson topology and thus *a fortiori* in the Scott topology.

11.2 Minkowski Duality (à la Keimel and Plotkin) gives rise to a palatable description of Kleisli Composition

In their paper on “Predicate Transformers for Convex Powerdomains” Keimel and Plotkin (2007) have made use of a certain “duality” between $\mathcal{P}_U(C)$, the convex compact saturated subsets of a d -cone C , and $[C^*, \mathbb{R}_+]$, the set of continuous superlinear⁵ functions from C^* to \mathbb{R} where $C^* = \mathbb{R}_+^C$, the d -cone of continuous linear functionals from C to \mathbb{R}_+ . This duality is established by sending $Q \in \mathcal{P}_U(C)$ to the functional

$$\min_Q : C^* \rightarrow \mathbb{R}_+ : f \mapsto \inf_{x \in Q} f(x)$$

and $F \in [C^*, \mathbb{R}_+]$ to the convex compact saturated subset

$$Q_F = \{x \in C \mid \eta(x) \sqsupseteq F\} = \{x \in C \mid \forall f \in C^*. f(x) \geq F(f)\}$$

This extends to the “bounded” case where \mathbb{R}_+ is replaced by $\mathcal{I} = [0, 1]$ at least for $C = \mathcal{V}(S)$. In this case $\mathcal{V}(S)^* = \mathcal{E}(S) = \mathcal{I}^S$ where $\langle A \mid \mu \rangle = \sum_{s \in S} A(s)\mu(s)$. Then $\mathcal{P}_U(\mathcal{V}(S))$ gets identified with the set $[\mathcal{E}(S), \mathcal{I}]$ of (continuous) functions $F : \mathcal{E}(S) \rightarrow \mathcal{I}$ such that for all $A < B \in \mathcal{E}(S)$ and $a, b \geq 0$ with $a + b \leq 1$ it holds that

- (1) $F(aA + bB) \geq aF(A) + bF(B)$
- (2) $F(aA + \underline{b}) \leq aF(A) + b$

where \underline{b} is the constant function with value b . A $Q \in \mathcal{P}_U(\mathcal{V}(S))$ again is sent to \min_Q defined as above and an $F \in [\mathcal{E}(S), \mathcal{I}]$ is sent to

$$Q_F = \{\mu \in \mathcal{V}(S) \mid \forall A \in \mathcal{E}(S). \langle A \mid \mu \rangle \geq F(A)\}$$

since $\eta(\mu)(A) = \langle A \mid \mu \rangle$.

This identification of $\mathcal{P}_U(\mathcal{V}(S))$ with $[\mathcal{E}(S), \mathcal{I}]$ allows one to define for $f : S \rightarrow \mathcal{P}_U(\mathcal{V}(S))$ its associated predicate transformer $\mathbf{wp}(f) \in [\mathcal{E}(S), \mathcal{E}(S)]$ as

$$\mathbf{wp}(f)(B)(s) = f(s)(B)$$

reminding one of the correspondence between $X \rightarrow R^{R^Y}$ and $R^Y \rightarrow R^X$.

As one certainly wants \mathbf{wp} to preserve composition this requirement allows one to reconstruct the appropriate (Kleisli) composition for maps $S \rightarrow \mathcal{P}_U(\mathcal{V}(S))$ as follows. Let $f, g : S \rightarrow \mathcal{P}_U(\mathcal{V}(S))$. Then the Kleisli composition $f;g$ should satisfy $\mathbf{wp}(f;g) = \mathbf{wp}(f) \circ \mathbf{wp}(g)$ from which it follows that

$$\begin{aligned} (f;g)(s)(C) &= \mathbf{wp}(f;g)(C)(s) \\ &= \mathbf{wp}(f)(\mathbf{wp}(g)(C))(s) \\ &= f(s)(\mathbf{wp}(g)(C)) \\ &= f(s)(\lambda s. g(s)(C)) \end{aligned}$$

⁵ F is “superlinear” iff $F(af + bg) \geq aF(f) + bF(g)$ for all $f, g \in C^*$ and $a, b \geq 0$.

which is a most palatable version of composition avoiding lifting twice as is necessary when one works with $\mathcal{P}_U(\mathcal{V}(S))$ instead of the isomorphic $[\mathcal{E}(S), \mathcal{I}]$.

Analogously, a map $g : S \rightarrow [\mathcal{E}(S), \mathcal{I}]$ lifts to $g^\dagger : [\mathcal{E}(S), \mathcal{I}] \rightarrow [\mathcal{E}(S), \mathcal{I}]$ sending F to $g^\dagger(F)(C) = F(\lambda s, g(s)(C))$.

Now given $f : S \rightarrow \mathcal{P}(\mathcal{V}(S))$ this gives rise to $g : S \rightarrow [\mathcal{E}(S), \mathcal{I}] : s \mapsto \min_{f(s)}$ which lifts to $g^\dagger : [\mathcal{E}(S), \mathcal{I}] \rightarrow [\mathcal{E}(S), \mathcal{I}]$ with $g^\dagger(F)(B) = F(\lambda s, \inf_{\mu \in f(s)} \langle B | \mu \rangle)$. Now we may define $f^\dagger : \mathcal{P}_U(\mathcal{V}(S)) \rightarrow \mathcal{P}_U(\mathcal{V}(S))$ as $f^\dagger = \min \circ g^\dagger \circ \min^{-1}$. For $Q \in \mathcal{P}_U(\mathcal{V}(S))$ we have

$$g^\dagger(\min_Q)(B) = \min_Q(\lambda s, \inf_{\mu \in f(s)} \langle B | \mu \rangle) = \inf_{\nu \in Q} \int \inf_{\mu \in f(s)} \langle B | \mu \rangle d\nu(s)$$

from which it follows that

$$\mu \in f^\dagger(Q) \quad \text{iff} \quad \forall B \in \mathcal{E}(S). \quad \inf_{\nu \in Q} \int \inf_{\mu \in f(s)} \langle B | \mu \rangle d\nu(s) \leq \langle B | \mu \rangle$$

Thus, in the particular case where $Q = \uparrow\mu$ we have $\mu' \in f^\dagger(\uparrow\mu)$ iff

$$\forall B \in \mathcal{E}(S). \quad \int \inf_{\nu \in f(s)} \langle B | \nu \rangle d\mu(s) \leq \langle B | \mu' \rangle \quad (\dagger)$$

which is equivalent to

$$\mu' \in \uparrow \{ \lambda t. \int g(s)(t) d\mu(s) \mid g \in \prod(f) \} \quad (\ddagger)$$

as claimed by A. Rosenbusch.

Proof: Clearly (\ddagger) implies (\dagger) . For the reverse direction suppose (\dagger) holds for μ' . Let $t \in S$. Let $B = \chi_{\{t\}}$. Since $f(s)$ is compact there exists $g_t(s) \in f(s)$ with $\inf_{\nu \in f(s)} \nu(t) = g_t(s)(t)$. Then (\dagger) entails that $\int g_t(s)(t) d\mu(s) \leq \mu'(t)$. Let $r_t = \frac{\mu(t)}{\sum_{t \in S} \mu(t)}$. We may assume w.l.o.g. that $\sum_{t \in S} \mu(t) > 0$ since otherwise (\ddagger) is trivially true. Now let $g(s) = \sum_{t \in S} r_t \cdot g_t$. Since $f(s)$ is convex closed (w.r.t. the Euclidean topology) it follows that $g(s) = \sum_{t \in S} r_t \cdot g_t(s) \in f(s)$. Then for arbitrary $t' \in S$ we have $\int g(s)(t') d\mu(s) = \int \sum_{t \in S} r_t \cdot g_t(s)(t') d\mu(s) = \sum_{t \in S} r_t \cdot \int g_t(s)(t') d\mu(s) \leq \sum_{t \in S} r_t \cdot \mu'(t') = \mu'(t')$, i.e. (\ddagger) . \square

It remains to show that

- (1) $f^\dagger : \mathcal{P}_U(\mathcal{V}(S)) \rightarrow \mathcal{P}_U(\mathcal{V}(S))$ preserves sub-convex combinations and
- (2) $f^\dagger(\uparrow\eta(s)) = f(s)$ for all $s \in S$.

The isomorphism $\min : \mathcal{P}_U(\mathcal{V}(S)) \rightarrow [\mathcal{E}(S), \mathcal{I}]$ preserves sub-convex combinations (which are defined pointwise on $[\mathcal{E}(S), \mathcal{I}]$). Thus it suffices to show that g^\dagger preserves sub-convex combinations which, however, is immediate from the definition of g^\dagger . Thus (1) holds.

For (2) we argue as follows

$$\begin{aligned}
\mu' \in f^\dagger(\uparrow\eta(s)) & \quad \text{iff} \\
\forall B \in \mathcal{E}(S). \int \inf_{\nu \in f(t)} \langle B \mid \nu \rangle d\eta(s)(t) & \leq \langle B \mid \mu' \rangle \quad \text{iff} \\
\forall B \in \mathcal{E}(S). \inf_{\nu \in f(s)} \langle B \mid \mu' \rangle & \quad \text{iff} \\
\mu' \in f(s) &
\end{aligned}$$

where the last equivalence follows from \min being an isomorphism.

11.3 Quintessence of Keimel and Plotkin (2007)

Let T be some monad on the category \mathcal{C} of d -cones, as e.g. the upper convex powercone monad. For cones C and D we write $[C, D]$ for the cone of linear continuous maps from C to D . For every cone C there is a morphism

$$\Lambda : TC \rightarrow T\overline{\mathbb{R}}_+^{[C, T\overline{\mathbb{R}}_+]} : \gamma \mapsto (f \mapsto f^\dagger(\gamma))$$

in case of the upper convex powercone monad we have $T\overline{\mathbb{R}}_+ \cong \overline{\mathbb{R}}_+$.

For all cones C and D we have an isomorphism

$$t : [C, T\overline{\mathbb{R}}_+^{[D, T\overline{\mathbb{R}}_+]}] \cong [C, T\overline{\mathbb{R}}_+]^{[D, T\overline{\mathbb{R}}_+]}$$

with $t(m)(f)(x) = m(x)(f)$. Now using this isomorphism t we can define

$$W_{C,D} : [C, TD] \rightarrow [C, T\overline{\mathbb{R}}_+]^{[D, T\overline{\mathbb{R}}_+]} : m \mapsto t(\Lambda \circ m)$$

or more explicitly $W(m)(f)(x) = t(\lambda \circ m)(f)(x) = (\Lambda \circ m)(x)(f) = \Lambda(m(x))(f) = f^\dagger(m(x))$, i.e.

$$W(m)(f) = f^\dagger \circ m$$

which allows one to show that W is functorial w.r.t. the Kleisli composition. For $m = \eta_C$ we have $W(\eta_C)(f) = f^\dagger \circ \eta_C = f$ and for $m \in [C, TD]$ and $n \in [D, TE]$ we have $W(m; n)(f) = f^\dagger \circ (m; n) = f^\dagger \circ n^\dagger \circ m = (f^\dagger \circ n)^\dagger \circ m = W(m)(W(n)(f)) = (W(m) \circ W(n))(f)$ and thus $W(m; n) = W(m) \circ W(n)$ as desired.

12 Derivation of the Left Implication Rule

From hypothetical derivations of the sequents $? \Gamma^{*\perp}, A^*$ and $? \Delta^{*\perp}, ? B^{*\perp}, \Pi^*$ a derivation of $? \Gamma^{*\perp}, ? \Delta^{*\perp}, ? (!A^* \otimes ? B^{*\perp}), \Pi^*$ can be obtained as follows

$$\frac{\frac{\frac{? \Gamma^{*\perp}, A^*}{? \Gamma^{*\perp}, !A^*} ! \quad \frac{}{B^{*\perp}, B^*} \text{(ax)}}{? \Gamma^{*\perp}, !A^* \otimes B^{*\perp}, B^*} \otimes}{? \Gamma^{*\perp}, ? (!A^* \otimes B^{*\perp}), B^*} \text{(deril)}}{\frac{? \Gamma^{*\perp}, ? (!A^* \otimes B^{*\perp}), !B^*}{? \Gamma^{*\perp}, ? \Delta^{*\perp}, ? (!A^* \otimes B^{*\perp}), \Pi^*} ! \quad ? \Delta^{*\perp}, ? B^{*\perp}, \Pi^*} \text{(cut)}$$

where applications of the exchange rule are omitted.

We observe that the translation of a cut-free proof in IL to LL does not directly give rise to a cut-free proof in LL. But, that's not required anyway!

Translating natural deduction proofs in intuitionistic logic to linear logic is easier and details can be found on pp 32–35 of Braüner's survey article.

13 Derivation of $\neg\neg A \rightarrow \neg\neg B \vdash \neg\neg(A \rightarrow B)$

First observe that from $\Gamma, A \vdash B$ it follows that $\Gamma, \neg B \vdash \neg A$. Thus, from $\Gamma, A \vdash B$ it also follows that $\Gamma, \neg\neg A \vdash \neg\neg B$. Now we have

$$\frac{\frac{\frac{A, \neg A \vdash \perp}{A, \neg A \vdash B}}{\neg A \vdash A \rightarrow B} \quad \frac{\frac{A, B \vdash B}{B \vdash A \rightarrow B}}{\neg(A \rightarrow B), B \vdash \perp}}{\frac{\neg(A \rightarrow B), \neg A \vdash \perp \quad \neg(A \rightarrow B), \neg\neg B \vdash \perp}{\neg(A \rightarrow B) \vdash \neg\neg A} \quad \neg(A \rightarrow B), \neg\neg B \vdash \perp} \text{(\rightarrow L)}}{\frac{\neg\neg A \rightarrow \neg\neg B, \neg(A \rightarrow B) \vdash \perp}{\neg\neg A \rightarrow \neg\neg B \vdash \neg\neg(A \rightarrow B)}}$$

14 A Tait calculus for classical logic which is deterministic for its \exists -free fragment

The formulas are built up from literals, i.e. negated and unnegated prime formulas, by \wedge , \forall , \vee and \exists where negation is defined à la de Morgan and involutive on literals. We use P, Q, \dots as metavariables for literals and $A, B, C \dots$ as metavariables for formulas.

There is an axiom

$$\frac{}{\Gamma, P, \neg P} \text{ (ax)}$$

for every literal P and a single rule for every connector and quantor

$$\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} (\wedge) \qquad \frac{\Gamma, A}{\Gamma, \forall x A} (\forall)$$

$$\frac{\Gamma, A, B}{\Gamma, A \vee B} (\vee) \qquad \frac{\Gamma, \exists x A, A[t/x]}{\Gamma, \exists x A} (\exists)$$

where, of course, in (\forall) the variable x is assumed not to occur freely in Γ .

Notice that weakening is built into the formulation of (ax) by allowing arbitrary side contexts. Contraction is built into the formulation of (\exists) by repeating $\exists x A$ in the premiss.

Most importantly, notice that all rules are deterministic with the single exception of the rule for \exists . Thus, from any (cut-free) proof of $\exists x A$ one can easily read off the corresponding Herbrand disjunction.

The cut rule

$$\frac{\Gamma, A \quad \Gamma, \neg A}{\Gamma} \text{ (cut)}$$

can be shown to be admissible.

15 Exponentiation is weaker than Subset Collection

We know that $\Delta \dashv \Gamma : \mathbf{Sh}(\mathbb{R}) \rightarrow \mathbf{Set}$ is a connected, locally connected geometric morphism, i.e. Δ is full and faithful and preserves the locally cartesian closed structure. Moreover, the Dedekind reals \mathbb{R}_D are not in the image of Δ . Thus, if $\mathbf{Sh}(\mathbb{R})$ were a model of \mathbf{CZF}_0 , i.e. Aczel's \mathbf{CZF} without Subset Collection, and \mathbf{Exp} , the exponentiation axiom, whose sets are precisely the objects in the image of Δ then this would provide a model of $\mathbf{CZF}_0 + \mathbf{Exp}$ not validating Subset Collection as otherwise \mathbb{R}_D were a set which it isn't as \mathbb{R}_D is not locally constant.

Let U be a Grothendieck universe in \mathbf{Set} then⁶

$$\mathbf{El} = \Delta(\in_U \dashv \rightarrow U \times U \xrightarrow{\pi_2} U) : E \rightarrow U$$

should give rise to a notion of smallness in the following way: a map $b : B \rightarrow A$ in $\mathbf{Sh}(\mathbb{R})$ is in \mathcal{S}_U , i.e. U -small, iff there is an epi $e : C \rightarrow A$ such that $e^*b \cong f^*\mathbf{El}$ for some $f : C \rightarrow U$.

In 2003 R. Lubarsky has shown via a forcing argument that Exponentiation does not imply Subset Collection !

⁶We don't distinguish syntactically between U and $\Delta(U)$ and write E for $\Delta(\in_U)$.

16 Regular Extension Axiom (REA) and Axiom of Multiple Choice (AMC)

For sets a and b we write $\text{mv}(a, b)$ for the class of total relations from a to b . A set A is called *regular* iff A is inhabited, transitive and for all $a \in A$ and $R \in \text{mv}(a, A)$ there exists a set $b \in A$ such that $(\forall x \in a \exists y \in b \ xRy) \wedge (\forall y \in b \exists x \in a \ xRy)$. We write $\text{Reg}(A)$ to express that A is regular. Now the *Regular Extension Axiom* REA says that

$$\forall a \exists A a \subseteq A \wedge \text{Reg}(A)$$

If X is a set we write $\text{mv}(X)$ for the class of relations R such that $\forall x \in X \exists y \ xRy$. For $R \in \text{mv}(X)$ we say that a set C *covers* R iff $(\forall x \in X \exists y \in C \ xRy) \wedge (\forall y \in C \exists x \in X \ xRy)$. We say that a class \mathcal{Y} is a *cover base* for X iff for every $R \in \text{mv}(X)$ there exists a cover which is an image of a set in \mathcal{Y} . If \mathcal{Y} is a set then we say that \mathcal{Y} is a *small cover base* for X . One can show that

Proposition

\mathcal{Y} is a cover base for X iff for every surjection $e : Z \twoheadrightarrow X$ there exists a surjection $Y \twoheadrightarrow X$ with $Y \in \mathcal{Y}$ that factors through e .

The *Axiom of Multiple Choice* (AMC) says that every set is element of a *collection family*, i.e. a set which is a cover base for all of its elements.

Suppose \mathcal{Y} is a collection family with $X \in \mathcal{Y}$. Let $E = \{e : C \twoheadrightarrow X \mid C \in \mathcal{Y}\}$. Then $(\text{dom}(e))_{e \in E}$ is a collection family. Suppose $e : C \twoheadrightarrow X$ is in E and $e' : Z \twoheadrightarrow C$. Since \mathcal{Y} is a collection family with $C \in \mathcal{Y}$ there is a $C' \in \mathcal{Y}$ and a map $f : C' \rightarrow Z$ with $e'f : C' \twoheadrightarrow C$. Thus also $ee'f : C' \twoheadrightarrow X$, i.e. $ee'f \in E$. Moreover, we have $\text{id}_X \in E$. Thus $(\text{dom}(e))_{e \in E}$ is an inhabited collection family containing X together with the family $(e : \text{dom}(e) \twoheadrightarrow X)_{e \in E}$ of epis to X .

On the other hand suppose $(C_i)_{i \in I}$ is an inhabited collection family together with a family $(e_i : C_i \twoheadrightarrow X)_{i \in I}$ of epis to X . Let $\mathcal{Y} = \{C_i \mid i \in I\} \cup \{X\}$. Then \mathcal{Y} is a collection family since $(C_i)_{i \in I}$ is a collection family and for an arbitrary epi $e : Z \twoheadrightarrow X$ we may consider for some $i \in I$ the pullback

$$\begin{array}{ccc} Z' & \xrightarrow{g} & Z \\ e' \downarrow & \lrcorner & \downarrow e \\ C_i & \xrightarrow{e_i} & X \end{array}$$

for which there exists $f : C_j \rightarrow Z'$ with $e'f : C_j \twoheadrightarrow C_i$ and thus $egf : C_j \twoheadrightarrow X$.

Thus, we have shown that AMC is equivalent to the requirement that for every X there exists an inhabited collection family $(C_i)_{i \in I}$ together with a family $(e_i : C_i \twoheadrightarrow X)_{i \in I}$ of epis to X . It is this latter form in which Moerdijk and Palmgren stated AMC originally.

16.1 TTCA_f as a Weakening of AMC

For the sake of verifying subset collection in the proof-irrelevant version of the Aczel interpretation T. Streicher has identified a weakening of AMC called TTCA_f claiming that every set x is element of a set y which is a cover base for x . This is weaker than AMC which claims that every set x is element of a set y which is a cover base for all elements of y and not just for x .

17 There are no nontrivial sup-lattices in the van den Berg – Lubarsky – Streicher model for CZF + REA

B. van den Berg has shown that the *hereditarily countable* sets within the Friedman – McCarty realizability model $V(K_1)$ for IZF give rise to a model for CZF + REA (REA holds since the universe $\mathbf{Mod}(K_1)$ is closed under W -types). Moreover, since all hereditarily countable sets are *quasimodest*⁷ objects of the effective topos $\mathcal{E}ff$ (which is equivalent to $V(K_1)$) the following *uniformity principle*

$$\text{UP} \quad (\forall p \in \mathcal{P}(1) \exists x \in a \phi(p, x)) \rightarrow \exists x \in a \forall p \in \mathcal{P}(1) \phi(p, x)$$

holds in the hereditarily countable sets of $V(K_1)$.

Th. Coquand has observed that in CZF + REA + UP one can show that all set based \bigvee -lattices are trivial. His argument proceeds as follows.

Suppose there is a set L with a partial order \leq on L which itself is a set and where every subset A of L has a supremum $\bigvee A$. Let $\top = \bigvee L$. Then we have

$$\forall p \in \mathcal{P}(1) \exists a \in A \ a = \bigvee \{\top \mid 0 \in p\}$$

from which it follows by the uniformity principle UP that there is an $a \in A$ with

$$\forall p \in \mathcal{P}(1) \ a = \bigvee \{\top \mid 0 \in p\}$$

Instantiating p as \emptyset and 1 we get from this that

$$\perp = \bigvee \{\top \mid 0 \in \emptyset\} = a = \bigvee \{\top \mid 0 \in 1\} = \top$$

from which it follows that all elements of A are equal, i.e. that (A, \leq) is trivial.

Thus in CZF one cannot prove the existence of nontrivial complete lattices or bounded complete dcpos.

⁷Recall that an object A of $\mathcal{E}ff$ is quasimodest iff A appears as subquotient of the nno N .

18 Infinity Axiom in CZF

In CZF (e.g. the introductory notes of P.Aczel and M. Rathjen) infinity is axiomatized in a somewhat nonstandard way as

$$(\text{Inf}) \quad \exists a (\exists x. x \in a) \wedge (\forall x \in a. \exists y \in a. x \in y)$$

as opposed to the more common infinity axiom stating the existence of a set a with $0 \in a$ and closed under $x \mapsto x^+ = x \cup \{x\}$.

The reason why Inf works nevertheless is somewhat intricate. Let Γ be the monotone class function sending a set x to $\Gamma(x) = \{0\} \cup \{y^+ \mid y \in x\}$. In CZF one may show that there is a class J such that⁸ for all sets a , $J^a = \Gamma(J^{\in a})$ where $J^{\in a} = \bigcup_{x \in a} J^x$. Now one can show that for any set a satisfying $(\exists x. x \in a) \wedge (\forall x \in a. \exists y \in a. x \in y)$ the set J^a is the least fixpoint of Γ w.r.t. set inclusion.

One easily shows by set induction over x that J^x is a subset of any fixpoint of Γ . Thus, it suffices to show that $J^a = \Gamma(J^a)$.

Lemma 18.1 *For all sets x it holds that $J^{\in x} \subseteq J^x$ and thus $J^x \subseteq \Gamma(J^x)$.*

Proof: We prove the first claim by set induction on x . Suppose as IH that $J^{\in y} \subseteq J^y$ for all $y \in x$. Then for all $y \in x$ we have $J^y \stackrel{\text{IH}}{\subseteq} \Gamma(J^y) \subseteq \Gamma(J^{\in x}) = J^x$ and thus $J^{\in x} \subseteq J^x$.

Thus, also $J^x = \Gamma(J^{\in x}) \subseteq \Gamma(J^x)$. □

Lemma 18.2 *For any a satisfying the requirement of the infinity axiom it holds that $\Gamma(J^{\in a}) \subseteq J^{\in a}$.*

Proof: Suppose $b \in \Gamma(J^{\in a})$. Then either $b = 0$ or $b = z^+$ for some $z \in J^{\in a}$.

In the first case $b = 0 \subseteq \Gamma(J^{\in x}) = J^x \subseteq J^{\in a}$ where x is some element of a .

In the second case we argue as follows. As $z \in J^{\in a}$ there exists an $x \in a$ with $z \in J^x$. Then there exists a $y \in a$ with $y \in x$. Thus $z \in J^{\in y}$ and we have $b = z^+ \in \Gamma(J^{\in y}) = J^y \subseteq J^{\in a}$. □

Theorem 18.1

For every set a satisfying the requirement of Inf we have $J^a = \Gamma(J^a)$.

Proof: By Lemma 5.1 we have $J^{\in a} \subseteq J^a = \Gamma(J^{\in a})$ and by Lemma 5.2 we have $J^{\in a} \subseteq \Gamma(J^{\in a})$. Thus $J^{\in a} = \Gamma(J^{\in a}) = J^a$ from which it follows that $J^a = \Gamma(J^a)$. □

⁸If X is a class (of ordered pairs) and a is a set then we write X^a for $\{b \mid \langle a, b \rangle \in X\}$.

19 Algebraicity of cpos of fixpoints

Let $\Phi : \mathcal{P}(2) \rightarrow \mathcal{P}(2)$ with $\Phi(\emptyset) = \Phi(\{0\}) = \{0\}$, $\Phi(\{1\}) = \{1\}$ and $\Phi(\{0, 1\}) = \{0, 1\}$. Obviously, the map Φ is inflationary, i.e. $X \subseteq \Phi(X)$ and $\text{FP}(\Phi) = \{X \in \mathcal{P}(2) \mid X = \Phi(X)\}$ is not a complete lattice.

Let L be the complete algebraic lattice as given by $\omega+1 \cup \{a, b, \top\}$ where $\omega+1$ is ordered as usual, a and b are two incomparable elements above ω and \top is the greatest element. Let $\Phi : L \rightarrow L$ be the identity on $L \setminus \{\omega\}$ and $\Phi(\omega) = a$. Obviously, the map Φ is inflationary and $\text{FP}(\Phi) = L \setminus \{\omega\}$ is not a cpo since $\omega \subseteq L$ has two different minimal upper bounds a and b .

If S is a set and $\Phi : \mathcal{P}(S) \rightarrow \mathcal{P}(S)$ is monotone then $\text{FP}(\Phi)$ is a complete lattice where $\bigcap_{i \in I} X_i = \bigcap_{\alpha \in \text{ON}} \Phi^\alpha(\bigcap_{i \in I} X_i)$.

The map $\Phi : \mathcal{P}(\mathbb{N}) \rightarrow \mathcal{P}(\mathbb{N}) : A \mapsto A \cup \{n \in \mathbb{N} \mid A \text{ infinite}\}$ is monotone, idempotent and inflationary. Thus, its set $\text{FP}(\Phi)$ of fixpoints is a complete lattice w.r.t. \subseteq . However, the lattice $\text{FP}(\Phi)$ is not algebraic since \emptyset is the only compact elements of $\text{FP}(\Phi)$ which can be seen as follows. Obviously \mathbb{N} is not compact. If A is a nonempty finite subset of \mathbb{N} with greatest element n_0 then $A \subseteq \mathbb{N} = \bigsqcup_{n \in \mathbb{N}} A_n$ where $A_n = \{k \in \mathbb{N} \mid n_0 < k \leq n\}$ but $A \not\subseteq A_n$ for all $n \in \mathbb{N}$.

Let $\{q_n \mid n \in \mathbb{N}\}$ be an enumeration of $\mathbb{Q} \cap [0, 1]$. Let $\Phi : \mathcal{P}(\mathbb{N}) \rightarrow \mathcal{P}(\mathbb{N})$ be the Scott continuous with $\Phi(X) = \{n \in \mathbb{N} \mid \exists k \in X. q_n < q_k\}$. Obviously, the map Φ is idempotent (but not inflationary) and $(\text{FP}(\Phi), \subseteq)$ is isomorphic to $([0, 1], \leq)$ which is not algebraic (since 0 is the only compact element).

Let D be a cpo and $f : D \rightarrow D$ be Scott continuous. Then $\text{FP}(f)$ is a cpo w.r.t. the order inherited from D . If f is moreover inflationary and idempotent, i.e. $\text{id}_D \sqsubseteq f = f \circ f$, then $\text{FP}(f)$ is an algebraic cpo which can be seen as follows. Let $e \in D$ be compact. Then $f(e)$ is compact in $\text{FP}(f)$ because if $X \subseteq \text{FP}(f)$ is directed with $f(e) \sqsubseteq \bigsqcup X$ then $e \sqsubseteq \bigsqcup X$ from which it follows that there exists an $x \in X$ with $e \sqsubseteq x$ (since e is compact) and thus $f(e) \sqsubseteq f(x) = x$. If $x \in \text{FP}(f)$ then $\{f(e) \mid f(e) \sqsubseteq x \text{ and } e \text{ compact}\}$ is directed with supremum x .

If $g : D \rightarrow D$ is Scott continuous and inflationary the map $f = \bigsqcup_{n \in \mathbb{N}} g^n$ is Scott continuous, inflationary and idempotent since it sends every $x \in D$ to the least fixpoint $f(x) = \bigsqcup_{n \in \mathbb{N}} g^n(x)$ of g above x . Obviously, we have $\text{FP}(f) = \text{FP}(g)$ which is an algebraic cpo by the previous paragraph.

20 Gabriel-Ulmer and Bunge Duality

A *locally finitely presentable (lfp)* category is one of the form $\mathbf{Lex}(\mathbb{C}, \mathbf{Set})$, i.e. the category of finite limit preserving functors from the finite limit category \mathbb{C} to \mathbf{Set} . If $F : \mathbb{C} \rightarrow \mathbb{D}$ in \mathbf{Lex} then $\mathbf{Lex}(F, \mathbf{Set})$ preserves cofiltered colimits and has a left adjoint (i.e. preserves small limits). Let us denote this category by \mathbf{LFP} . *Gabriel-Ulmer duality* says that

$$\mathbf{Lex}(-, \mathbf{Set}) : \mathbf{Lex}^{\mathrm{op}} \rightarrow \mathbf{LFP}$$

is an equivalence. The quasi-inverse of this functor is given by $\mathbf{LFP}(-, \mathbf{Set})$.

In her Thesis *Generalised Algebraic Models* (Louvain-la-Neuve, 2004) C. Centazzo has generalized Gabriel-Ulmer Duality by replacing the category of finite categories (implicit in the notion of finite limit category and cofiltered colimit) to more general classes \mathcal{D} of diagram shapes. If \mathcal{D} is the empty class one obtains the *Bunge duality* between the category \mathbf{Cat} of small categories and arbitrary functors between them and the category \mathbf{Psh} of (covariant) presheaf categories with continuous and cocontinuous functors between them by sending $F : \mathbb{C} \rightarrow \mathbb{D}$ to $\mathbf{Set}^F : \mathbf{Set}^{\mathbb{D}} \rightarrow \mathbf{Set}^{\mathbb{C}}$. The quasi-inverse is given up to equivalence by $\mathbf{Psh}(-, \mathbf{Set})$.

In her Thesis M. Bunge has characterized presheaf toposes as cocomplete locally small regular categories \mathcal{E} whose atoms (up to isomorphism) form a small set of generators. Here atoms are objects $A \in \mathcal{E}$ such that $\mathcal{E}(A, -) : \mathcal{E} \rightarrow \mathbf{Set}$ preserves small colimits, i.e. A is indecomposable and projective.

21 Conditional pca's give rise to triposes

A *conditional pca* (c-pca) is a tuple $\mathcal{A} = (A, \cdot, k, s)$ where A is a set, $\cdot : A \times A \rightarrow A$ and k and s are elements of A satisfying

$$kab = a \qquad sabc \succeq ac(bc)$$

where $t_1 t_2$ stands for $t_1 \cdot t_2$ and $t_1 \succeq t_2$ is an abbreviation for the statement $t_2 \downarrow$ implies $t_1 = t_2$. Instead of postulating elements k and s with the above properties one might equivalently postulate the following more *liberal form of functional completeness*: whenever t is a polynomial in the variables x, \vec{z} then there exists a polynomial $\Lambda x.t$ in the variables \vec{z} such that $(\Lambda x.t)[\vec{a}/\vec{z}]a \succeq t[a, \vec{a}/x, \vec{z}]$ for all a, \vec{a} in A .

Recall that pcs's are c-pca's satisfying the further requirement that sab is always defined. In c-pca's it is *not* guaranteed that $(\Lambda x.t)[\vec{a}/\vec{z}] \downarrow$ unless $t[a, \vec{a}/x, \vec{z}] \downarrow$ for some $a \in A$. In particular, for terms t their abstraction $\Lambda x.t$ need not be defined in general. Thus, the usual trick of *thunking* is not available. However, there is a slightly more sophisticated way of achieving the same goal. Define a binary operation $\mathbf{thunk}(t, c)$ where t is a term and $c \in A$ such that $\mathbf{thunk}(t, c)$ is a term satisfying

$$\mathbf{thunk}(t, c) \cdot \mathit{true} \succeq t \qquad \mathbf{thunk}(t, c) \cdot \mathit{false} \succeq c$$

where $\mathit{true} \equiv \Lambda x.\Lambda y.x$ and $\mathit{false} \equiv \Lambda x.\Lambda y.y$. Such an operation \mathbf{thunk} can be defined by recursion on the structure of t as follows

$$\begin{aligned} \mathbf{thunk}(t, c) &\equiv \Lambda z.ztc \quad \text{if } t \text{ is a constant or variable} \\ \mathbf{thunk}(t_1 t_2, c) &\equiv s \mathbf{thunk}(t_1, kc) \mathbf{thunk}(t_2, c) \end{aligned}$$

where in the second clause it is necessary to modify the second argument as otherwise $\mathbf{thunk}(t_1 t_2, c) \mathit{true} \succeq cc$ and not $\mathbf{thunk}(t_1 t_2, c) \mathit{true} \succeq kcc = c$ as under the correct definition.

Given a c-pca \mathcal{A} one may now define a fibred poset $\mathcal{P}_{\mathcal{A}}$ over \mathbf{Set} where $\mathcal{P}_{\mathcal{A}}(I) = (\mathcal{P}(A))^I, \vdash_I$ with

$$\varphi \vdash_I \psi \quad \text{iff} \quad \exists e \in A. \forall i \in I. \forall a \in \varphi(i). ea \in \psi(i)$$

and $\mathcal{P}_{\mathcal{A}}(u)(\varphi) = \varphi \circ u$. It is a priori not clear whether $\mathcal{P}(\mathcal{A})$ is a tripos as \mathcal{A} is not a pca. However, one may define a pca \mathcal{A}^* from \mathcal{A} such that $\mathcal{P}_{\mathcal{A}}$ and $\mathcal{P}_{\mathcal{A}^*}$ are equivalent as fibred posets. The underlying set of \mathcal{A}^* is again A but application in \mathcal{A}^* is defined as

$$a \cdot_{\mathcal{A}^*} b \simeq a \cdot k \cdot b$$

for $a, b \in A$.

Before showing that \mathcal{A}^* is actually a pca we first verify that $\mathcal{P}_{\mathcal{A}}$ and $\mathcal{P}_{\mathcal{A}^*}$ are equivalent as fibred posets. The equivalence is the identity on objects in each fibre. Thus, it remains to show that the following two conditions are equivalent

$$(1) \exists e \in A. \forall i \in I. \forall a \in \varphi(i). ea \in \psi(i)$$

$$(2) \exists e \in A. \forall i \in I. \forall a \in \varphi(i). eka \in \psi(i)$$

for all $\varphi, \psi \in \mathcal{P}(A)^I$. Suppose that e satisfies $\forall i \in I. \forall a \in \varphi(i). ea \in \psi(i)$ then $e' = \mathbf{thunk}(e, k)$ satisfies $\forall i \in I. \forall a \in \varphi(i). e'ka \in \psi(i)$ because $\mathbf{thunk}(e, k)k \simeq \mathbf{thunk}(e, k)\mathbf{true} = e$. Suppose e' satisfies $\forall i \in I. \forall a \in \varphi(i). e'ka \in \psi(i)$ then $e = e'k$ satisfies $\forall i \in I. \forall a \in \varphi(i). ea \in \psi(i)$ if $e'ka$ is defined for some $a \in \varphi(i)$ and otherwise choose e arbitrarily.⁹

Finally, we have to show that for $a \cdot_{\mathcal{A}^*} b \simeq a \cdot k \cdot b$ there exist appropriate k^* and s^* satisfying

$$k^*kabb \simeq a \quad \text{and} \quad s^*kabbkc \succeq akck(bkc)$$

for all $a, b, c \in A$. An appropriate choice for k^* is $\Lambda xyzu.y$ and an appropriate choice for s^* is $\Lambda uvxyz.\mathbf{thunk}(xkz(ykz), \mathbf{false})w$ as we have¹⁰

$$s^*kabbkc \succeq \mathbf{thunk}(akc(bkc), \mathbf{false})k \succeq akc(bkc)$$

and for $k' = \mathbf{false}$ we have

$$s^*kabbk'c \succeq \mathbf{thunk}(akc(bkc), \mathbf{false})\mathbf{false} \succeq \mathbf{false}$$

from which it follows that s^*kabb is defined.

⁹Notice that this argument uses classical logic as e is constructed according to a case analysis whether $\bigcup_{i \in I} \varphi(i)$ is inhabited or not.

¹⁰where for the first step we use the fact that $\mathbf{thunk}(t[\vec{x}], c)[\vec{a}/\vec{x}] \succeq \mathbf{thunk}(t[\vec{a}/\vec{x}], c)$ for all lists \vec{a} of elements in A of appropriate length

22 J. R. Moschovakis' G -realizability

Let \mathcal{A} be a pca and $\mathcal{A}_\#$ be a subpca of \mathcal{A} . Typical examples are $\text{R.E.} \subseteq \mathcal{P}\omega$ and $K_{2, \text{eff}} \subseteq K_2$. It's the latter example that was considered by J. R. Moschovakis in 1971 in [Mo71].

Let $\text{RT}(\mathcal{A}, \mathcal{A}_\#)$ be the topos originating via the tripos-to-topos construction from the tripos $\mathcal{P}(\mathcal{A}, \mathcal{A}_\#)$ over \mathbf{Set} which is defined as follows. Objects of $\mathcal{P}(\mathcal{A}, \mathcal{A}_\#)(I)$ are functions $\varphi = \langle \varphi_a, \varphi_p \rangle : I \rightarrow \mathcal{P}(\mathcal{A}_\#) \times \mathcal{P}(\mathcal{A})$ with $\varphi_a(i) \subseteq \varphi_p(i)$ for all $i \in I$. The preorder \vdash_I on $\mathcal{P}(\mathcal{A}, \mathcal{A}_\#)(I)$ is defined as follows: $\varphi \vdash_I \psi$ iff $\bigcap_{i \in I} [\varphi_a(i) \rightarrow \psi_a(i)] \cap \bigcap_{i \in I} [\varphi_p(i) \rightarrow \psi_p(i)] \cap \mathcal{A}_\#$ is nonempty (where $[A \rightarrow B] = \{e \in \mathcal{A} \mid \forall a \in A. ea \in B\}$ for $A, B \subseteq \mathcal{A}$).

Let $u = \langle \emptyset, \{a_0\} \rangle$ with $a_0 \in \mathcal{A}_\#$. Then $u \vee (-)$ defines a topology on the tripos $\mathcal{P}(\mathcal{A}, \mathcal{A}_\#)$ giving rise to the subtripos $\mathcal{M}(\mathcal{A}, \mathcal{A}_\#)$ of $\mathcal{P}(\mathcal{A}, \mathcal{A}_\#)$ where $\varphi \in \mathcal{M}(\mathcal{A}, \mathcal{A}_\#)(I)$ iff $\mathcal{A}_\# \cap \bigcap_{i \in I} \varphi_p(i)$ is nonempty.¹¹ Then the topos obtained from $\mathcal{M}(\mathcal{A}, \mathcal{A}_\#)$ is called the *G -realizability topos induced by $\mathcal{A}_\# \subseteq \mathcal{A}$* and also denoted by $\mathcal{M}(\mathcal{A}, \mathcal{A}_\#)$.

Recall that $\perp \in \mathcal{M}(\mathcal{A}, \mathcal{A}_\#)$ is given by u . As usual we write $\neg A$ for $A \rightarrow \perp = A \rightarrow u$. Let $A \in \mathcal{M}(\mathcal{A}, \mathcal{A}_\#)$ be a G -realizability proposition. One easily checks that $(\neg A)_p = A_p \rightarrow \{a_0\}$ and $(\neg A)_a = \mathcal{A}_\# \cap (A_a \rightarrow \emptyset) \cap (\neg A)_p$. Thus we have $(\neg \neg A)_p = (A_p \rightarrow \{a_0\}) \rightarrow \{a_0\}$ and $(\neg \neg A)_a = (\neg \neg A)_p \cap \{e \in \mathcal{A}_\# \mid A_a \neq \emptyset\}$.

Next we characterize the $\neg \neg$ -separated objects of $\mathcal{M}(\mathcal{A}, \mathcal{A}_\#)$. Using the above explication of $\neg \neg$ in $\mathcal{M}(\mathcal{A}, \mathcal{A}_\#)$ it is straightforward to check that the category $\mathbf{Sep}_{\neg \neg}(\mathcal{M}(\mathcal{A}, \mathcal{A}_\#))$ is equivalent to the category $\mathbf{Asm}_G = \mathbf{Asm}_G(\mathcal{A}, \mathcal{A}_\#)$ which is defined as follows. Objects of \mathbf{Asm}_G are triples $X = (|X|, P_X, \|\cdot\|_X)$ where $|X|$ is a set, $P_X \subseteq \mathcal{A}$ with $\mathcal{A}_\# \cap P_X \neq \emptyset$ and $\|\cdot\|_X : |X| \rightarrow \mathcal{P}(\mathcal{A}_\# \cap P_X) \setminus \{\emptyset\}$. For objects X and Y of \mathbf{Asm}_G the hom-set $\mathbf{Asm}_G(X, Y)$ consists of all maps $f : |X| \rightarrow |Y|$ with

$$e \in [P_X \rightarrow P_Y] \cap \bigcap_{x \in |X|} [\|x\|_X \rightarrow \|f(x)\|_Y]$$

for some $e \in \mathcal{A}_\#$.

At first sight \mathbf{Asm}_G looks quite similar to $\mathbf{Asm}(\mathcal{A}_\#)$ as f is an $\mathbf{Asm}(\mathcal{A}_\#)$ -morphism from $(|X|, \|\cdot\|_X)$ to $(|Y|, \|\cdot\|_Y)$ iff $f : |X| \rightarrow |Y|$ and

$$e \in \bigcap_{x \in |X|} [\|x\|_X \rightarrow \|f(x)\|_Y]$$

for some $e \in \mathcal{A}_\#$. However, such an e need not satisfy that $ea \in P_Y$ whenever $a \in P_X$ as is required for \mathbf{Asm}_G -morphisms!

Notice also that $\mathbf{Asm}_G(\mathcal{A}, \mathcal{A}_\#)$ contains as a full subcategory the category $\mathbf{Sep}_{\neg \neg}(\mathbf{Mod}(\mathcal{A}_\#))$ of $\neg \neg$ -separated objects of the *modified realizability topos $\mathbf{Mod}(\mathcal{A}_\#)$* over $\mathcal{A}_\#$, namely as those $(|X|, P_X, \|\cdot\|_X) \in \mathbf{Asm}_G$ with $P_X \subseteq \mathcal{A}_\#$.

¹¹Notice that $u \rightarrow (-)$ also defines a topology on $\mathcal{P}(\mathcal{A}, \mathcal{A}_\#)$ giving rise to the so-called relative realizability tripos $\mathcal{R}(\mathcal{A}, \mathcal{A}_\#)$ where $\mathcal{R}(\mathcal{A}, \mathcal{A}_\#)(I)$ consist of all $\varphi \in \mathcal{P}(\mathcal{A}, \mathcal{A}_\#)(I)$ with $\varphi_a(i) = \varphi_p(i) \cap \mathcal{A}_\#$. The ensuing relative realizability topos has been investigated intensively by Bauer, Birkedal and Scott.

References

- [Mo71] J.R. Moschovakis *Can there be no non-recursive functions?* J. Symbolic Logic 36 (1971), pp. 309–315.

23 Moschovakis’ G -realizability refutes MP

Notice that classically already FAN_D entails the existence of a nonrecursive $\alpha \in 2^{\mathbb{N}}$ (using the Kleene tree). Thus $\text{INT} + \text{MP}$ proves that $\neg\neg\exists\alpha\neg\exists e\{e\} = \alpha$. But this is not the case intuitionistically since by Moschovakis’ G -realizability INT is consistent with the principle $\forall\alpha\neg\neg\exists e\{e\} = \alpha$, i.e. $\neg\exists\alpha\neg\exists e\{e\} = \alpha$ saying that “there exist no non-recursive functions”. Thus G -realizability refutes MP which is no surprise after all since it is a kind of modified realizability.

24 Markov’s Principle fails in $\text{Sh}(\mathbb{N}^{\mathbb{N}})$

Though MP holds in all realizability models it does not hold in all sheaf models. For example in $\text{Sh}(\mathbb{N}^{\mathbb{N}})$, i.e. sheaves over Baire space, Markov’s principle does not hold which can be seen as follows. First recall that $N^N(U) = \text{Cont}(U, \mathbb{N}^{\mathbb{N}})$, $N(U) = \text{Cont}(U, \mathbb{N})$ and $\text{ev} : N^N \times N \rightarrow N$ is given by $\text{ev}_U(f, g)(x) = f(x)(g(x))$. The interpretation of the predicate $P(f) \equiv \exists n:N.f(n) = 0$ on N^N is given by $\llbracket P \rrbracket_U(\Phi) = \{x \in U \mid \exists n \in \mathbb{N}. \Phi(x)(n) = 0\}$. Now putting $U = \mathbb{N}^{\mathbb{N}}$ and $\Phi = \text{id}_{\mathbb{N}^{\mathbb{N}}}$ we get $\llbracket P \rrbracket(\text{id}_{\mathbb{N}^{\mathbb{N}}}) = \{f \in \mathbb{N}^{\mathbb{N}} \mid \exists n:\mathbb{N}. f(n) = 0\}$. Since this open set is dense in $\mathbb{N}^{\mathbb{N}}$ we have $\neg\neg\llbracket P \rrbracket(\text{id}_{\mathbb{N}^{\mathbb{N}}}) = \mathbb{N}^{\mathbb{N}}$ although $\llbracket P \rrbracket(\text{id}_{\mathbb{N}^{\mathbb{N}}}) \neq \mathbb{N}^{\mathbb{N}}$.

25 “Muchnik topos” as subtopos of “Medvedev topos”

Let \mathcal{A} be a pca and $\mathcal{A}_\#$ a subpca. Typical examples are the second Kleene algebra K_2 with its subpca $K_{2,eff}$ of effective elements and $\mathcal{P}\omega$ (Scott’s graph model) and its subalgebra RE of recursively enumerable sets.

A *mass problem* is a subset of \mathcal{A} . On the set $\mathcal{P}(\mathcal{A})$ of mass problems Medvedev considered the following reducibility relation: B is reducible to A , notation $A \vdash B$, iff there exists a $\gamma \in \mathcal{A}_\#$ with $\forall \alpha \in A. \gamma \cdot \alpha \in B$. Notice that this relation \vdash is the entailment relation of the tripos from which one constructs the *relative realizability* topos $\text{RT}(\mathcal{A}, \mathcal{A}_\#)$ which for this reason might as well be called *Medvedev topos*.

On \mathcal{A} one may consider the relation \leq_T (“Turing reducibility”) where $\alpha \leq_T \beta$ iff $\exists \gamma \in \mathcal{A}_\#. \alpha = \gamma \cdot \beta$. A subset A of \mathcal{A} is upward closed w.r.t. \leq_T iff $\alpha \in A$ and $\alpha \leq_T \beta$ implies $\beta \in A$. We write $\mathcal{P}_\uparrow(\mathcal{A})$ for the set of upward closed subsets of \mathcal{A} .

Lemma 25.1 *For $A, B \in \mathcal{P}_\uparrow(\mathcal{A})$ we have $A \vdash B$ iff $A \subseteq B$.*

Proof: Suppose $A \vdash B$, i.e. there exists a $\gamma \in \mathcal{A}_\#$ such that $\gamma \cdot \alpha \in B$ for all $\alpha \in A$. Suppose $\alpha \in A$. Then $\gamma \cdot \alpha \leq_T \alpha$. Since $\gamma \cdot \alpha \in B$ and B is upward closed it follows that $\alpha \in B$. Thus we have shown that $A \subseteq B$. On the other hand if $A \subseteq B$ then $A \vdash B$ is realized by a $\gamma \in \mathcal{A}_\#$ with $\gamma \cdot \alpha = \alpha$ for all $\alpha \in \mathcal{A}$. \square

Let $j : \mathcal{P}(\mathcal{A}) \rightarrow \mathcal{P}(\mathcal{A})$ with $j(A) = \{\beta \in \mathcal{A} \mid \exists \alpha \in A. \alpha \leq_T \beta\}$, i.e. j performs upward closure w.r.t. \leq_T . One easily shows that j is a local operator on the Medvedev tripos, i.e. the one inducing the Medvedev topos. The corresponding subtripos/subtopos is known as the “Muchnik” tripos/topos where entailment between upward closed subsets of \mathcal{A} is given by inclusion. Muchnik originally defined a notion of reducibility on subsets of \mathcal{A} where $A \vdash B$ iff for all $\alpha \in A$ exists a $\beta \in B$ with $\beta \leq_T \alpha$ which is easily seen to be equivalent to $j(A) \subseteq j(B)$.

26 Fan vs. Heine-Borel

In [FH] there has been considered the localic topos $\mathbf{Sh}(K(\mathbb{R}^2))$ of sheaves over the pointless locale $K(\mathbb{R}^2)$ of *coperfect* subsets of \mathbb{R}^2 (i.e. those open subsets of \mathbb{R}^2 (w.r.t. the Euclidean topology) whose complement has no isolated points).¹² As $K(\mathbb{R}^2)$ is locally connected the functor $\Delta : \mathbf{Set} \rightarrow \mathbf{Sh}(K(\mathbb{R}^2))$ (left adjoint to $\Gamma = \mathbf{Sh}(K(\mathbb{R}^2))(1, -)$) is known to preserve first order logic and exponentials. Thus, this topos validates FAN_D and *weak* compactness of $[0, 1]^c$ (the latter meaning that every countable covering of $[0, 1]^c$ by open intervals with rational endpoints admits a finite subcovering). Recently, in [Loe] it has been shown that these two principles are actually equivalent over Troelstra's EL where number choice is replaced by unique number choice.

The last theorem of [FH] shows that, however, the unrestricted fan principle FAN does not hold in $\mathbf{Sh}(K(\mathbb{R}^2))$. In [Moe] it has been shown that nevertheless $\mathbf{Sh}(K(\mathbb{R}^2))$ validates the compactness of $[0, 1]^d$. In [Gr1] it has been shown that $\mathbf{Sh}(K(\mathbb{R}^2))$ does not validate compactness of $[0, 1]^c$ as this would entail that $\mathcal{O}(\mathbb{R}^d) \cong \mathcal{O}(\mathbb{R}^c)$ (by sending U to $U \cap \mathbb{R}^c$) and the latter does not hold in $\mathbf{Sh}(K(\mathbb{R}^2))$. Thus, we have

- (1) $\text{FAN} \Rightarrow \text{compactness of } [0, 1]^c$
- (2) $\text{compactness of } [0, 1]^c \Rightarrow \text{FAN}_D$
- (3) $\text{compactness of } [0, 1]^c \Rightarrow \text{compactness of } [0, 1]^d$

where (1) is well known, (2) follows from Loeb's result (as, obviously, compactness of $[0, 1]^c$ implies weak compactness of $[0, 1]^c$) and (3) is a result of [Gr1].

The topos $\mathbf{Sh}(K(\mathbb{R}^2))$ shows that the implications (2) and (3) cannot be strengthened to equivalences. It is open whether (1) can be strengthened to an equivalence. The penultimate theorem of [FH] provides a locally connected localic topos where compactness of $[0, 1]^d$ fails. Obviously, this topos validates FAN_D and the statement that all continuous functions on $[0, 1]^c$ are uniformly continuous (because this property is expressible in the language of HA^ω and holds in \mathbf{Set}) although $[0, 1]^c$ is not compact in it (as otherwise $[0, 1]^d$ were compact in it by (3) above). It remains as an open problem, however, whether compactness of $[0, 1]^d$ entails FAN_D .

¹²Recall that a *perfect* set is a closed set without isolated points. For a space X there is an endomap P on the set $\mathcal{C}(X)$ of closed subsets of X sending A to the greatest perfect set $P(A) \subseteq A$. $P(A)$ is obtained from A by transfinitely iterating the operation $(-)'$ sending a closed set A to the set A' consisting of all points in A which are not isolated in A . Obviously, the operation P is deflationary ($P(A) \subseteq A$), idempotent, monotone and preserves \cup . The latter follows from the fact that for closed A and B we have $(A \cup B)' = A' \cup B'$ since a point is isolated in $A \cup B$ iff it is isolated in A or isolated in B . Thus $P^{\text{op}} : \mathcal{C}(X)^{\text{op}} \rightarrow \mathcal{C}(X)^{\text{op}}$ is a nucleus on $\mathcal{O}(X) \cong \mathcal{C}(X)^{\text{op}}$.

If X is a T_1 -space then all singleton subsets of X are closed and the map $F : \mathcal{O}(X) \rightarrow \mathcal{O}(X) : U \mapsto X \setminus (X \setminus U)'$ can be described more explicitly as $F(U) = \bigcup \{V \in \mathcal{O}(X) \mid \exists x \in V. V \setminus \{x\} \subseteq U\}$. The set $K(X)$ of coperfect subsets of X is then given by $\{U \in \mathcal{O}(X) \mid U = F(U)\}$. It is the sublocale of $\mathcal{O}(X)$ given by the least nucleus identifying U and $U \setminus \{x\}$ for all open sets U and points x .

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27 Kripke-Joyal Semantics for Toposes

Let \mathcal{E} be an elementary topos. If X is an object of \mathcal{E} , ϕ is a predicate on X and $\alpha : I \rightarrow X$ is a generalised element of X then we write $I \Vdash \phi(\alpha)$ iff α factors vthrough the subobject $\llbracket \phi \rrbracket$ of X . The “forcing” relation \Vdash is define inductively as follows

- (\wedge) $I \Vdash \phi(\alpha) \wedge \psi(\alpha)$ iff $I \Vdash \phi(\alpha)$ and $I \Vdash \psi(\alpha)$
- (\rightarrow) $I \Vdash \phi(\alpha) \rightarrow \psi(\alpha)$ iff for all $f : J \rightarrow I$ from $J \Vdash \phi(\alpha f)$ it follows that $J \Vdash \psi(\alpha f)$
- (\perp) $I \Vdash \perp$ iff $I \cong 0$
- (\vee) $I \Vdash \phi(\alpha) \vee \psi(\alpha)$ iff there exist jointly epic maps $p : J \rightarrow I$ and $q : K \rightarrow I$ such that $J \Vdash \phi(\alpha p)$ and $K \Vdash \psi(\alpha q)$
- (\forall) $I \Vdash \forall y:Y. \phi(y, \alpha)$ iff $J \Vdash \phi(\beta, \alpha f)$ for all $f : J \rightarrow I$ and $\beta : J \rightarrow Y$
- (\exists) $I \Vdash \exists y:Y. \phi(y, \alpha)$ iff $J \Vdash \phi(\beta, \alpha e)$ for some epi $e : J \twoheadrightarrow I$ and $\beta : J \rightarrow Y$

Notice that $I \Vdash \forall y:Y. \phi(y, \alpha)$ iff $Y \times I \Vdash \phi(\text{id}_Y \times \alpha)$ iff $Y \times I \Vdash \phi(\pi_1, \alpha \pi_2)$.

For the basic predicates we have

- ($=$) $I \Vdash \alpha = \beta$ iff $\alpha = \beta$
- (\in) $I \Vdash \alpha \in S$ iff $\langle \alpha, S \rangle$ factors through $\in_X \twoheadrightarrow X \times \mathcal{P}(X)$.

Let $\mathcal{E} = \text{Sh}(\mathcal{C}, \mathcal{J})$ where \mathcal{J} is a subcanonical¹³ coverage (Grothendieck topology) on a small category \mathcal{C} . Let X be an object in \mathcal{E} and $\phi(x)$ be a predicate on X . We write $C \Vdash \phi(\alpha)$ for $\alpha \in \llbracket \phi \rrbracket \subseteq X$. This “forcing” relation \Vdash is defined inductively as follows

- (\wedge) $C \Vdash \phi(\alpha) \wedge \psi(\alpha)$ iff $C \Vdash \phi(\alpha)$ and $C \Vdash \psi(\alpha)$
- (\rightarrow) $C \Vdash \psi(\alpha) \rightarrow \phi(\alpha)$ iff for all $f : D \rightarrow C$ from $D \Vdash \psi(\alpha f)$ it follows that $D \Vdash \phi(\alpha f)$
- (\perp) $C \Vdash \perp$ iff $\emptyset \in \mathcal{J}(C)$
- (\vee) $C \Vdash \phi(\alpha) \vee \psi(\alpha)$ iff for some $\{f_i : D_i \rightarrow C\}_{i \in I} \in \mathcal{J}(C)$ for all $i \in I$ $D_i \Vdash \phi(\alpha f_i)$ or $D_i \Vdash \psi(\alpha f_i)$
- (\forall) $C \Vdash \forall y:Y. \phi(y, \alpha)$ iff $D \Vdash \phi(\beta, \alpha f)$ for all $f : D \rightarrow C$ and $\beta \in Y(D)$
- (\exists) $C \Vdash \exists y:Y. \phi(y, \alpha)$ iff there exist $\{f_i : D_i \rightarrow C\}_{i \in I} \in \mathcal{J}(C)$ and $(\beta_i \in Y(D_i))_{i \in I}$ with $D_i \Vdash \phi(\beta_i, \alpha f_i)$ for all $i \in I$.

For equality and elementhood the conditions are

- ($=$) $C \Vdash \alpha = \beta$ iff $\alpha = \beta$

¹³i.e. all representable objects of $\widehat{\mathcal{C}}$ are \mathcal{J} -sheaves

(\in) $C \Vdash \alpha \in S$ iff $\langle \text{id}_I, \alpha \rangle \in S$.

If $\mathcal{E} = \widehat{\mathcal{C}}$, i.e. \mathcal{J} is the minimal topology, then the clauses (\perp), (\vee) and (\exists) can be simplified as follows

(\perp) $C \Vdash \perp$ never holds

(\vee) $C \Vdash \phi(\alpha) \vee \psi(\alpha)$ iff $C \Vdash \phi(\alpha)$ or $C \Vdash \psi(\alpha)$

(\exists) $C \Vdash \exists y:Y. \phi(y, \alpha)$ iff $C \Vdash \phi(\beta, \alpha)$ for some $\beta \in Y(C)$.

\exists_X and \forall_X

Let X be an object of $\mathcal{E} = \text{Sh}(\mathcal{C}, \mathcal{J})$. Then $\exists_X, \forall_X : \Omega^X \rightarrow \Omega$ are the characteristic maps of the predicates $\exists x:X. x \in S$ and $\forall x:X. x \in S$ on $\mathcal{P}(X) = \Omega^X$, respectively. Thus, we have for $S \in \text{Sub}_{\mathcal{E}}(Y_{\mathcal{E}}(C) \times X)$ and $f : D \rightarrow C$ in \mathcal{C} that

(1) $f \in \exists_X(S)$ iff there exists a cover $(g_i : E_i \rightarrow D)_{i \in I} \in \mathcal{J}(D)$ and a family $(\alpha_i \in X(E_i))_{i \in I}$ with $\langle fg_i, \alpha_i \rangle \in S$ for all $i \in I$

(2) $f \in \forall_X(S)$ iff $\langle fg, \alpha \rangle \in S$ for all $g : E \rightarrow D$ and $\alpha \in X(E)$.

Notice that for $\mathcal{E} = \widehat{\mathcal{C}}$ condition (1) simplifies to $f \in \exists_X(S)$ iff $\langle f, \alpha \rangle \in S$ for some $\alpha \in X(D)$.

28 Kripke-Joyal Semantics for Fibred Categories and Stacks

Suppose $P : \mathcal{X} \rightarrow \mathcal{C}$ is a fibration not necessarily split. We write \mathcal{X}_I for the fibre of P at I . The language of \mathcal{X}_I is a dependently typed first order language where the atomic formulas are equalities between morphisms. We assume P to be endowed with some cleavage choosing a reindexing functor $u^* : \mathcal{X}_I \rightarrow \mathcal{X}_J$ for every $u : J \rightarrow I$ in \mathcal{C} . More generally, we consider the situation where \mathcal{C} is endowed with a Grothendieck topology \mathcal{J} and P is a \mathcal{J} -stack, i.e. for all \mathcal{J} -covers $i : S \hookrightarrow Y_{\mathcal{C}}(I)$ the functor $\mathbf{Fib}(\mathcal{C})(i, P) : \mathbf{Fib}(\mathcal{C})(Y_{\mathcal{C}}(I), P) \rightarrow \mathbf{Fib}(\mathcal{C})(S, P)$ is an equivalence of categories. As suggested by M. Shulman (2008) we extend the Kripke-Joyal semantics for $\mathbf{Sh}(\mathcal{C}, \mathcal{J})$ as follows

$$I \Vdash f = g \text{ iff } \mathcal{X}_I \models f = g$$

$$I \Vdash \forall X. \varphi(X) \text{ iff for every } u : J \rightarrow I \text{ and every object } X \text{ in } \mathcal{X}_J \text{ we have } J \Vdash u^* \varphi(X)$$

$$I \Vdash \forall f : X \rightarrow Y. \varphi(f) \text{ iff for every } u : J \rightarrow I \text{ and every morphism } f : u^* X \rightarrow u^* Y \text{ in } \mathcal{X}_J \text{ we have } J \Vdash u^* \varphi(f)$$

$$I \Vdash \exists X. \varphi(X) \text{ iff there is a } \mathcal{J}\text{-cover } S \text{ of } I \text{ and for every } u \in S \text{ there is an object } X_u \text{ in } \mathcal{X}_{\partial_0(u)} \text{ with } \partial_0(u) \Vdash u^* \varphi(X_u)$$

$$I \Vdash \exists f : X \rightarrow Y. \varphi(f) \text{ iff there is a } \mathcal{J}\text{-cover } S \text{ of } I \text{ and for every } u \in S \text{ there is a morphism } f_u : u^* X \rightarrow u^* Y \text{ in } \mathcal{X}_{\partial_0(u)} \text{ with } \partial_0(u) \Vdash u^* \varphi(f_u)$$

where $u^* \varphi$ is the formula obtained from φ by applying u^* to its parameters. One can show that this definition is independent of the choice of cleavage which were not the case if we had included equality of objects into the language. Moreover, as usual this extended Kripke-Joyal semantics satisfies the conditions

- (stable) if $I \Vdash \varphi$ and $u : J \rightarrow I$ then $J \Vdash u^* \varphi$
- (local) if S is a \mathcal{J} -cover of I and $\partial_0(u) \Vdash u^* \varphi$ for all $u \in S$ then $I \Vdash \varphi$.

The latter requirement holds for atomic formulas since P is assumed to be a \mathcal{J} -stack. Actually, a necessary and sufficient condition for this would be that P is \mathcal{J} -separated, i.e. for every \mathcal{J} -cover $i : S \hookrightarrow Y_{\mathcal{C}}(I)$ the functor $\mathbf{Fib}(\mathcal{C})(i, P)$ is faithful. But then the above Kripke-Joyal clauses would rather correspond to reasoning about the \mathcal{J} -stackification of P . Thus, it seems to be more appropriate to assume already beforehand that P is a \mathcal{J} -stack.

A most important case is when \mathcal{C} is a pretopos and \mathcal{J} is the finite cover topology in which case the clauses for existential quantifiers can be simplified as follows

$$I \Vdash \exists X. \varphi(X) \text{ iff there is a regular epi } e : J \rightarrow I \text{ and an object } X \text{ in } \mathcal{X}_J \text{ with } J \Vdash e^* \varphi(X)$$

$$I \Vdash \exists f : X \rightarrow Y. \varphi(f) \text{ iff there is a regular epi } e : J \rightarrow I \text{ and a morphism } f : e^* X \rightarrow e^* Y \text{ in } \mathcal{X}_J \text{ with } J \Vdash e^* \varphi(f)$$

and (local) can be reformulated as

(local) if $e : J \rightarrow I$ is a regular epi and $J \Vdash e^* \varphi$ then $I \Vdash \varphi$.

Notice that P is separated w.r.t. the finite cover topology whenever for P equality of morphisms is definable in the sense of Bénabou.

The above Kripke-Joyal semantics for the (first order) language of a fibration provides the linguistic framework missing (?) in Bénabou's fibrational account of category theory relative to (fairly) arbitrary base categories.

In his draft paper *Unbounded quantification and strong axioms in topos theory* from 2008 M. Shulman considered the particular case where \mathcal{C} is a (locally cartesian closed pre)topos and P is the fundamental fibration $P_{\mathcal{C}} = \partial_1 : \mathcal{C}^2 \rightarrow \mathcal{C}$ which is a stack w.r.t. the finite cover topology, i.e. $e^* : \mathcal{X}_I \rightarrow \mathcal{X}_J$ is an equivalence for all regular epis $e : J \rightarrow I$.¹⁴ Thus, in the language associated with the fundamental fibration $P_{\mathcal{C}}$ one can quantify over (families of) objects and morphisms in \mathcal{C} , i.e. this language provides unbounded quantification. Shulman also introduces there the notion of *logical completeness* of \mathcal{C} which requires that for every formula φ with parameters over I the subpresheaf

$$\llbracket \varphi \rrbracket = \{u : J \rightarrow I \mid J \Vdash u^* \varphi\}$$

of $\mathbf{Y}_{\mathcal{C}}(I)$ is representable, i.e. there exists a mono $m : I_0 \rightarrow I$ such that $m \in \llbracket \varphi \rrbracket$ and every $u \in \llbracket \varphi \rrbracket$ factors through m .¹⁵ This may be seen as a very strong version of Bénabou's notion of definability for $P_{\mathcal{C}}$. From a set-theoretic point of view it generalizes the *full separation* principle, since if \mathcal{C} is logically complete then the model of set theory in $\mathbf{Idl}(\mathcal{C})$ validates the full separation scheme. This model always validates (strong) collection which, however, gets strong only in presence of full separation because then one can construct families of sets by recursion as e.g. $(\mathcal{P}^n(X))_{n \in \omega}$.

¹⁴Thus $P_{\mathcal{C}}$ is in particular separated w.r.t. the finite cover topology.

¹⁵Later, in 2010, following a suggestion of P. Freyd he rebaptized this property as \mathcal{C} being *autological*.

29 Grothendieck toposes validate all true sentences of first order arithmetic

In a presheaf topos $\widehat{\mathcal{C}}$ the nno N is given by $\Delta(\mathbb{N})$. Using this it is easy to show that a first order arithmetical formulas $\varphi(\vec{n})$ (with parameters \vec{n}) is true iff $I \Vdash \varphi(\vec{n})$ for all $I \in \mathcal{C}$.

Now let \mathcal{J} be a Grothendieck topology on \mathcal{C} and j be the corresponding local operator on $\widehat{\mathcal{C}}$. Let $\eta_N : N \rightarrow \mathbf{a}(N)$ be the reflection map from N to its \mathcal{J} -sheafification $\mathbf{a}(N)$ which is the nno in $\mathbf{Sh}(\mathcal{C}, \mathcal{J})$. Since N is \mathcal{J} -separated η_N is monic. It is known that subsheaves of $\mathbf{a}(N)$ correspond to j -closed subobjects $m : P \rightarrow N$ as indicated in

$$\begin{array}{ccc}
 P & \xrightarrow{j\text{-closed}} & N \\
 \eta_P \downarrow & \lrcorner & \downarrow \eta_N \\
 \mathbf{a}(P) & \xrightarrow{\mathbf{a}(m)} & \mathbf{a}(N)
 \end{array}$$

Thus, for first order arithmetical formulas $\varphi(\vec{n})$ we have

$$\begin{array}{ccc}
 N^k & \xrightarrow{[\![\varphi(\vec{n})]\!]_{\widehat{\mathcal{C}}}} & \Omega \\
 \eta_N^k \downarrow & & \downarrow \eta_\Omega \\
 \mathbf{a}(N)^k & \xrightarrow{[\![\varphi(\vec{n})]\!]_{\mathbf{Sh}(\mathcal{C}, \mathcal{J})}} & \Omega_j \hookrightarrow \Omega
 \end{array}$$

where $k = |\vec{n}|$.

Using this fact validity of first order arithmetical formulas with parameters in N in $\mathbf{Sh}(\mathcal{C}, \mathcal{J})$ can be expressed in Kripke Joyal form by adapting the one for $\widehat{\mathcal{C}}$ by changing the clauses (\perp) , (\vee) and (\exists) appropriately. Let us write $\Vdash_{\mathcal{J}}$ for the thus adapted forcing relation.

Then one can prove by induction on the structure of first order arithmetical formulas that $\varphi(\vec{n})$ is true iff $I \Vdash_{\mathcal{J}} \varphi(\vec{n})$ for all $I \in \mathcal{C}$.

30 Local operators induced by subobjects of Ω

Let \mathcal{E} be a topos and $D \subseteq \Omega$. Following 3.5 of Johnstone's *Topos Theory* (1977) we will construct the least local operator $j : \Omega \rightarrow \Omega$ such that D is contained in J , the subobject of Ω classified by j .

Consider the binary relation $\Theta \subseteq \Omega \times \Omega$ where $\Theta(p, q) \equiv (p \supset q) \supset q$. The relation Θ induces a Galois connection

$$D^r = \{q \in \Omega \mid \forall p \in D. \Theta(p, q)\} \quad D^\ell = \{p \in \Omega \mid \forall q \in D. \Theta(p, q)\}$$

i.e. $(-)^r$ and $(-)^\ell$ are order reversing endomaps of $\mathcal{P}(\Omega)$ such that $D_1 \subseteq D_2^\ell$ iff $D_2 \subseteq D_1^r$.

Johnstone shows that

- (1) For a local operator j we have $J^r = \Omega_j$ and $\Omega_j^\ell = J$.
- (2) For $D \subseteq \Omega$ we have $D = D^{r\ell}$ iff the characteristic map for D is a local operator.

Thus the least local operator j with $D \subseteq J$ is given by the characteristic map of $D^{r\ell}$. Explicitating the construction of $D^{r\ell}$ we get

$$j(p) \equiv \forall q: \Omega ((\forall r \in D ((r \supset q) \supset q)) \supset (p \supset q) \supset q)$$

Thus, for a mono $m : P \rightarrow X$ in \mathcal{E} the least topology j on \mathcal{E} for which m is j -dense is given by $\{x \in P \mid x \in X\}^{rl}$, i.e.

$$j(p) \equiv \forall q: \Omega ((\forall x: X ((x \in P) \supset q) \supset q) \supset (p \supset q) \supset q)$$

If $f : \mathcal{F} \rightarrow \mathcal{E}$ is a geometric morphism and j is a local operator on \mathcal{E} then the pullback of $\mathcal{E}_j \hookrightarrow \mathcal{E}$ along f is given by \mathcal{F}_k where k is the least topology on \mathcal{F} for which the mono $f^*(1 \xrightarrow{\top} J)$ is dense.

31 Fixpoint objects need not be Ω -discrete

An object X of a topos is called Ω -discrete iff $X^{! \Omega} : X \cong X^1 \rightarrow X^\Omega$ is an isomorphism. In an unpublished note Peter Johnstone has shown that fixpoint objects in toposes need not be Ω -discrete.

His counterexample starts from a nontrivial extensional total pca \mathcal{A} as given by e.g. Scott's D_∞ model for the untyped λ -calculus. Since \mathcal{A} admits a fixpoint operator represented by some element of \mathcal{A} . Therefore, in $\mathcal{C} = \mathbf{PER}(\mathcal{A})$ the object $\Delta(\mathcal{A})$ is a fixpoint object in \mathcal{C} . Accordingly, the object $F = y(\Delta(\mathcal{A}))$ is a fixpoint object in $\mathcal{E} = \mathbf{Set}^{C^{op}}$. We will show now that F is not Ω -discrete in \mathcal{E} . For this purpose first observe that $i = [\perp, \top] : 1 + 1 \rightarrow \Omega$ is a split mono, i.e. there exists $p : \Omega \rightarrow 1 + 1$ with $p \circ i = \text{id}_{1+1}$.¹⁶ Let a_1 and a_2 be two distinct elements of \mathcal{A} . Then the map $f = [a_1, a_2] \circ p : \Omega \rightarrow F$ is not constant and, accordingly, the fixpoint object F is not Ω -discrete in $\mathcal{E} = \mathbf{Set}^{C^{op}}$.

¹⁶This p exists since in $\mathbf{Set}^{C^{op}}$ nonempty sieves are stable under reindexing because a nonempty sieve on I always contains the map $0 \rightarrow I$.

32 A Non-Inductive but Well-founded Relation on N in \mathbf{Set}^ω

Consider the topos \mathbf{Set}^ω of “sets developing in discrete time”. The natural numbers object N in \mathbf{Set}^ω is given by $\Delta(\mathbb{N})$. Consider the relation $R \subseteq N \times N$ with $R_k = \{(i+1, i) \mid i < k\}$ for $k \in \omega$. Obviously, in \mathbf{Set}^ω it holds that $\neg \exists f: N^N. \forall n: N. f(n+1) R f(n)$, i.e. R is well-founded. But, nevertheless, the relation R on N is not inductive, i.e. does not validate

$$\forall \mathcal{P}(N). \left[(\forall n: N. (\forall m: N. (m R n \rightarrow \mathcal{P}(m))) \rightarrow \mathcal{P}(n)) \rightarrow \forall n: N. \mathcal{P}(n) \right]$$

This can be seen when instantiating \mathcal{P} with the predicate $P_k = \{i \mid i < k\}$ for $k \in \omega$. Obviously $\forall n: N. P(n)$ fails at all $k \in \omega$ whereas $\forall n: N. (\forall m: N. (m R n \rightarrow P(m))) \rightarrow P(n)$ is universally valid which can be seen as follows. Let $k \in \omega$ and $n \in \mathbb{N}$. Since $P_k(n)$ holds for $n < k$ we just have to consider $n \geq k$ in which case $P_k(n)$ is false. We have to show that in this case also $\forall m: N. (m R n \rightarrow P(m))$ fails at k . For this purpose consider node $n+1 \geq k$ and instantiate m by $n+1$ and observe that $n+1 \notin P_{n+1}$ but $(n+1, n) \in R_{n+1}$ rendering the implication false.

Of course, in $\mathcal{E}ff$ the Kleene tree gives rise to a relation on N which is well-founded but not inductive.

33 Cauchy Completeness of Cauchy Reals

For locally connected locales A the functor $\Delta : \mathbf{Set} \rightarrow \mathbf{Sh}(A)$ preserves first order logic and exponentials. Thus, for every locally connected locale A in $\mathbf{Sh}(A)$ the Cauchy reals are Cauchy complete as they are Cauchy complete in \mathbf{Set} and the property of Cauchy completeness of Cauchy reals can be formulated in first order logic over higher types.

As \mathbb{R} is a locally connected space *the Cauchy reals are Cauchy complete* in $\mathbf{Sh}(\mathbb{R})$ although they are properly contained in the Dedekind reals. Thus in $\mathbf{Sh}(\mathbb{R})$ the Cauchy completion of Cauchy reals within Dedekind reals gives the Cauchy reals and not the Dedekind reals. In realizability toposes which validate number choice the Dedekind reals and Cauchy reals coincide.

M. Escardó and A. Simpson have asked whether there is a topos with NNO in which the Cauchy reals are not Cauchy complete. From the above considerations we know that such a topos can neither be a realizability topos nor a locally connected localic topos over \mathbf{Set} . Of course, if there is any topos with NNO in which the Cauchy reals are not Cauchy complete then this holds in particular for the free topos with NNO.

Moreover, there naturally arises the question whether there are toposes with NNO where the Cauchy completion of the Cauchy reals lies properly in between the Cauchy and the Dedekind reals.

33.1 Model of EL in sheaves over a space

Let T be a topological space. Then every topological space X induces a sheaf X_T in $\mathbf{Sh}(T)$ with $X_T(U) = \{f : U \rightarrow X \mid f \text{ continuous}\}$ for $U \in \mathcal{O}(X)$. For every set S the constant sheaf $\Delta(S)$ is given by S_T where S is considered as a discrete topological space. Thus $\Delta(S)(U)$ consists of all locally constant functions from U to S . The natural and rational numbers in $\mathbf{Sh}(T)$ are given by $N = \Delta(\mathbb{N})$ and $Q = \Delta(\mathbb{Q})$, respectively. We write \hat{s} for the constant function on T with value $s \in S$ and \hat{s}_U for $\hat{s} \upharpoonright U$. It can be shown that $\Delta(\mathbb{N})^{\Delta(\mathbb{N})}$ is isomorphic to $B = \mathbb{B}_T$ where $\mathbb{B} = \mathbb{N}^{\mathbb{N}}$ is Baire space, i.e. infinite sequences of natural numbers with initial segment topology. The application map $\text{App} : \mathbb{B}_T \times \mathbb{N}_T \rightarrow \mathbb{N}_T$ is given by $\text{App}_U(F, f)(x) = F(x)(f(x))$. We now give the clauses of Kripke-Joyal semantics needed for determining the validity of **EL** formulas in $\mathbf{Sh}(T)$.

$U \Vdash a = b$	iff	$a(x) = b(x)$ for all $x \in U$
$U \Vdash \perp$	never holds	
$U \Vdash \phi \wedge \psi$	iff	$U \Vdash \phi$ and $U \Vdash \psi$
$U \Vdash \phi \rightarrow \psi$	iff	for all $V \subseteq U$ from $V \Vdash \phi$ it follows that $V \Vdash \psi$
$U \Vdash \forall x:A. \phi(x, \vec{c})$	iff	for all $V \subseteq U$ and $a \in A(U)$ it holds that $V \Vdash \phi(a, \vec{c} \upharpoonright V)$
$U \Vdash \phi \vee \psi$	iff	there exist U_1 and U_2 with $U_1 \cup U_2 = U$ such that $U_1 \Vdash \phi$ and $U_2 \Vdash \psi$
$U \Vdash \exists x:A. \phi(x, \vec{c})$	iff	there exists an open covering $(U_i)_{i \in I}$ of U and $a_i \in A(U_i)$

such that $U_i \Vdash \phi(a_i, \vec{c}\upharpoonright U_i)$ for all $i \in I$

One easily checks that $Sh(T)$ validates the extensionality principle

$$\forall \alpha, \beta: B \alpha = \beta \leftrightarrow (\forall n: N \alpha(n) = \beta(n))$$

and that

$$U \Vdash \neg \phi \quad \text{iff} \quad V = \emptyset \text{ whenever } V \subseteq U \text{ with } V \Vdash \phi$$

where as usual $\neg \phi$ is defined as $\phi \rightarrow \perp$. Moreover, we define a binary relation $<$ on N as

$$U \Vdash f < g \quad \text{iff} \quad f(x) < g(x) \text{ for all } x \in U$$

for $f, g \in N(U)$. This suffices for giving an interpretation of **EL** in $Sh(T)$.

It can be shown that the forcing relation \Vdash has the following two properties

$$\begin{aligned} \text{(stable)} \quad & \text{if } U \Vdash \phi(\vec{c}) \text{ and } V \subseteq U \text{ then } V \Vdash \phi(\vec{c}\upharpoonright V) \\ \text{(local)} \quad & \text{if } U = \bigcup_{i \in I} U_i \text{ and } U_i \Vdash \phi(\vec{c}\upharpoonright U_i) \text{ for all } i \in I \text{ then } U \Vdash \phi(\vec{c}) \end{aligned}$$

from which it follows that $U \Vdash \forall x: N. \phi(x, \vec{c})$ iff for all $V \subseteq U$ and $n \in \mathbb{N}$ it holds that $V \Vdash \phi(\hat{n}_V, \vec{c}\upharpoonright V)$.

33.2 Dedekind reals in $Sh(T)$

The Dedekind reals in $Sh(T)$ appear as a subobject $R_d \subseteq \mathcal{P}(Q) \times \mathcal{P}(Q)$. Recall that for $U \in \mathcal{O}(T)$ we have $\mathcal{P}(Q)(U) = \text{Sub}_{Sh(X)}(Q\upharpoonright U)$, i.e. $\mathcal{P}(Q)(U)$ consists of all subsheaves of $Q\upharpoonright U \cong y(U) \times Q$.

In $Sh(T)$ the sheaf R_d is isomorphic to \mathbb{R}_T by the following construction. Given $(L, R) \in R_d(U)$ the corresponding continuous function $f_{(L,R)} : U \rightarrow \mathbb{R}$ is given by

$$q < f_{(L,R)}(t) < r \quad \text{iff} \quad \begin{aligned} & \hat{q}_V \in L(V) \text{ and } \hat{r}_V \in R(V) \\ & \text{for some open } V \subseteq U \text{ with } t \in V \end{aligned}$$

for $t \in U$ and $q, r \in \mathbb{Q}$.

For the reverse direction suppose $f : U \rightarrow \mathbb{R}$ is continuous. Then the corresponding $(L_f, R_f) \in R_d(U)$ is given by

$$\begin{aligned} p \in L_f(V) \quad & \text{iff} \quad p(t) < f(t) \text{ for all } t \in V \\ p \in R_f(V) \quad & \text{iff} \quad p(t) > f(t) \text{ for all } t \in V \end{aligned}$$

for all open $V \subseteq U$ and $p \in Q(V)$, i.e. locally constant $q : V \rightarrow \mathbb{Q}$.

Since \mathbb{B} is totally disconnected and \mathbb{R} is connected and locally connected we have $\Delta(\mathbb{N})^{\Delta(\mathbb{N})} \cong \Delta(\mathbb{N}^{\mathbb{N}}) \cong \mathbb{B}_T$ in $Sh(\mathbb{R})$. For this reason in $Sh(\mathbb{R})$ the Cauchy reals R_c are isomorphic to $\Delta(\mathbb{R})$ which is a proper subobject of R_d since for example $\text{id}_{\mathbb{R}} \in R_d(\mathbb{R})$ is not locally constant.

33.3 Lubarsky's counterexamples

For the purpose of answering Escardó and Simpson's question R. Lubarsky has suggested to consider an appropriate sheaf model $\mathbf{Sh}(T)$ in which there exists a Cauchy sequence of Cauchy sequences which does not converges to a Cauchy sequence.¹⁷

33.3.1 Lubarsky's 1st model

As a first approximation he has constructed a sheaf topos $\mathbf{Sh}(C)$ in which there exists a Cauchy sequence of rational numbers for which there does not exist a modulus of convergence. The underlying set of C are Cauchy sequences of rational numbers. Its topology is generated by basic open sets (p, I) where $p \in \mathbb{Q}^{<\omega}$ and I is an open interval in \mathbb{R} (with rational endpoints) where (p, I) consists of all $r \in C$ such that

- (1) $\bar{r}(|p|) = p$
- (2) $r_k \in I$ for all $k \geq |p|$
- (3) $\lim(r) \in I$.

Thus $U \subseteq C$ is open iff for all $r \in U$ there exist $p \in \mathbb{Q}^{<\omega}$ and an open interval I (with rational endpoints) such that $r \in (p, I) \subseteq U$.

At first sight one might think that every continuous $f : C \rightarrow \mathbb{N}$ is already continuous w.r.t. the relative Baire topology, i.e. that for every $r \in C$ there exists a $p \in \mathbb{Q}^{<\omega}$ such that $r \in (p, \mathbb{R})$ and f is constant on (p, \mathbb{R}) . This, however, is not the case as the following counterexample due to A. Simpson shows.

Let D be the subset of C consisting of those $x \in C$ such that

$$-(n+1) < x_n < n+1$$

for all $n \in \omega$. We will show that D is a clopen ("decidable") subset of C which, however, is not open w.r.t. the relative Baire topology and, therefore, the characteristic function $\chi_D : C \rightarrow \{0, 1\} \subseteq \mathbb{N}$ is continuous but not w.r.t. the relative Baire topology.

We first show that D is open. Suppose $x \in D$. Since x is bounded there exists a natural number n such that $-(n+1) < x_k < n+1$ for all k . Then $x \in (\bar{x}(n+1), (-(n+1), n+1)) \subseteq D$. If $x \in C \setminus D$ then there exists an n such that $x_n \notin (-(n+1), n+1)$ and thus $x \in (\bar{x}(n+1), \mathbb{R}) \subseteq D$. Thus $C \setminus D$ is also open and accordingly D is clopen. Suppose D were open in the relative Baire topology. Then, since the constant there function 0 is in D , there exists a natural number n such that $0 \in (\bar{0}(n), \mathbb{R}) \subseteq D$. This, however, is impossible since for the Cauchy sequence x with $x_n = n+1$ and $x_k = 0$ for $k \neq n$ it holds that $x \in (\bar{0}(n), \mathbb{R})$ but $x \notin D$.

¹⁷Notice that this is slightly stronger than exhibiting a Cauchy sequence of Cauchy reals which does not converge to a Cauchy real.

34 Lubarsky's Forcing with Settling Down

34.1 Forcing with Settling Down over \mathbb{R}

In their recent paper “On the Constructive Dedekind Reals” (2007) Lubarsky and Rathjen have introduced a new (kind of) forcing model for CZF where “names” are inductively defined as sets of the form $\sigma = \{\langle \sigma_i, J_i \rangle \mid i \in I\} \cup \{\langle \sigma_h, r_h \rangle \mid h \in H\}$ where the σ_i and σ_h are previously generated names, the J_i are non-empty open subsets of \mathbb{R} and the r_h are elements of \mathbb{R} . For every $r \in \mathbb{R}$ and name σ they define σ^r , the “settling down” of σ at r , inductively as

$$\sigma^r = \{\langle \sigma_i^r, \mathbb{R} \rangle \mid \langle \sigma_i, J_i \rangle \in \sigma \wedge r \in J_i\} \cup \{\langle \sigma_h^r, \mathbb{R} \rangle \mid \langle \sigma_h, r \rangle \in \sigma\}$$

which is an ordinary set. One may think of the operation $(-)^r$ as “localizing at r ”. Next they define a forcing relation $J \Vdash \phi$ between open subsets J of \mathbb{R} and formulas ϕ possible involving names as parameters. For the basic predicates = and \in forcing is defined by mutual recursion as follows

- (=) $J \Vdash \sigma = \tau$ iff for all $\langle \sigma_i, J_i \rangle \in \sigma$ $J \cap J_i \Vdash \sigma_i \in \tau$ and *vice versa* and for all $r \in J$ $\sigma^r = \tau^r$
- (\in) $J \Vdash \sigma \in \tau$ iff for all $r \in J$ there is an $\langle \tau_i, J_i \rangle \in \tau$ and $J' \subseteq J$ with $r \in J' \cap J_i \Vdash \sigma = \tau_i$ and for all $r \in J$ $\langle \sigma^r, \mathbb{R} \rangle \in \tau^r$.

For a formula ϕ in the language of set theory ϕ^r is the formula obtained from ϕ by replacing each parameter σ by σ^r . The clauses for logical connectives and quantifiers are as follows

- (\wedge) $J \Vdash \phi \wedge \psi$ iff $J \Vdash \phi$ and $J \Vdash \psi$
- (\vee) $J \Vdash \phi \vee \psi$ iff for all $r \in J$ there is an open set J' with $r \in J' \subseteq J$ such that $J' \Vdash \phi$ or $J' \Vdash \psi$
- (\rightarrow) $J \Vdash \phi \rightarrow \psi$ iff for all open $J' \subseteq J$ if $J' \Vdash \phi$ then $J' \Vdash \psi$ and for all $r \in J$ if $\mathbb{R} \Vdash \phi^r$ then $\mathbb{R} \Vdash \psi^r$
- (\exists) $J \Vdash \exists x \phi(x)$ iff for all $r \in J$ there is an open J' with $r \in J' \subseteq J$ and a name σ such that $J' \Vdash \phi(\sigma)$
- (\forall) $J \Vdash \forall x \phi(x)$ iff for all $r \in J$ and all names σ there is an open J' with $r \in J' \subseteq J$ such that $J' \Vdash \phi(\sigma)$ and $J' \Vdash \phi^r(\sigma)$.

Lubarsky and Rathjen show that every axiom of CZF_{Exp} is forced in a neighbourhood of 0 and that there does not exist a name σ such that a neighbourhood of 0 forces “all Dedekind reals are elements of σ ”. Thus SubSet Collection is not forced either because it would entail that Dedekind reals form a set.

34.2 Forcing with Settling Down for arbitrary spaces

More recently Lubarsky has come up with a generalisation to arbitrary spaces T instead of the very particular space \mathbb{R} . Again they define “names” inductively as sets of the form $\sigma = \{\langle \sigma_i, J_i \rangle \mid i \in I\} \cup \{\langle \sigma_h, r_h \rangle \mid h \in H\}$ where the σ_i and σ_h are previously generated names, the J_i are non-empty open subsets of T and the r_h are elements of T . Again for every $r \in T$ and name σ they define σ^r , the “settling down” of σ at r , inductively as

$$\sigma^r = \{\langle \sigma_i^r, T \rangle \mid \langle \sigma_i, J_i \rangle \in \sigma \wedge r \in J_i\} \cup \{\langle \sigma_h^r, T \rangle \mid \langle \sigma_h, r \rangle \in \sigma\}$$

which is the image of an ordinary set. One may think of the operation $(-)^r$ as “localizing at r ”. Next they define a forcing relation $J \Vdash \phi$ between open subsets J of T and formulas ϕ possible involving names as parameters. For the basic predicates $=$ and \in forcing is defined by mutual recursion as follows

- (=) $J \Vdash \sigma = \tau$ iff for all $\langle \sigma_i, J_i \rangle \in \sigma$ $J \cap J_i \Vdash \sigma_i \in \tau$ and *vice versa* and for all $r \in J$ $\sigma^r = \tau^r$
- (\in) $J \Vdash \sigma \in \tau$ iff for all $r \in J$ there is an $\langle \tau_i, J_i \rangle \in \tau$ and $J_r \subseteq J_i \cap J$ with $r \in J_r$ such that $J_r \Vdash \sigma = \tau_i$.

For a formula ϕ in the language of set theory ϕ^r is the formula obtained from ϕ by replacing each parameter σ by σ^r . The clauses for logical connectives and quantifiers are as follows

- (\wedge) $J \Vdash \phi \wedge \psi$ iff $J \Vdash \phi$ and $J \Vdash \psi$
- (\vee) $J \Vdash \phi \vee \psi$ iff for all $r \in J$ there is an open set J_r with $r \in J_r \subseteq J$ such that $J_r \Vdash \phi$ or $J_r \Vdash \psi$
- (\rightarrow) $J \Vdash \phi \rightarrow \psi$ iff for all open $J' \subseteq J$ if $J' \Vdash \phi$ then $J' \Vdash \psi$ and for all $r \in J$ there exists an open J_r with $r \in J_r \subseteq J$ such that for all opens $K \subseteq J_r$ if $K \Vdash \phi^r$ then $K \Vdash \psi^r$
- (\exists) $J \Vdash \exists x \phi(x)$ iff for all $r \in J$ there is an open J_r with $r \in J_r \subseteq J$ and a name σ such that $J_r \Vdash \phi(\sigma)$
- (\forall) $J \Vdash \forall x \phi(x)$ iff for all names σ $J \Vdash \phi(\sigma)$ and for all $r \in J$ there exists an open J_r with $r \in J_r \subseteq J$ such that $J_r \Vdash \phi^r(\sigma)$ for all names σ .

It can be shown that for a formula ϕ if $J \Vdash \phi$ then for all $r \in J$ exists an open J_r with $r \in J_r \subseteq J$ such that $J_r \Vdash \phi^r$.

34.3 Simplification in case of homogeneous spaces T

For general spaces T from $J \Vdash \phi$ it just follows that for every $r \in J$ there exists an open J_r with $r \in J_r \subseteq J$ and $J_r \Vdash \phi^r$. However, if T is a homogeneous space, i.e. for all $r, s \in |T|$ there exist open neighbourhoods J_r and J_s of r and s , respectively, such that J_r and J_s are homeomorphic (subspaces of T)

from $J \Vdash \phi$ it follows that $T \Vdash \phi^r$ for all $r \in J$. A most typical example of a homogeneous space is \mathbb{R} .

For general spaces T Lubarsky's model doesn't validate full separation though it validates bounded separation. However, if T is homogeneous then one may interpret full separation as follows: if σ is a name and ϕ is a predicate in the language of set theory then a τ with $T \Vdash \forall x. x \in \tau \leftrightarrow x \in \sigma \wedge \phi(x)$ is given by $\{\langle \sigma_i, J \cap J_i \rangle \mid \langle \sigma_i, J_i \rangle \in \sigma \text{ and } J \Vdash \phi(\sigma_i)\} \cup \{\langle \hat{x}, r \rangle \mid T \Vdash \phi^r(\hat{x})\}$.

34.4 The logic of forcing with settling down

There arises the question to which extent the various forcing conditions can be derived from the universal properties of the connectives and quantifiers and how Lubarsky's new notion is related to $\mathcal{O}(T)$ -valued sets. For this purpose we try to abstract from the particular case of names and the settling down operations $(\cdot)^r$ defined for names.

Whereas $\mathcal{O}(T)$ -valued predicates on a set I are just functions from I to $\mathcal{O}(T)$ with entailment defined as $\phi \vdash_I \psi$ iff $\phi(i) \subseteq \psi(i)$ for all $i \in I$ we have to assume some additional structure on I , namely a $|T|$ -indexed family of operations $(\cdot)^r : I \rightarrow I$ ($r \in T$) such that $(i^r)^s = i^r$ for all $i \in I$ and $r, s \in T$. In particular this means that all $(\cdot)^r$ are retractions with the same image. Let us write \mathcal{G}_T for the category of such structures and structure preserving¹⁸ maps between them. Notice that \mathcal{G}_T is a presheaf topos (actually a topos of monoid actions) which in case T has 2 elements coincides with Lawvere's topos of reflexive graphs.

Now for such I endowed with settling down operations $(\cdot)^r$ for $r \in T$ we define a class of predicates \mathcal{P}^I consisting of all functions $\phi \in \mathcal{O}(T)^I$ satisfying the condition

$$(*) \text{ for all } i \in I \text{ and } r \in \phi(i) \text{ it holds that } r \in \phi(i^r)$$

which is definitely not of a local character, i.e., it can't be described as Ω^I for some $\Omega \subseteq \mathcal{O}(T)$. Now if $h : J \rightarrow I$ is a morphism in \mathcal{G}_T and $\phi \in \mathcal{P}^I$ then its reindexing $h^*\phi = \phi \circ h \in \mathcal{P}^J$. Defining entailment on \mathcal{P}^I pointwise as for Ω^I it is evident that reindexing preserves entailment. One can show that the poset \mathcal{P} indexed over \mathcal{G}_T is a posetal hyperdoctrine where connectives and quantifiers (along projections) are uniquely determined by their universal properties. Lubarsky's forcing conditions make these constructions implicit in the sense that the interpretation of $\phi(i)$ is the union of all opens J with $J \Vdash \phi(i)$. But it seems worthwhile to give here the following direct construction of connectives and quantifiers

$$\begin{aligned} (\phi \wedge \psi)(i) &= \phi(i) \wedge \psi(i) \\ (\phi \vee \psi)(i) &= \phi(i) \vee \psi(i) \\ (\phi \rightarrow \psi)(i) &= \text{int}\{r \in (\phi(i) \rightarrow \psi(i)) \mid r \in (\phi(i^r) \rightarrow \psi(i^r))\} \\ \exists j. \phi(i, j) &= \bigcup_{j \in J} \phi(i, j) \end{aligned}$$

¹⁸ $h : I \rightarrow J$ is structure preserving iff $h(i^r) = h(i)^r$ for all $i \in I$ and $r \in T$

$$\forall j. \phi(i, j) = \text{int}\{r \in \bigwedge_{j \in J} \phi(i, j) \mid r \in \bigwedge_{j \in J} \phi(i^r, j)\}$$

where \bigwedge refers to infimum in $\mathcal{O}(T)$ and int is the interior operation sending a subset X of T to the open set $\text{int}X = \{x \in X \mid \exists U \in \mathcal{O}(X). x \in U \subseteq X\}$.

Notice that the usual posetal hyperdoctrine of $\mathcal{O}(T)$ -valued sets is obtained from \mathcal{P} by change of base along $\Delta : \mathbf{Set} \rightarrow \mathcal{G}_T$ sending I to the set I where all $(\cdot)^r$ are equal to id_I . In this sense the logic of Lubarsky forcing is a generalisation and not a variation of $\mathcal{O}(T)$ -valued logic.

34.5 Relation to the ‘‘Joyal topos’’

Let T be a topological space. Then one may consider the topological space \tilde{T} with $|\tilde{T}| = |T| + |T| = \{0, 1\} \times |T|$ and $U \in \mathcal{O}(\tilde{T})$ iff $U_1 \supseteq U_0 \in \mathcal{O}(T)$ (where we write U_i for $\{x \in |T| \mid \langle i, x \rangle \in U\}$). One may now consider the topos $\mathbf{Sh}(\tilde{T})$ which comes equipped with a geometric morphism $i : \mathbf{Sh}(|T|) \hookrightarrow \mathbf{Sh}(\tilde{T})$ induced by the inclusion $|T| \hookrightarrow \tilde{T} : x \mapsto \langle 1, x \rangle$.¹⁹ The inverse image part of i is given by $i^*(A)(J) = A(\{1\} \times J)$ and its direct image part is given by $i_*(X)(U) = X(U_1)$. Moreover, there is a left adjoint $i_!$ to i^* with $i_!(X)(U) = \{x \in X(U_1) \mid U_0 = \emptyset\}$. Apparently, both $i_!$ and i_* are full and faithful and $i_*i_!$ and i^*i_* are both isomorphic to the identity functor on $\mathbf{Sh}(|T|)$. The functor $i^* : \mathbf{Sh}(\tilde{T}) \rightarrow \mathbf{Sh}(|T|) \simeq \mathbf{Set}^{|T|}$ is logical which follows from the fact that $\mathbf{Sh}(\tilde{T})$ is equivalent to $\mathbf{Sh}(T) \downarrow F$, the (Artin) glueing of F , where F is the direct image part of the geometric morphism induced by the continuous map $|T| \rightarrow T : x \mapsto x$ (which, however, is not a subspace inclusion). More explicitly, we have $F(X)(U) = X(U)$ for $U \in \mathcal{O}(T)$. Under this equivalence i^* is equivalent to $\partial_1 : \mathbf{Sh}(T) \downarrow F \rightarrow \mathbf{Sh}(|T|)$ which is known to be logical since F preserves finite limits. The right adjoint of ∂_1 sends X to $(X, \text{id}_{F(X)})$ and the left adjoint sends X to $(X, 0 \mapsto F(X))$.

Apparently, the ‘‘names’’ in Lubarsky’s model correspond to $V^{\mathcal{O}(\tilde{T})}$, the $\mathcal{O}(\tilde{T})$ -valued model of IZF. Moreover, the forcing conditions for this model **seem** to coincide with those of Lubarsky’s forcing with settling down. When Lubarsky writes $J \Vdash \phi$ this seems to be equivalent to $\{0\} \times J \subseteq \llbracket \phi \rrbracket$. There is also a relation $r \Vdash \phi$ abbreviating $\{\langle 1, r \rangle\} \subseteq \llbracket \phi \rrbracket$ corresponding to validity of ϕ^r . To establish this relation one first observes that $r \Vdash \widehat{x^r} = x$. Thus, the operation $(-)^r$ corresponds to $\mathbf{Sh}(\tilde{T}) \xrightarrow{i^*} \mathbf{Set}^{|T|} \xrightarrow{r^*} \mathbf{Set}$ where r^* is evaluation at point r .

Actually, the clauses for both notions of forcing all coincide with the exception of universal quantification. In case of $\mathcal{O}(\tilde{T})$ -valued sets we have $\llbracket \forall x. \phi(\vec{z}, x) \rrbracket = \bigwedge_{x \in V^{\mathcal{O}(\tilde{T})}} \llbracket \phi(\vec{z}, x) \rrbracket$ from which it follows that

$$\{0\} \times J \subseteq \llbracket \forall x. \phi(\vec{z}, x) \rrbracket \text{ iff } \{0\} \times J \subseteq \llbracket \phi(\vec{z}, x) \rrbracket \text{ for all } x \in V^{\mathcal{O}(\tilde{T})}.$$

¹⁹ André Joyal has used $\mathbf{Sh}(\tilde{R})$ for proving the Continuous Choice Rule for HAH. He never published this result but an account of it can be found in R. Grayson’s survey article *Heyting Valued Semantics* published in the Proceedings of Logic Colloquium 1982.

Therefore, we refer to $\mathbf{Sh}(\tilde{R})$ and more generally $\mathbf{Sh}(\tilde{T})$ as ‘‘Joyal topos’’.

Thus, if $\{0\} \times J \subseteq \llbracket \forall x. \phi(\bar{z}, x) \rrbracket$ then for all $r \in J$ we have $\langle 1, r \rangle \in \llbracket \forall x. \phi(\bar{z}, x) \rrbracket$ and thus $r \Vdash \forall x. \phi(\bar{z}^r, x)$ but not (necessarily) that $\{0\} \times J_r \subseteq \llbracket \forall x. \phi(\bar{z}^r, x) \rrbracket$ for some $J_r \in \mathcal{O}(T)$ with $r \in J_r$. This latter condition is added as an “ammendment” in the clause for universal quantifications in forcing with settling down. It is precisely this “ammendment” which prevents the model from validating full IZF.

35 J. Berger's characterization of $\text{UC}(2^{\mathbb{N}})$ relative to $\text{HA}_{\omega} + \text{AUC}$

We consider HA_{ω} augmented by the *Axiom of Unique Choice*

$$(\text{AUC}) \quad \forall x \in \sigma. \exists! y \in \tau. A(x, y) \rightarrow \exists f: \sigma \rightarrow \tau. \forall x: \sigma. A(x, f(x))$$

where σ and τ are arbitrary finite types and A is an arbitrary formula in the language of HA_{ω} . In order to avoid too much coding we have adopted a base type 2 containing just the elements 0 and 1. We refer to the type $2^{\mathbb{N}} = \mathbb{N} \rightarrow 2$ as *Cantor space* since its elements are infinite sequences of 0's and 1's.

In Brouwer's intuitionistic mathematics the following principle of *uniform continuity* plays a prominent role

$$\begin{aligned} \text{UC}(2^{\mathbb{N}}) \quad & (\forall \alpha \in 2^{\mathbb{N}}. \exists n \in \mathbb{N}. \forall \beta \in 2^{\mathbb{N}}. \bar{\alpha}(n) = \bar{\beta}(n) \rightarrow F(\alpha) = F(\beta)) \\ & \rightarrow \exists n \in \mathbb{N}. \forall \alpha, \beta \in 2^{\mathbb{N}}. \bar{\alpha}(n) = \bar{\beta}(n) \rightarrow F(\alpha) = F(\beta) \end{aligned}$$

where F ranges over $2^{\mathbb{N}} \rightarrow N$. In the following we write $\text{pc}(F)$ and $\text{uc}(F)$ as abbreviations for the premiss and conclusion of $\text{UC}(2^{\mathbb{N}})$, respectively.

Brouwer restricted attention to those continuous functionals from $2^{\mathbb{N}}$ to N which are given by a so-called “neighbourhood function”, i.e. a function $\gamma \in N^{\mathbb{N}}$ such that $\forall \alpha \in 2^{\mathbb{N}}. \exists n \in \mathbb{N}. \gamma(\bar{\alpha}(n)) > 0$ and $\forall u, v \in 2^{\mathbb{N}}. \gamma(u) > 0 \rightarrow \gamma(u) = \gamma(uv)$. The functional F induced by γ is given by $F(\alpha) = n$ iff $\exists k \in \mathbb{N}. \gamma(\bar{\alpha}(k)) = n+1$. When restricting attention to continuous functionals on $2^{\mathbb{N}}$ induced by neighbourhood functions the principle $\text{UC}(2^{\mathbb{N}})$ is equivalent to the principle

$$\text{FAN}_D \quad \forall \alpha \in 2^{\mathbb{N}}. \exists n \in \mathbb{N}. f(\bar{\alpha}(n)) > 0 \rightarrow \exists n \in \mathbb{N}. \forall \alpha \in 2^{\mathbb{N}}. f(\bar{\alpha}(n)) > 0$$

called “detachable fan”.

In $\text{HA}_{\omega} + \text{AUC}$ from $\text{UC}(2^{\mathbb{N}})$ one easily derives FAN_D and the principle²⁰

$$(\text{DECF}) \quad \forall F, G : 2^{\mathbb{N}} \rightarrow N. \text{pc}(F) \wedge \text{pc}(G) \rightarrow F = G \vee \neg F = G$$

where $F = G$ stands for extensional equality, i.e. $\forall \alpha \in 2^{\mathbb{N}}. F(\alpha) = G(\alpha)$. That the reverse implication holds as well can be seen as follows. Suppose $\text{pc}(F)$. By DECF we get that $\forall u \in 2^{\mathbb{N}}. \exists! n. n \leq 1 \wedge (n = 1 \leftrightarrow \forall v \in 2^{\mathbb{N}}. F(u * v * 0^{\infty}) = F(u * v * 0^{\infty}))$ from which we get by AUC the existence of an $f \in N^{\mathbb{N}}$ such that $\forall u \in 2^{\mathbb{N}}. f(u) = 1 \leftrightarrow \forall v \in 2^{\mathbb{N}}. F(u * v * 0^{\infty}) = F(u * v * 0^{\infty})$. Now applying FAN_D to this f we get $\exists n \in \mathbb{N}. \forall \alpha \in 2^{\mathbb{N}}. f(\bar{\alpha}(n)) = 1$ from which it follows that $\exists n \in \mathbb{N}. \forall \alpha, \beta \in 2^{\mathbb{N}}. F(\bar{\alpha}(n)) = F(\bar{\beta}(n))$ as desired.

J. Berger has further shown that in $\text{HA}_{\omega} + \text{AUC}$ there is a 1-1-correspondence between continuous $F : 2^{\mathbb{N}} \rightarrow N$ and functions $f : 2^* \rightarrow N$ satisfying

$$\text{pc}(f) \equiv \forall \alpha \in 2^{\mathbb{N}}. \exists n \in \mathbb{N}. \forall u \in 2^*. f(\bar{\alpha}(n)) = f(\bar{\alpha}(n) * u)$$

²⁰ DECF stands for “decidable equality for continuous functionals (on Cantor space)”

With a continuous F one associates $f : 2^* \rightarrow N$ with $f(u) = F(u * 0^\infty)$. Obviously, we have $\text{pc}(f)$. Given $f : 2^* \rightarrow N$ satisfying $\text{pc}(f)$ one may define a relation R_f between 2^N and N as follows

$$R_f(\alpha, n) \equiv \forall k \in N. ((\forall u \in 2^*. f(\bar{\alpha}(k)) = f(\bar{\alpha}(k) * u)) \rightarrow n = f(\bar{\alpha}(k)))$$

for which one easily verifies (using assumption $\text{pc}(f)$) that $\forall \alpha \in 2^N. \exists! n. R_f(\alpha, n)$. Thus, by AUC we get a functional $F : 2^N \rightarrow N$ satisfying $\forall \alpha \in 2^N. R_f(\alpha, F(\alpha))$ and, accordingly, is continuous and satisfies $F(u * 0^\infty) = f(u)$ for all $u \in 2^*$. Notice, however, that this equivalence is not provable in HA_ω without AUC because it does not hold in the HEO model for HA_ω with classical logic (in this model one can define (using the Kleene tree) the graph of a continuous functional on 2^N which is not computable). This observation is interesting because it (is unexpected and) allows one to reduce the question whether $\text{HA}_\omega + AUC + \text{FAN}_D \vdash \text{UC}(2^N)$ to the (possibly) simpler question whether $\mathbf{EL} + \text{FAN}_D \vdash \forall f \in N^{2^*} (\text{pc}(f) \rightarrow \text{uc}(f))$ where $\text{uc}(f)$ stands for $\exists n \in N. \forall \alpha \in 2^N. \forall u \in 2^*. f(\bar{\alpha}(n)) = f(\bar{\alpha}(n) * u)$.

A bit later (2006) in his paper *The logical strength of the uniform continuity theorem* J. Berger has shown that over $\text{HA}_\omega + AUC$ the principle $\text{UC}(2^N)$ is equivalent to FAN_c , i.e.

$$\forall \alpha \in 2^N. \exists n \in \mathbb{N}. B(\bar{\alpha}(n)) \rightarrow \exists n \in \mathbb{N}. \forall \alpha \in 2^N. B(\bar{\alpha}(n))$$

for so-called c -predicates B , i.e. predicates of the form

$$B(u) \equiv \forall v \in 2^*. C(u * v)$$

for some decidable predicate C on 2^* . Obviously FAN_c is in between FAN_D and the $\text{FAN}_{\Pi_1^0}$, i.e.

$$\forall \alpha \in 2^N. \exists n \in \mathbb{N}. B(\bar{\alpha}(n)) \rightarrow \exists n \in \mathbb{N}. \forall \alpha \in 2^N. B(\bar{\alpha}(n))$$

for Π_1^0 -predicates B , i.e. predicates of the form

$$B(u) \equiv \forall n \in \mathbb{N}. C(u, n)$$

for some decidable predicate C on $2^* \times \mathbb{N}$. Obviously, we have

$$\text{FAN}_D \Leftarrow \text{FAN}_c \Leftarrow \text{FAN}_{\Pi_1^0}$$

but is an open problem to which extent these implications can be reversed relative to $\text{HA}_\omega + AUC$. The problem is that in most realizability models either Church's Thesis holds or there holds enough of continuous choice to guarantee $\text{FAN}_{\Pi_1^0}$ in presence of FAN_D .

Can one weaken AUC in J. Berger's argument?

Actually, in his 2006 paper J. Berger claimed something stronger, namely that relative to $\text{HA}_\omega + QF - AUC^{1,0}$ the principle $\text{UC}(2^N)$ were equivalent to FAN_c . This, however, is wrong for the following reasons.

- (1) U. Kohlenbach in his paper *Foundational and mathematical uses of higher types* (2002) has shown that $\mathbf{E-PRA}_\omega^c + QF-AC^{0,0} + \text{FAN}_D \vdash \text{UC}(2^{\mathbb{N}})$ (Prop. 4.10).²¹
- (2) $\mathbf{E-PRA}_\omega^c + \text{FAN}_c$ proves $\text{FAN}_{\Pi_1^0}$ and actually arithmetic comprehension, i.e. that every arithmetically definable predicate has a characteristic function. This can be seen as follows. By contraposition FAN_c is equivalent to WKL_c , i.e.

$$(\forall n \in \mathbb{N}. \exists u \in 2^*. T(u)) \rightarrow \exists \alpha \in 2^{\mathbb{N}}. \forall n \in \mathbb{N}. T(\bar{\alpha}(n))$$

where $T(u) \equiv \exists v \in 2^*. P(u*v)$ for some decidable predicate P on 2^* . Now instantiating WKL_c by the predicate

$$P(u) \equiv \forall n < |u|. (u_n = 0 \leftrightarrow \forall k < |u|. \alpha(\langle n, k \rangle) = 0)$$

one obtains that there exists a $\chi \in 2^{\mathbb{N}}$ with

$$\forall n. \exists v \in 2^*. P(\bar{\chi}(n)*v)$$

from which it follows that $\chi(n) = 0$ iff $\forall k. \alpha(\langle n, k \rangle) = 0$. Now using the existence of characteristic functions for Π_1^0 predicates $\text{FAN}_{\Pi_1^0}$ follows from FAN_D . Since this holds in presence of parameters one easily shows that arithmetic predicates have characteristic functions.

Now if $\text{HA}_\omega + QF-AUC^{1,0} + \text{UC}(2^{\mathbb{N}}) \vdash \text{FAN}_c$ as claimed by J. Berger it follows using (1) and (2) that $\mathbf{E-PA}_\omega + QF-AUC^{1,0} + \text{FAN}_D$ proves arithmetic comprehension in contradiction with Th. 10.47 of U. Kohlenbach's book *Applied Proof Theory* (2008).

We now will spot the subtle mistake in J. Berger's tentative proof of $\text{HA}_\omega + QF-AUC^{1,0} + \text{UC}(2^{\mathbb{N}}) \vdash \text{FAN}_c$.

He starts from the assumption that for a decidable predicate C on 2^* (w.l.o.g. we assume $\neg C(\langle \rangle)$) for the associated predicate $B(u) \equiv \forall v \in 2^*. C(u*v)$ it holds that $\forall \alpha \in 2^{\mathbb{N}}. \exists n. B(\bar{\alpha}(n))$. He then claims to construct a function $F : 2^{\mathbb{N}} \rightarrow \mathbb{N}$ assigning to α the greatest n such that $\bar{\alpha}(n) \notin C$. He then shows that this F is continuous and thus by $\text{UC}(2^{\mathbb{N}})$ there is an n with $F(\alpha) < n$ for all $\alpha \in 2^{\mathbb{N}}$. Then for all $u \in 2^*$ with $|u| \geq n$ we have $B(u)$.

²¹From this it follows by $\neg\neg$ -translation and the fact that $\mathbf{E-PRA}_\omega + \text{MP}$ (where MP is Markov's principle) proves the $\neg\neg$ -translation of FAN_D that $\mathbf{E-PRA}_\omega + \text{MP} + \text{FAN}_D$ proves

$$\begin{aligned} &(\forall \alpha \in 2^{\mathbb{N}}. \exists n \in \mathbb{N}. \forall \beta \in 2^{\mathbb{N}}. \bar{\alpha}(n) = \bar{\beta}(n) \rightarrow F(\alpha) = F(\beta)) \\ &\rightarrow \neg\neg \exists n \in \mathbb{N}. \forall \alpha, \beta \in 2^{\mathbb{N}}. \bar{\alpha}(n) = \bar{\beta}(n) \rightarrow F(\alpha) = F(\beta) \end{aligned}$$

Since there is no (obvious) way to remove the double negation in the conclusion it remains an open problem whether $\mathbf{E-PRA}_\omega + \text{FAN}_D \vdash \text{UC}(2^{\mathbb{N}})$ and even whether $\mathbf{E-PRA}_\omega + \text{MP} + \text{FAN}_D \vdash \text{UC}(2^{\mathbb{N}})$.

I first thought that the classical equivalence of FAN_c and $\text{FAN}_{\Pi_1^0}$ would allow one to show their equivalence in constructive logic together with MP. But I was mistaken since although under MP the negative translation of FAN_D is equivalent to FAN_D this is not the case for FAN_c and $\text{FAN}_{\Pi_1^0}$.

The mistake, of course, is made when arguing for the existence of the desired functional F . Using the assumption $\forall \alpha \in 2^N . \exists n . B(\bar{\alpha}(n))$ one easily shows that $\forall \alpha \exists ! n R(\alpha, n)$ where $R(\alpha, n) \equiv \neg C(\bar{\alpha}(n)) \wedge \forall k > n . C(\bar{\alpha}(k))$. Now if R were quantifier free one could conclude using $QF-AUC^{1,0}$ that there exists a functional $F : 2^N \rightarrow N$ with $\forall \alpha \exists ! n R(\alpha, F(\alpha))$. But, alas, the relation R is not quantifier free but Π_1^0 .

This mistake is interesting because it shows how strong AUC is compared with $QF-AUC^{1,0}$.

Summary of the Current State of Knowledge

In the formal system $FS \equiv HA_\omega + QF-AC^{0,0} + QF-AUC^{1,0}$ one can show that

$$FAN_D \Leftarrow UC(2^\omega) \Leftarrow FAN_c$$

where the first implication can be reversed when adding classical logic to FS and the second implication can be reversed when adding AUC (for Π_1^0 relations) to FS . Thus, under different strengthenings of FS the principle $UC(2^\omega)$ gets once equivalent to FAN_D and once to FAN_c though in FS with classical logic the principle FAN_c is significantly stronger than FAN_D . However, when adding both AUC and classical logic to FS both fan principles imply the fan principle for bars of arbitrary logical complexity.

36 Open locales constructively

A locale map $f : A \rightarrow B$ is called *open* iff $f_! \dashv f^* \dashv f_* : A \rightarrow B$ validating the Frobenius reciprocity law $f_!(a) \wedge b = f_!(a \wedge f^*b)$. A locale A is called *open* iff the unique map $t : A \rightarrow \Omega = \mathcal{P}(1)$ is open. One can show that in this case Frobenius reciprocity is automatic provided t^* has a left adjoint $t_!$.

Thus, it remains to characterise in logical terms when such a left adjoint $t_!$ exists. First recall that $t^* : \Omega \rightarrow A : u \mapsto \bigvee \{\top \mid * \in u\}$. Then, provided it exists, the left adjoint $t_!$ is constructed as follows

$$t_!(a) = \bigwedge \{u \mid a \leq t^*u\}$$

One easily checks that $t_!$ is monotonic and $t_!t^*u \leq u$. Thus, a left adjoint to t^* exists iff $a \leq t^*t_!a = \bigvee \{\top \mid * \in t_!(a)\}$ for all $a \in A$. Now $* \in t_!(a)$ is equivalent to $\forall u : \Omega. a \leq t^*(u) \Rightarrow * \in u$ which latter condition we abbreviate as $\text{Pos}(a)$. Obviously $\text{Pos}(a)$ holds iff every cover of a is nonempty. Thus $a \leq t^*t_!a$ iff $a \leq \bigvee \{\top \mid \text{Pos}(a)\}$. But the latter is equivalent to $a = \bigvee \{x \in A \mid x \leq a \text{ and } \text{Pos}(a)\}$. Thus A is open iff every $a \in A$ is the supremum of all $x \leq a$ with $\text{Pos}(a)$, i.e. every element of A is the supremum of positive elements below it.²²

Notice that $\text{Pos}(a)$ is an impredicative notion. Therefore, in formal topology one adds a positivity predicate Pos as additional structure and calls a formal space with a positivity predicate open iff every formal neighbourhood is the supremum the positive neighbourhoods contained in it.

37 For every countably based T_0 space X there is an open surjection from a subspace C of $\mathbb{N}^{\mathbb{N}}$ to X

Let X be a countably based T_0 space.²³ Let $\{B_n \mid n \in \mathbb{N}\}$ be an enumeration of a basis for $\mathcal{O}(X)$. The map $i_X : X \rightarrow \mathcal{P}(\mathbb{N}) : x \mapsto \{n \in \mathbb{N} \mid x \in B_n\}$ exhibits X as a subspace of $\mathcal{P}(\mathbb{N})$ endowed with the Scott topology. This space is isomorphic to $S^{\mathbb{N}}$, the countable product of the Sierpiński space S . The map $e : \mathbb{N}^{\mathbb{N}} \rightarrow \mathcal{P}(\mathbb{N}) : \alpha \mapsto \{n \in \mathbb{N} \mid \exists k \in \mathbb{N} \alpha(\langle n, k \rangle) = 0\}$ is an open surjection. Now consider the pullback

$$\begin{array}{ccc} C & \xrightarrow{c} & X \\ i_C \downarrow \lrcorner & & \downarrow i_X \\ \mathbb{N}^{\mathbb{N}} & \xrightarrow{e} & \mathcal{P}(\mathbb{N}) \end{array}$$

where i_X and i_C are subspace inclusions and e and c are open surjections. Thus c is an open surjection from a subspace C of $\mathbb{N}^{\mathbb{N}}$ to the space X .

²²See also Lemma C3.1.17 of PTJ's *Elephant*.

²³Sometimes such spaces are called *completely separable*.

38 Formal Spaces (Fourman and Grayson 1982)

Formal spaces may be thought of as infinitary propositional geometric theories. In their 1982 paper *Formal Spaces* M. Fourman and R. Grayson they define such theories to be given by a conditional \wedge -semilattice (\mathbb{P}, \leq) called the language of the theory and a set T of sequents $\theta \vdash \Gamma$ where $\theta \in \mathbb{P}$ and $\Gamma \subseteq \mathbb{P}$. The induced entailment relation \vdash_T is the least set of sequents which contains T and is closed under the rules

$$\frac{p \leq q}{p \vdash_T q} \quad \frac{p \leq \Gamma \quad \Gamma \subseteq \Delta}{p \vdash_T \Delta} \text{ (weakening)}$$

$$\frac{p \vdash_T \Gamma}{p \wedge q \vdash_T \Gamma \wedge p} \text{ (localisation)} \quad \frac{p \vdash_T \Delta \quad q \vdash_T \Gamma (q \in \Delta)}{p \vdash_T \Gamma} \text{ (composition)}$$

Such an entailment relation \vdash_T is nothing but a Grothendieck topology on the poset \mathbb{P} .

A T -closed sieve is a subset K of \mathbb{P} such that $p \in K$ whenever $p \vdash_T K$. Thus closed sieves are in particular downward closed w.r.t. \leq . For $U \subseteq \mathbb{P}$ let $\overline{U} := \{p \in \mathbb{P} \mid p \vdash_T U\}$. Obviously, we have $U \subseteq \overline{U} = \overline{\overline{U}}$. Moreover, one can show that that $\overline{U \wedge V} = \overline{U} \cap \overline{V}$. We write $M(T)$ for the set of T -closed sieves ordered by \subseteq . It is a complete lattice where

$$\bigvee_{i \in I} U_i = \overline{\bigcup_{i \in I} U_i} \quad \text{and} \quad \bigwedge_{i \in I} U_i = \bigcap_{i \in I} U_i$$

for $(U_i)_{i \in I}$. Using $\overline{U \wedge V} = \overline{U} \cap \overline{V}$ one easily checks that $M(T)$ is a frame, i.e. validates the infinite distributivity law $V \wedge \bigvee_{i \in I} U_i = \bigvee_{i \in I} V \wedge U_i$.

Suppose T is countable. Let $(\theta_i \vdash \Gamma_i)_{i \in \mathbb{N}}$ be an enumeration of T where each axiom shows up infinitely often. Let \mathbb{T} be the tree defined inductively as follows

$$\langle \rangle \in \mathbb{T}$$

$$\langle \phi \rangle \in \mathbb{T} \text{ for all } \phi \in \mathbb{P}$$

if $a \in \mathbb{T}$ of length $i + 1$ then

$$a * \langle a(i) \rangle \in \mathbb{T} \quad \text{whenever } a(i) \not\leq \theta_i$$

$$a * \langle a(i) \wedge \phi \rangle \in \mathbb{T} \quad \text{whenever } a(i) \leq \theta_i \text{ and } \phi \in \Gamma_i$$

If $a \in \mathbb{T}$ has length $i + 1$ then we write ϕ_a for $a(i)$. Apparently $a(i) \geq a(j)$ whenever $i \leq j$. We say that $a \in \mathbb{T}$ *secures* $\psi \in \mathbb{P}$ iff $\phi_a \leq \psi$ and a *secures* $\Gamma \subseteq \mathbb{P}$ iff $\phi_a \leq \psi$ for some $\psi \in \Gamma$.

For $a, b \in \mathbb{T}$ we write $b \leq a$ iff a is a prefix of b . The inductive topology on \mathbb{T} is the one generated by the immediate successor relation, i.e. sequents of the form $a \vdash \{a * \langle \phi \rangle \mid \phi \in \mathbb{P} \text{ and } a * \langle \phi \rangle \in \mathbb{T}\}$. We write \vdash_I for the induced entailment relation and say that C is an *inductive bar* for a iff $a \vdash_I C$. We write $P(\mathbb{T})$ for the formal space induced by \vdash_I .

Fourman and Grayson have shown that

Theorem 1

- (1) Every infinite path α in \mathbb{T} induces a model \mathfrak{m} of \mathbb{T} where $\psi \in \mathfrak{m}$ iff $\phi_a \leq \psi$ for some $i \in \mathbb{N}$.
- (2) $\theta \vdash_T \gamma$ iff for all $a \in \mathbb{T}$ which secure θ the set $\{b \in \mathbb{T} \mid b \preceq a \text{ and } b \text{ secures } \Gamma\}$ is an inductive bar of a .

For every $U \in M(T)$ let $f^*(U) = \{a \in \mathbb{T} \mid \phi_a \vdash_T U\}$. For showing that $f^*(U) \in P(\mathbb{T})$ suppose that $a \in \mathbb{T}$ of length $i + 1$ such that all successors of a are in $f^*(U)$. If $\phi_a \not\leq \theta_i$ then $a * \langle \phi_a \rangle$ is a successor of a and thus $\phi_a \vdash_T U$, i.e. $a \in f^*(U)$. Otherwise $\phi_a \wedge \phi \vdash_T U$ for all $\phi \in \Gamma_i$ and, therefore, since $\phi_a = \phi_a \wedge \theta_i \vdash_T \phi_a \wedge \Gamma_i$ it follows that $\phi_a \vdash_T U$. Thus, we have shown that $f^* : M(T) \rightarrow P(\mathbb{T})$. It is easy to see that f^* is 1-1 and preserves meets (since meets in $M(T)$ are given by intersections). The map f^* preserves arbitrary joins since by Theorem 1 we have $\phi_a \vdash_T U$ iff for every $b \in \mathbb{T}$ with $\phi_b \leq \phi_a$ the set $\{c \in \mathbb{T} \mid c \preceq b \text{ and } c \text{ secures } U\}$ is an inductive cover of b . Thus f^* has a right adjoint f_* . Since f^* is 1-1 the locale map $f : P(\mathbb{T}) \rightarrow M(T)$ as given by $f^* \dashv f_*$ is a surjection.

Since f^* preserves arbitrary meets it has a left adjoint $f_!$ sending $V \in P(\mathbb{T})$ to $f_!(V) = \{\phi_a \mid a \in V\}$. The adjunction $f_! \dashv f^*$ satisfies the Frobenius reciprocity condition $f_!(V) \cap U = f_!(V \cap f^*(U))$ which can be seen as follows: if $a \in V$ with $\phi_a \in U$ then $\phi_a \vdash_T U$, i.e. $a \in f^*(U)$, and thus $a \in V \cap f^*(U)$ from which it follows that $\phi_a \in f_!(V \cap f^*(U))$. Thus f^* preserves also Heyting implication. Thus, we have shown

Theorem 2

There is an open surjection $f : P(\mathbb{T}) \rightarrow M(T)$ given by

$$f^*(U) = \{a \in \mathbb{T} \mid \phi_a \vdash_T U\} \quad \text{and} \quad f_!(V) = \overline{\{\phi_a \mid a \in V\}}$$

This theorem tells us that for *countably presented* formal spaces A there is an open surjection from formal Baire space to A and thus assuming bar induction such an A has enough points. Here “countably presented” means that the underlying language has countably many propositions (“countably generated”) and that the formal space is axiomatized by a theory with countably many sequents.

There arises the question whether “countably generated” is equivalent to “countably presentable”. As pointed out by M. Schröder this is not the case already for subspaces of Baire space $\mathcal{B} = \mathbb{N}^{\mathbb{N}}$ since a subset X of \mathcal{B} is a continuous image of \mathcal{B} iff X is analytic and not all subsets of \mathcal{B} are analytic. Thus, for any subset X of \mathcal{B} which is not analytic the corresponding subspace X of \mathcal{B} is a countably based Hausdorff space and thus sober but since X is not analytic there cannot exist an open continuous surjection $f : \mathcal{B} \rightarrow X$ since there does not even exist a continuous surjection from \mathcal{B} to X since X was assumed as not analytic.

39 $\mu n (F(\bar{f}(n)*0^\infty) < n)$ is computable

Let $F \in Tp(2)$ and $f \in Tp(1)$. We write f_n as an abbreviation for $\bar{f}(n)*0^\infty$. Due to an argument by T. Grilliot one may decide whether $F(f) = \lim_{n \rightarrow \infty} F(f_n)$.

For $m \in \mathbb{N}$ consider the functions $g_m \in Tp(1)$ defined as

$$g_m(k) = \begin{cases} f_n(k) & \text{if } m \leq n < k \text{ and } F(f) \neq F(f_n) \\ & \text{where } n \text{ is chosen minimal with this property} \\ f(k) & \text{otherwise.} \end{cases}$$

Then $F(f) = F(g_m)$ iff $\forall m \geq n (F(f) = F(f_n))$.

Now define $h \in Tp(1)$ as

$$h(m) = \begin{cases} 0 & \text{if } F(f) = F(g_m) \\ f(k) & \text{otherwise} \end{cases}$$

for which it holds that

- (1) if $F(f) = \lim_{n \rightarrow \infty} F(f_n)$ then $h = f_n$ where n is the greatest number with $F(f) \neq F(f_n)$
- (2) if $F(f) \neq \lim_{n \rightarrow \infty} F(f_n)$ then $h = f$.

Thus $F(f) = \lim_{n \rightarrow \infty} F(f_n)$ iff $F(f) \neq F(h)$.

A consequence of Grilliot's decidability result is that the functional $M : Tp(2) \times Tp(1) \rightarrow \mathbb{N}$ with

$$M(F, f) = \begin{cases} n & \text{if } n \text{ is the least number with } F(\bar{f}(n)*0^\infty) < n \\ 0 & \text{if such an } n \text{ does not exist} \end{cases}$$

is (S1)-(S9) computable. Given F and f first check whether $F(f) = \lim_{n \rightarrow \infty} F(f_n)$. If yes then there exists a least n with $F(\bar{f}(n)*0^\infty) < n$ and compute it by terminating μ -search. If $F(f) \neq \lim_{n \rightarrow \infty} F(f_n)$ then 2E is computable relative to F and f . Use this program for 2E to check whether $\exists n (F(\bar{f}(n)*0^\infty) < n)$. If yes then compute the desired n by terminating μ -search and otherwise output 0.

Thus, we can interpret $T + M$ programs P in Tp and in \mathcal{M} , the hereditary majorizable functionals, and know that $\llbracket P \rrbracket^{Tp}$ and $\llbracket P \rrbracket^{\mathcal{M}}$ are related by the canonical logical relation R between Tp and \mathcal{M} which is identity at type 0. At type 2 this relation is the inclusion of $\mathcal{M}(2)$ into $Tp(2)$. Thus, $T + M$ programs of type 2 receive the same interpretation in Tp and in \mathcal{M} .

40 U. Berger's Program for the Fan Functional

Let ι be the type of natural numbers and o the type of booleans (both with \perp included). We will exhibit a functional E of type $((\iota \rightarrow o) \rightarrow o) \rightarrow o$ such that for total $t \in (\iota \rightarrow o) \rightarrow o$ it holds that $E(t) = \text{true}$ if $t(f) = \text{true}$ for some total f and $E(t) = \text{false}$ otherwise.

First consider the auxiliary functions $\text{left}, \text{right} : ((\iota \rightarrow o) \rightarrow o) \rightarrow (\iota \rightarrow o) \rightarrow o$ which are defined as

$$\text{left}(t)(f) = t(\lambda n^t. \text{if } n=0 \text{ then true else } f(n-1))$$

$$\text{right}(t)(f) = t(\lambda n^t. \text{if } n=0 \text{ then false else } f(n-1))$$

respectively. Then we define $E : ((\iota \rightarrow o) \rightarrow o) \rightarrow o$ and $B : ((\iota \rightarrow o) \rightarrow o) \rightarrow \iota \rightarrow o$ as

$$E(t) = t(B(t))$$

$$B(t)(n) = \text{if } n=0 \text{ then } E(\text{left}(t)) \\ \text{else if } E(\text{left}(t)) \text{ then } B(\text{left}(t))(n-1) \\ \text{else } B(\text{right}(t))(n-1)$$

by mutual recursion.

For verifying that E behaves as specified first observe that for every total and continuous $t : (\iota \rightarrow o) \rightarrow o$ there exists a least number n such that $t(f) = t(g)$ whenever f and g are total functions with $f(i) = g(i)$ for $i < n$. This n is called the *modulus of uniform continuity of t* and denoted as $\text{muc}(t)$. Now by induction on $\text{muc}(t)$ we show that

- (i) $B(t)$ is total and
- (ii) if $t(B(t)) = \text{false}$ then $t(f) = \text{false}$ for all total f .

Obviously $\text{muc}(\text{left}(t)), \text{muc}(\text{right}(t)) < \text{muc}(t)$. Thus, by induction hypothesis the functions $B(\text{left}(t))$ and $B(\text{right}(t))$ are both total. Accordingly, the function $B(t)$ is total as well. Suppose $t(B(t)) = \text{false}$ but $t(f) = \text{true}$ for some total f . Then for some total f we have $\text{left}(t)(f) = \text{true}$ or $\text{right}(t)(f) = \text{true}$. Thus by induction hypothesis we have $E(\text{left}(t)) = \text{true}$ or $E(\text{right}(t)) = \text{true}$. Suppose $E(\text{left}(t)) = \text{true}$ then $\text{false} = t(B(t)) = \text{left}(t)(B(\text{left}(t))) = E(\text{left}(t)) = \text{true}$. On the other hand if $E(\text{left}(t)) = \text{false}$ and $E(\text{right}(t)) = \text{true}$ then $\text{false} = t(B(t)) = \text{right}(t)(B(\text{right}(t))) = E(\text{right}(t)) = \text{true}$. Thus, we have shown that from $t(B(t)) = \text{false}$ it follows in any case that $t(f) = \text{true}$ for all total f .

Using E we can define A of type $((\iota \rightarrow o) \rightarrow o) \rightarrow o$ as

$$A(t) = \neg E(\lambda f^{\iota \rightarrow o}. \neg t(f))$$

which has the property that for total t it holds that $A(t) = \text{true}$ if $t(f) = \text{true}$ for all total f and $A(t) = \text{false}$ otherwise.

Now given A we can define the *Fan functional* $\text{muc} : ((\iota \rightarrow o) \rightarrow o) \rightarrow \iota$ recursively as follows

```

muc(t) =  if A( $\lambda f^{t \rightarrow t}.t(f) = t(\lambda n^t.\text{true})$ )
           then 0
           else 1 + max(muc(left(t)), muc(right(t)))

```

Notice, however, that the above programs do not work in the hierarchy of total continuous functionals because already the mutual recursive definition of B and E has the constant \perp functions as least fixpoint! In the hierarchy of partial continuous functionals this is not the case since $t(B(t))$ may be defined even if $B(t) = \perp$.

41 Universal quantification on Cantor space not definable in Gödel's T

In the previous section we have seen that in the model \mathcal{C} of continuous functionals there exists a functional $A : 2^{\mathcal{C}} \rightarrow 2$ with $A(p) = 0$ iff $p(f) = 0$ for all $f \in \mathcal{C}$. The reason is that such a functional A can be implemented in PCF. Already Gandy and Hyland have shown that such an A is not S1-S9 definable in \mathcal{C} and thus not definable in Gödel's T.

D. Normann has given the following slick proof based on domain theory. Suppose A were S1-S9 definable in \mathcal{C} then $A(p) = 0$ whenever $p(f) = 0$ for all recursive $f \in 2^{\mathcal{N}}$. Then there exists a functional Φ in the Scott model implementing A . Due to the Kleene tree there exists a functional F in the Scott model with $F(f) = 0$ for all recursive f but $F(f) = \perp$ for some nonrecursive total f . By continuity there exists a compact $F_0 \sqsubseteq F$ with $\Phi(F_0) = 0$. But then there exists a total computable G extending F_0 with $G(f) = 1$ for some recursive f . But then by continuity $\Phi(G) = 0$ and thus $A(G) = 0$ although G is not constantly 0 on total arguments.

Let \mathcal{T} be the extensional collapse of closed Gödel T terms. Since Gödel T definable elements are dense in \mathcal{C} at type levels of degree ≤ 2 if A were Gödel T definable in \mathcal{T} then it were Gödel T definable also in \mathcal{C} . Thus in \mathcal{T} the functional A is not definable in Gödel's T.

In HEO we have $2^{\mathcal{N}} \cong \mathcal{N}^{\mathcal{N}}$ and thus $2^{2^{\mathcal{N}}} \cong 2^{\mathcal{N}^{\mathcal{N}}}$. Suppose A exists in HEO. Then also universal quantification on $2^{\mathcal{N}^{\mathcal{N}}}$ would exist in HEO. Since \mathcal{N} is a retract of $\mathcal{N}^{\mathcal{N}}$ in HEO there would exist also universal quantification over \mathcal{N} as a functional $2^{\mathcal{N}} \rightarrow 2$ which is not the case. Thus A does not exist in HEO.

41.1 Fan functional not Gödel T definable from A

The functional A is hereditarily majorisable (in \mathcal{C}) and so are all element definable in $\text{T} + \text{A}$ (since Gödel T definability preserves the property of being hereditarily majorizable). Since the fan functional is not hereditarily majorizable it cannot be defined in $\text{T} + \text{A}$.

42 Gödel T definable elements are dense in \mathcal{C} and HEO

Due to a result by Plotkin all compact elements e of the Scott model can be denoted by a PCF + por term t_e not using the fixpoint operator but constants Ω_σ for denoting $\perp \in D_\sigma$. Now replacing the Ω_σ 's by 0_σ 's gives rise to a T + por term t'_e denoting a total element d' extending e . Let \tilde{t}_e be the Gödel T term obtained from t'_e replacing por by or. Then \tilde{t}_e denotes a total element d *equivalent* to d' . This shows that Gödel T denotable elements are dense in the continuous functionals model \mathcal{C} and in HEO (since the latter is isomorphic to the effective Scott model factored by equivalence of total elements).

43 Esakia's Representation Theorem for Heyting Algebras

43.1 Preliminary Observations

Lemma 43.1 *Let L be a distributive lattice. If F is a filter in L and I is an ideal in L with $F \cap I = \emptyset$ then there exists a prime filter ∇ with $F \subseteq \nabla$ and $\nabla \cap I = \emptyset$.*

Proof: Let \mathcal{F} be the set of filters Φ in L with $F \subseteq \Phi$ and $\Phi \cap I = \emptyset$. By Zorn's lemma there is a maximal element ∇ (w.r.t. \subseteq) in \mathcal{F} . Suppose ∇ were not prime. Then there exist $a_1, a_2 \in L$ with $a_1 \vee a_2 \in \nabla$ and $a_1, a_2 \notin \nabla$. Then by maximality of ∇ there exist $b_1, b_2 \in I$ and $c_1, c_2 \in \nabla$ such that $c_i \wedge a_i \leq b_i$. Then $b = b_1 \vee b_2 \in I$ $c = c_1 \wedge c_2 \in \nabla$. We have

$$b \geq (a_1 \wedge c) \vee (a_2 \wedge c) = (a_1 \vee a_2) \wedge c \in \nabla$$

and thus $b \in \nabla \cap I$ contradicting $\nabla \cap I = \emptyset$. Thus, we have shown by contradiction that ∇ is a prime filter in L . \square

Lemma 43.2 *Every Heyting algebra (ha) \mathfrak{A} appears as sub-Ha of a complete Ha.*

Proof: Let \mathfrak{A} be a Ha. Let X be the set of all prime filters on \mathfrak{A} and $\mathcal{O}(X)$ be the least topology on X containing as basic opens all sets of the form $P_a = \{\nabla \in X \mid a \in \nabla\}$ for some $a \in \mathfrak{A}$. Obviously, we have $P_{a \wedge b} = P_a \cap P_b$ and $P_{a \vee b} = P_a \cup P_b$. Moreover, for $c \in \mathfrak{A}$ we have $P_c \subseteq P_{a \rightarrow b}$ iff $c \leq a \rightarrow b$ iff $c \wedge a \leq b$ iff $P_c \cap P_a = P_{c \wedge a} \subseteq P_b$. From this it follows that $P_{a \rightarrow b} = P_{a \rightarrow b}$. \square

43.2 Descriptive Frames

A *generalized intuitionistic frame* (gif) is a triple $\mathfrak{F} = (W, R, P)$ where (W, R) is a partial order and P is a subHa of $\text{Up}(W, R)$, the Ha of upward closed subsets of (W, R) . We write \mathfrak{F}^* for the Ha P . The *Vietoris topology* on W is the least topology on W such that all elements of P and their complements are open sets.

With a Heyting algebra \mathfrak{A} one may associate the generalized intuitionistic frame $\mathfrak{A}_* = (W_{\mathfrak{A}}, R_{\mathfrak{A}}, P_{\mathfrak{A}})$ where $W_{\mathfrak{A}}$ is the set of prime filters on \mathfrak{A} , the relation $R_{\mathfrak{A}}$ is given by \subseteq and $P_{\mathfrak{A}} = \{P_a \mid a \in \mathfrak{A}\}$ with $P_a = \{\nabla \in W_{\mathfrak{A}} \mid a \in \nabla\}$. Thus, we have $\mathfrak{A} \cong (\mathfrak{A}_*)^*$ via the map $y_{\mathfrak{A}} : \mathfrak{A} \rightarrow P_{\mathfrak{A}} : a \mapsto P_a$.

Next we want to characterize those gif's $\mathfrak{F} = (W, R, P)$ where the map $y_{\mathfrak{F}} : W \rightarrow W_P : x \mapsto P_x = \{U \in P \mid x \in U\}$ establishes an isomorphism between \mathfrak{F} and $(\mathfrak{F}^*)^*$. Such gif's are called *descriptive*. Obviously, gif's $(W, R, \{\emptyset, W\})$ with $|W| > 2$ are not descriptive.

A gif $\mathfrak{F} = (W, R, P)$ is called *refined* iff for all $x, y \in W$ we have xRy iff $\forall U \in P(x \in U \Rightarrow y \in U)$ and \mathfrak{F} is called *compact* iff the Vietoris topology on W is compact.

Lemma 43.3 (Prop. 8.48 of [ChZa] p.255) *A gif $\mathfrak{F} = (W, R, P)$ is compact iff all prime filters in P are of the form $P_x = \{U \in P \mid x \in U\}$ for some $x \in W$.*

Proof: (\Rightarrow) Suppose that the Vietoris topology on W is compact and ∇ is a prime filter in P . Then $\Delta = W \setminus \nabla$ is a prime ideal and, therefore, closed under finite unions.

Let $\mathcal{X} = \nabla$ and $\mathcal{Y} = \{-P \mid P \in \Delta\}$. Suppose $X_1, \dots, X_n \in \mathcal{X}$ and $Y_1, \dots, Y_m \in \mathcal{Y}$. Then $X = X_1 \cap \dots \cap X_n \in \mathcal{X}$ and $Y = Y_1 \cup \dots \cup Y_m \in \mathcal{Y}$. Now if $X \cap Y = \emptyset$ then $X \subseteq W \setminus Y \in \nabla$ which is impossible since $Y \in \mathcal{Y}$ and thus $W \setminus Y \in \Delta$ and $\Delta \cap \nabla = \emptyset$. Thus, since \mathfrak{F} is compact it follows that $\bigcap \mathcal{X} \cup \mathcal{Y}$ is nonempty, i.e. $x \in \bigcap \mathcal{X} \cup \mathcal{Y}$ for some $x \in X$.

We show now that $P_x = \nabla$. Clearly, we have $\nabla \subseteq P_x$ since $x \in \bigcap \mathcal{X} = \bigcap \nabla$. Suppose $x \in X \in P$ but $X \notin \nabla$. But then $X \in \Delta$ and thus $W \setminus X \in \mathcal{Y}$ from which it follows that $x \notin W \setminus X \in \mathcal{Y}$ which is impossible since $x \in \bigcap \mathcal{X} \cup \mathcal{Y}$.

(\Leftarrow) Suppose every prime filter in P is of the form P_x for some $x \in W$. Suppose $\mathcal{X} \subseteq P$ and $\mathcal{Y} \subseteq \bar{P} = \{W \setminus X \mid X \in P\}$ such that $\bigcap \mathcal{Z} \neq \emptyset$ for all finite $\mathcal{Z} \subseteq \mathcal{X} \cup \mathcal{Y}$. We must show that $\bigcap \mathcal{X} \cup \mathcal{Y} \neq \emptyset$.

Let ∇ be the filter generated by \mathcal{X} and Δ be the ideal generated by $\{W \setminus Y \mid Y \in \mathcal{Y}\}$. We show that $\nabla \cap \Delta = \emptyset$. Otherwise there exist $\mathcal{X}' \subseteq_{\text{fin}} \mathcal{X}$ and $\mathcal{Y}' \subseteq_{\text{fin}} \mathcal{Y}$ with $\bigcap \mathcal{X}' = \bigcup_{Y \in \mathcal{Y}'} W \setminus Y$ from which it follows that $\emptyset = \bigcap \mathcal{X}' \cap W \setminus (\bigcup_{Y \in \mathcal{Y}'} W \setminus Y) = \bigcap \mathcal{X}' \cap \mathcal{Y}' = \bigcap \mathcal{X}' \cup \mathcal{Y}'$ which is impossible since $\mathcal{X}' \cup \mathcal{Y}' \subseteq_{\text{fin}} \mathcal{X} \cup \mathcal{Y}$. Let ∇' be a prime filter with $\nabla \subseteq \nabla'$ and $\nabla' \cap \Delta = \emptyset$. Then due to our assumption about \mathfrak{F} there is an $x \in W$ with $\nabla' = P_x$. Then $x \in Z$ for all $Z \in \nabla$ and $x \notin Z$ for all $Z \in \Delta$ from which it follows that $x \in \bigcap \mathcal{X} \cup \mathcal{Y}$. \square

Theorem 43.1 *A frame $\mathfrak{F} = (W, R, P)$ is descriptive iff \mathfrak{F} is refined and compact.*

Proof: Obviously, a frame \mathfrak{F} is refined iff $y_{\mathfrak{F}} : W \rightarrow W_{\mathfrak{F}^*}$ preserves and reflects the order, i.e. $y_{\mathfrak{F}}(x) \subseteq y_{\mathfrak{F}}(y)$ iff xRy , and by Lemma 43.3 \mathfrak{F} is compact iff $y_{\mathfrak{F}}$ is surjective.

Thus, a descriptive frame is necessarily refined and compact. On the other hand if $\mathfrak{F} = (W, R, P)$ is refined and compact then by the above consideration $y_{\mathfrak{F}} : W \rightarrow W_{\mathfrak{F}^*}$ is an isomorphism of partial orders. Thus, it remains to show that for $X \subseteq W$ we have $X \in P$ iff $y_{\mathfrak{F}}[X] \in P_{\mathfrak{F}^*}$. Recall that $P_{\mathfrak{F}^*} = \{y_{\mathfrak{F}^*}(X) \mid X \in P\}$ and $y_{\mathfrak{F}^*}(X) = \{\nabla \in W_{\mathfrak{F}^*} \mid X \in \nabla\}$. So if $X \in P$ we have $y_{\mathfrak{F}}[X] = \{P_x \mid x \in X\} = \{P_x \mid x \in W \text{ and } X \in P_x\} = \{\nabla \in W_{\mathfrak{F}^*} \mid X \in \nabla\} = y_{\mathfrak{F}^*}(X) \in P_{\mathfrak{F}^*}$. Conversely, if $y_{\mathfrak{F}}[X] \in P_{\mathfrak{F}^*}$ then there exists a $Y \in P$ with $y_{\mathfrak{F}}[X] = \{\nabla \in W_{\mathfrak{F}^*} \mid Y \in \nabla\}$, i.e. $\{P_x \mid x \in X\} = \{P_x \mid x \in W \text{ and } Y \in P_x\} = \{P_x \mid x \in Y\}$, from which it follows that $X = Y \in P$ since \mathfrak{F} is refined (i.e. $P_x = P_y$ iff $x = y$). \square

43.3 Esakia Spaces

Recall that a *Stone space* is a compact, 0-dimensional Hausdorff space (where 0-dimensional means that clopen sets form a basis). An *Esakia space* (as intro-

duced in [Esa]) is a pair (X, R) where X is a Stone space and R is a partial order such that

(E1) for every $x \in X$ the set $R(x) = \{y \in X \mid xRy\}$ is closed and

(E2) for every clopen subset U of X the set $R^{-1}(U) = \{y \in X \mid \exists x \in U. yRx\}$ is clopen.

Alternatively, as done originally in [Esa], one may axiomatize Esakia spaces as pairs (X, R) where X is a Stone space and R is a partial order on X such that for every $x \in X$ the set $\rho(x) = R(x) = \{y \in X \mid xRy\}$ is closed and the map $\rho : X \rightarrow 2^X$ is perfect²⁴ where 2^X is the set of nonempty closed sets with the Vietoris²⁵ topology.

Lemma 43.4 *Let \mathfrak{A} be a Ha and $\mathcal{O}_{\mathfrak{A}}$ the Vietoris topology on $W_{\mathfrak{A}}$, i.e. the least topology such that P_a and $-P_a$ are open for all $a \in \mathfrak{A}$. Then $(W_{\mathfrak{A}}, \mathcal{O}_{\mathfrak{A}})$ is a Stone space, where the clopens are precisely the finite unions of sets of the form $P_a \cap -P_b$ for $a, b \in \mathfrak{A}$, and $(W_{\mathfrak{A}}, \mathcal{O}_{\mathfrak{A}}, \subseteq)$ is an Esakia space. Finally, the clopen upward closed subsets of $W_{\mathfrak{A}}$ are precisely those of the form P_a for some $a \in \mathfrak{A}$, i.e. \mathfrak{A} is isomorphic to the clopen upward closed subsets of $W_{\mathfrak{A}}$ ordered by \subseteq .*

Proof: First we show that $W_{\mathfrak{A}}$ is compact w.r.t. the Vietoris topology.²⁶ Let $X, Y \subseteq \mathfrak{A}$ such that $\bigcap_{a \in X} P_a \cap \bigcap_{b \in Y} -P_b = \emptyset$. Suppose that for all $X_0 \subseteq_{\text{fin}} X$ and $Y_0 \subseteq_{\text{fin}} Y$ the set $\bigcap_{a \in X_0} P_a \cap \bigcap_{b \in Y_0} -P_b$ is nonempty. Then the filter F generated by X and the ideal I generated by Y are disjoint. Thus, by Zorn's Lemma there exists a prime filter $\nabla \in W_{\mathfrak{A}}$ with $X \subseteq \nabla$ and $Y \cap \nabla = \emptyset$. But then $\nabla \in \bigcap_{a \in X} P_a \cap \bigcap_{b \in Y} -P_b = \emptyset$ contradicting the assumption.

That the space $W_{\mathfrak{A}}$ is Hausdorff can be seen as follows. Suppose ∇_1 and ∇_2 are distinct prime filters on \mathfrak{A} . Then w.l.o.g. there is an $a \in \mathfrak{A}$ with $a \in \nabla_1$ and $a \notin \nabla_2$. Thus, we have $\nabla_1 \in P_a$ but $\nabla_2 \in -P_a$ and, of course, the opens P_a and $-P_a$ are disjoint.

Every open set in the Vietoris topology appears as a union of sets of the form $P_a \cap -P_b$ for some $a, b \in \mathfrak{A}$. Since $W_{\mathfrak{A}} \setminus (P_a \cap -P_b) = -P_a \cup P_b$ is also open it follows that $P_a \cap -P_b$ is clopen. Thus, the clopen sets form a basis for the Vietoris topology, i.e. $W_{\mathfrak{A}}$ is 0-dimensional w.r.t. the Vietoris topology.

Suppose $C \subseteq W_{\mathfrak{A}}$ is clopen. Since C is closed and $W_{\mathfrak{A}}, \mathcal{O}_{\mathfrak{A}}$ is compact Hausdorff it follows that C is compact and, thus, the clopen C appears as finite union of sets of the form $P_a \cap -P_b$.

For showing (E1) suppose ∇ is a prime filter on \mathfrak{A} . We have to show that $\{\nabla' \in W_{\mathfrak{A}} \mid \nabla \subseteq \nabla'\}$ is clopen. A prime filter $\nabla' \not\subseteq \nabla$ iff there exists a $b \in \nabla$ such that $b \notin \nabla'$. Thus, we have $W_{\mathfrak{A}} \setminus \{\nabla' \in W_{\mathfrak{A}} \mid \nabla \subseteq \nabla'\} = \bigcup_{b \in \nabla} -P_b$ which is open and, accordingly, the set $\{\nabla' \in W_{\mathfrak{A}} \mid \nabla \subseteq \nabla'\}$ is closed.

²⁴A map $f : X \rightarrow Y$ is called *perfect* iff f is continuous and surjective and $f^{-1}\{y\}$ is compact for all $y \in Y$.

²⁵A basis for the Vietoris topology on 2^X is given by sets of the form $\langle \mathcal{U} \rangle = \{F \in 2^X \mid F \subseteq \bigcup \mathcal{U} \text{ and } \forall x \in F. \exists V \in \mathcal{U}. x \in V\}$ where \mathcal{U} is a finite set of open sets in X .

²⁶Notice that compactness is also an immediate consequence of Theorem 43.1.

For the rest of the proof we need the following observation. For $a, b \in \mathfrak{A}$ we have $R_{\mathfrak{A}}^{-1}(P_a \cap -P_b)$ iff $\nabla \subseteq \nabla'$ for some $\nabla' \in P_a \cap -P_b$ iff the filter generated by $\nabla \cup \{a\}$ does not contain b iff $a \wedge c \not\leq b$ for all $c \in \nabla$. Thus $\Delta \notin R_{\mathfrak{A}}^{-1}(P_a \cap -P_b)$ iff $a \wedge c \leq b$ for some $c \in \nabla$ iff $c \leq a \rightarrow b$ for some $c \in \nabla$ iff $a \rightarrow b \in \nabla$ iff $\nabla \in P_{a \rightarrow b}$. Thus, we have $R_{\mathfrak{A}}^{-1}(P_a \cap -P_b) = -P_{a \rightarrow b}$.

For showing (E5) suppose U is a clopen subset of $W_{\mathfrak{A}}$, i.e. $U = \bigcup_{i=1}^n P_{a_i} \cap -P_{b_i}$. Then $R_{\mathfrak{A}}^{-1}(U) = \bigcup_{i=1}^n R_{\mathfrak{A}}^{-1}(P_{a_i} \cap -P_{b_i}) = \bigcup_{i=1}^n -P_{a_i \rightarrow b_i}$ which is clopen.

Since $W_{\mathfrak{A}} \setminus \bigcup_{i=1}^n -P_{a_i \rightarrow b_i} = \bigcap_{i=1}^n P_{a_i \rightarrow b_i} = P_{(a_1 \rightarrow b_1) \wedge \dots \wedge (a_n \rightarrow b_n)}$ it follows that $W_{\mathfrak{A}} \setminus R_{\mathfrak{A}}^{-1}(U)$ is of the form P_a for some $a \in \mathfrak{A}$. Since upward closed clopen sets are complements of downward closed clopen subsets we conclude that downward closed clopen sets are of the form P_a . Since sets of the form P_a are always clopen and upward closed it follows that a subset U of $W_{\mathfrak{A}}$ is upward closed and clopen iff $U = P_a$ for some $a \in \mathfrak{A}$. \square

Lemma 43.5 *For an Esakia space (X, R) the upward closed open sets form a Ha $\mathfrak{A}_{(X, R)}$ w.r.t. set inclusion such that (X, R) is isomorphic to the Esakia space $(W_{\mathfrak{A}_{(X, R)}}, \mathcal{O}_{\mathfrak{A}_{(X, R)}}, R_{\mathfrak{A}_{(X, R)}})$ as constructed in Lemma 43.4.*

Proof: Given an Esakia space (X, R) we may consider the sets $\mathfrak{A}_{(X, R)}$ of upward closed subsets of X ordered by \subseteq . Let $(-)^{\circ} : \mathcal{P}(X) \rightarrow \mathcal{P}(X) : U \mapsto \{x \in U \mid R(x) \subseteq U\}$ which clearly is a kernel operator. Since $U^{\circ} = -R^{-1}(-U)$ it follows from (E2) that U° is clopen whenever U is clopen, i.e. $(-)^{\circ}$ restricts to a kernel operator on $\text{Clp}(X)$, from which it follows that the poset $\mathfrak{A}_{(X, R)}$ is a Ha.

Since X is a Stone space and by (E1) the set $R(x)$ is closed we have $R(x) = \bigcap \{C \in \text{Clp}(X) \mid R(x) \subseteq C\}$. If $R(x) \subseteq C \in \text{Clp}(X)$ then $R(x) \subseteq C^{\circ}$ since $C^{\circ} = \{x \in C \mid R(x) \subseteq C\}$ and, therefore, we have $R(x) = \bigcap \{U \in \mathfrak{A}_{(X, R)} \mid R(x) \subseteq U\} = \bigcap \{U \in \mathfrak{A}_{(X, R)} \mid x \in U\}$. Thus, we have xRy iff $y \in R(x)$ iff $\forall U \in \mathfrak{A}_{(X, R)}. (x \in U \Rightarrow y \in U)$.

Next we show that prime filters on $\text{Clp}(X)$ are in 1-1-correspondence to prime filters on $\mathfrak{A}_{(X, R)}$. Suppose ∇_1 and ∇_2 are prime filters on $\text{Clp}(X)$ with $\nabla_1 \cap \mathfrak{A}_{(X, R)} = \nabla_2 \cap \mathfrak{A}_{(X, R)}$. Since X is a Stone space there are unique $x_1, x_2 \in X$ with $\nabla_i = \eta(x_i) = \{C \in \text{Clp}(X) \mid x_i \in C\}$ for $i=1, 2$. Thus, from $\nabla_1 \cap \mathfrak{A}_{(X, R)} = \nabla_2 \cap \mathfrak{A}_{(X, R)}$ it follows that x_1 and x_2 inhabit the same elements of $\mathfrak{A}_{(X, R)}$ and, therefore, by the previous paragraph we have $x_1 = x_2$, i.e. $\nabla_1 = \nabla_2$. It remains to show that every prime filter on $\mathfrak{A}_{(X, R)}$ extends to a prime filter on $\text{Clp}(X)$ (without adding new elements from $\mathfrak{A}_{(X, R)}$). Suppose ∇_0 is a prime filter on $\mathfrak{A}_{(X, R)}$. Then $F = \{C \in \text{Clp}(X) \mid \exists U \in \nabla_0. U \subseteq C\}$ is a filter on $\text{Clp}(X)$ and $I = \{C \in \text{Clp}(X) \mid \exists U \in \mathfrak{A}_{(X, R)} \setminus \nabla_0. C \subseteq U\}$ is an ideal in $\text{Clp}(X)$. Then there exists a maximal filter ∇ on $\text{Clp}(X)$ with $F \subseteq \nabla$ and $\nabla \cap I = \emptyset$ and thus ∇ and ∇_0 contain the same elements of $\mathfrak{A}_{(X, R)}$, i.e. ∇ is a prime filter on $\text{Clp}(X)$ with $\nabla \cap \mathfrak{A}_{(X, R)} = \nabla_0$.

Let $\mathfrak{B}(X)$ be the least sub-boolean-algebra of $\mathcal{P}(X)$ containing all elements of $\mathfrak{A}_{(X, R)}$ ($\mathfrak{B}(X)$ consists of all sets of the form $\bigcup_{i=1}^n -C_i \cup D_i$ with $C_i, D_i \in \mathfrak{A}_{(X, R)}$). With an argument like in the previous paragraph one can see that prime filters on $\mathfrak{B}(X)$ correspond to prime filters on $\mathfrak{A}_{(X, R)}$. Thus, since both

$\mathfrak{B}(X)$ and $\mathfrak{A}_{(X,R)}$ are free boolean algebras over $\mathfrak{A}_{(X,R)}$ they are isomorphic and, therefore, the topology on X is generated by the elements of $\mathfrak{A}_{(X,R)}$ and their complements.

Thus, we have proved that (X, R) is isomorphic to $(W_{\mathfrak{A}_{(X,R)}}, \mathcal{O}_{\mathfrak{A}_{(X,R)}}, R_{\mathfrak{A}_{(X,R)}})$ as desired. \square

Notice that for an Esakia space (X, \leq) implication in $\mathfrak{A}_{(X,R)}$ is computed as follows. Let $a, b \in \mathfrak{A}_{(X,R)}$ then $P_{a \rightarrow b} = (-P_a \cup P_b)^\circ$, i.e. $\nabla \in P_{a \rightarrow b}$ iff $a \in \nabla'$ implies $b \in \nabla'$ for all $\nabla' \supseteq \nabla$. Thus, we have $\nabla \in W_{\mathfrak{A}_{(X,R)}} \setminus (P_{a \rightarrow b})$ iff there exists some $\nabla' \supseteq \nabla$ with $a \in \nabla'$ but $b \notin \nabla'$, i.e. iff $c \wedge a \not\leq b$ for all $c \in \nabla$. Thus, we have $\nabla \in P_{a \rightarrow b}$ iff $a \wedge c \leq b$ for some $c \in \nabla$ iff $c \leq a \rightarrow b$ for some $c \in \nabla$ iff $a \rightarrow b \in \nabla$ iff $\nabla \in P_{a \rightarrow b}$, i.e. we have shown that $P_{a \rightarrow b} = P_{a \rightarrow b}$. Thus, we have $x \in a \rightarrow b$ iff $y \in b$ whenever $x \leq y \in a$, i.e. implication is computed like in Kripke models.

If (W, R) is a partial order then the poset $(\text{ucl}(W, R), \subseteq)$ of upward closed subsets of W is a very particular kind of Ha, namely a complete Heyting algebra whose dual is also a (complete) Ha. Even complete Ha's, e.g. $(\mathcal{O}(\mathbb{R}), \subseteq)$, need not²⁷ be of this particular form. But this can be remedied by endowing W with the structure of a Stone space and restricting to the clopen upward closed sets of W .

References

- [ChZa] A. Chagrov, M. Zakharyashev *Modal Logic* Oxford Logic Guides 35, Clarendon Press, Oxford 1997.
- [Esa] L. Esakia *Topological Kripke Models* Sov. Math. Dokl. 15 No.1, pp. 147-151, 1974.

²⁷Let $U_n =]-\frac{1}{n+1}, \frac{1}{n+1}[$ for $n \in \mathbb{N}$ and $V = \mathbb{R} \setminus \{0\}$. Then $U_n \vee V = \mathbb{R}$ for all n and thus $\bigwedge_{n \in \omega} (U_n \vee V) = \mathbb{R}$ but $(\bigwedge_{n \in \omega} U_n) \vee V = \emptyset \vee V = V \neq \mathbb{R}$.

44 Completely Distributive Lattices

A *completely distributive lattice* (cdl) is a complete lattice L such that

$$(cd) \quad \bigwedge_{i \in I} \bigvee_{j \in J_i} x_{i,j} = \bigvee_{f \in \Pi(J)} \bigwedge_{i \in I} x_{i,f(i)}$$

for all double indexed sequences $((x_{i,j})_{j \in J_i})_{i \in I}$ in L where $\Pi(J)$ stands for $\prod_{i \in I} J_i$.

We show now that in cdl's it holds also that

$$(cd') \quad \bigvee_{i \in I} \bigwedge_{j \in J_i} x_{i,j} = \bigwedge_{f \in \Pi(J)} \bigvee_{i \in I} x_{i,f(i)}$$

for all double indexed families $((x_{i,j})_{j \in J_i})_{i \in I}$.²⁸

For all $i \in I$ and $f \in \Pi(J)$ we have $\bigwedge_{j \in J_i} x_{i,j} \leq x_{i,f(i)} \leq \bigvee_{i \in I} x_{i,f(i)}$ from which it follows that $\bigvee_{i \in I} \bigwedge_{j \in J_i} x_{i,j} \leq \bigwedge_{f \in \Pi(J)} \bigvee_{i \in I} x_{i,f(i)}$. For the reverse inequality first observe that $\bigwedge_{f \in \Pi(J)} \bigvee_{i \in I} x_{i,f(i)} \stackrel{(cd)}{=} \bigvee_{F: \Pi(J) \rightarrow I} \bigwedge_{f \in \Pi(F)} x_{f,F(f)}$. Suppose $F: \Pi(J) \rightarrow I$. We want to show that

$$(\dagger) \quad \text{there is some } i \in I \text{ with } \{i\} \times J_i \subseteq \{(F(f), f(F(f))) \mid f \in \Pi(J)\}$$

because then for such an i it holds that

$$\bigwedge_{f \in \Pi(F)} x_{f,F(f)} \leq \bigwedge_{j \in J_i} x_{i,j} \leq \bigvee_{i \in I} \bigwedge_{j \in J_i} x_{i,j}.$$

Suppose (\dagger) does not hold. Then by the axiom of choice there is an $f_0 \in \Pi(J)$ such that $(i, f_0(i)) \notin \{(F(f), f(F(f))) \mid f \in \Pi(J)\}$ for all $i \in I$. But this is impossible since for $i_0 = F(f_0) \in I$ we have $(i_0, f_0(i_0)) = (F(f_0), f_0(F(f_0))) \in \{(F(f), f(F(f))) \mid f \in \Pi(J)\}$.

Of course, by duality for complete lattices (cd') implies (cd). Thus, every cdl is in particular a *complete biHeyting algebra*.

Since the notion of completely distributive lattice is infinitary equational one can show the existence of free cdl's (because by (cd) one can restrict attention to terms of the form $\bigwedge_{S \in \mathcal{S}} \bigvee S$ with $\mathcal{S} \in \mathcal{P}^2(L)$ and there are only $2^{2^{|L|}}$ many of them). Based on this observation M. Bonsangue has shown in his PhD Thesis that the forgetful functor $U: \mathbf{CDL} \rightarrow \mathbf{Frm}$ has a left adjoint F where $F(A)$ is the free cdl over $|A|$ factored by the least equivalence containing all infinitary equations holding in A . Thus $F(A)$ can be described as a certain quotient of $\text{Fil}(A)^{\text{op}}$. However, as observed by C. Butz, it is unclear how to describe this quotient *explicitly*, i.e. how to *choose canonical representatives* for these equivalence classes of filters in A .

M. Bonsangue has shown that a space X is sober iff $\mathcal{Q}(X)$, the set of upward closed sets (w.r.t. the specialization order \leq_X) ordered by \supseteq , is isomorphic to $F(\mathcal{O}(X))$, the free cdl over the frame $\mathcal{O}(X)$ of open subsets of X .

²⁸This proof is taken from Johnstone's *Stone Spaces* Lemma VII 1.10. (pp.278-279).

45 Constructive Complete Distributivity

Let L be a completely distributive (cd) lattice. For $x \in L$ consider the family $D = (D_i)_{i \in I}$ of all downward closed sets whose supremum is x . By (cd) we have

$$x = \bigwedge_{i \in I} \bigvee_{x \in D_i} x = \bigvee_{f \in \prod(D)} \bigwedge_{i \in I} f(i)$$

where $\prod(D)$ stands for $\prod_{i \in I} D_i$. Obviously, for every $f \in \prod(D)$ and $i \in I$ we have $\bigwedge_{i \in I} f(i) \leq f(i) \in D_i$ and thus $\bigwedge_{i \in I} f(i) \in D_i$. Thus x is already the supremum of $\bigwedge_{i \in I} D_i$, the least downward closed set with supremum x which we denote as $\downarrow x$. Writing \mathcal{D} for the set of downward closed subset of L we have

$$\downarrow \dashv \bigvee \dashv \downarrow : L \rightarrow \mathcal{D}$$

where $\downarrow x = \{y \in L \mid y \leq x\}$. Posets L with $\downarrow \dashv \bigvee \dashv \downarrow : L \rightarrow \mathcal{D}$ are called *constructively completely distributive* (ccd) lattices and were introduced in a paper by Fawcett, Rosebrugh and Wood.

Notice that a complete lattice L is ccd iff $\bigvee : \mathcal{D} \rightarrow L$ preserves infima (since both L and \mathcal{D} are complete lattices where the latter is ordered by \subseteq). This allows us to show that ccd lattices are already cd. Suppose $((x_{i,j})_{j \in J_i})_{i \in I}$ is a double indexed family in L . W.l.o.g we may assume that all J_i are nonempty. Let $D_i = \downarrow \{x_{i,j} \mid j \in J_i\}$ for $i \in I$. Apparently, we have

$$\bigwedge_{i \in I} \bigvee_{j \in J_i} x_{i,j} = \bigwedge_{i \in I} \bigvee D_i = \bigvee_{i \in I} \bigcap_{i \in I} D_i \stackrel{(*)}{\leq} \bigvee_{f \in \prod J} \bigwedge_{i \in I} f(i) \leq \bigwedge_{i \in I} \bigvee_{j \in J_i} x_{i,j}$$

where step (*) uses axiom of choice implicitly: for $x \in \bigcap_{i \in I} D_i$ and $i \in I$ choose an $f(i) \in J_i$ with $x \leq x_{i,f(i)}$. It has been shown by Fawcett, Rosebrugh and Wood that the implication $(\text{ccd}) \supset (\text{cd})$ for arbitrary complete lattices is equivalent to the Axiom of Choice.

46 Artin glueing

Let X be a space and $S \in \mathcal{O}(X)$. We write A for $X \setminus S$. Let $i : S \hookrightarrow X$ and $j : A \hookrightarrow X$ be the corresponding subspace inclusions. When considering them as locale maps we have

$$\begin{aligned} i^*(V) &= S \cap V & i_*(U) &= \bigcup \{V \in \mathcal{O}(X) \mid S \cap V \subseteq U\} \\ j^*(V) &= A \cap V & j_*(U) &= \bigcup \{V \in \mathcal{O}(X) \mid A \cap V \subseteq U\} \end{aligned}$$

The composite $i^*j_* : \mathcal{O}(A) \rightarrow \mathcal{O}(S)$ is trivial since $i^*j_*(U) = S$ for all $U \in \mathcal{O}(A)$. But the composition $j^*i_* : \mathcal{O}(S) \rightarrow \mathcal{O}(A)$ sending $U \in \mathcal{O}(S)$ to $A \cap \bigcup_{S \cap V \subseteq U} V = A \setminus \bar{U}$ (\bar{U} is the closure of U in X) is nontrivial and called the *fringe* map.

One easily checks that $\mathcal{O}(X) \cong \mathcal{O}(S) \downarrow j^*i_*$ by identifying $V \in \mathcal{O}(X)$ with the pair $((A \cap U)^\circ, S \cap U)$.

If \mathcal{E} is a topos and $U \in \text{Sub}_{\mathcal{E}}(1_{\mathcal{E}})$, i.e. “ U is an open of \mathcal{E} ”, then $\mathcal{E}_o = \mathcal{E}/U$ is a(n open) subtopos of \mathcal{E} via the injective geometric morphism $U^* \dashv \Pi_U : \mathcal{E}/U \rightarrow \mathcal{E}$ induced by the topology $U \rightarrow (-)$ on \mathcal{E} . Its closed complement $j : \mathcal{E}_c \hookrightarrow \mathcal{E}$ is the subtopos induced by the topology $U \vee (-)$ on \mathcal{E} . Then one may consider the functor $F : j^*\Pi_U : \mathcal{E}_o \rightarrow \mathcal{E}_c$ for which one can show that $\mathcal{E} \simeq \mathcal{E}_c \downarrow F$. In case of $\mathcal{E} = \text{Sh}(X)$ and $U \in \mathcal{O}(X)$ this is precisely the topos formulation of the above topological fact.

Later it has been shown that for arbitrary finite limit preserving functors $F : \mathcal{E} \rightarrow \mathcal{F}$ between (elementary) toposes their *Artin glueing* $\mathcal{F} \downarrow F$ is also a(n elementary) topos. The functor $i^* = \partial_1 : \mathcal{F} \downarrow F \rightarrow \mathcal{E}$ is logical and has a right adjoint i_* sending $A \in \mathcal{E}$ to (id_{FA}, A) . This gives rise to the injective geometric morphism $i^* \dashv i_* : \mathcal{E} \hookrightarrow \mathcal{F} \downarrow F$ induced by the open $0 \rightarrow F1$ of $\mathcal{F} \downarrow F$. Its closed complement is given by the injective geometric morphism $j : \mathcal{F} \hookrightarrow \mathcal{F} \downarrow F$ where $j^* = \partial_0 : \mathcal{F} \downarrow F \rightarrow \mathcal{F}$ and its right adjoint j_* sends $B \in \mathcal{F}$ to $(!_B, 1)$.

If $\Delta \dashv \Gamma : \mathcal{E} \rightarrow \mathbf{Set}$ is a topos over \mathbf{Set} , i.e. in addition Δ preserves also finite limits, one may consider the glueing of Γ , i.e. $\mathbf{Set} \downarrow \Gamma$ aka *sconing*²⁹ of \mathcal{E} , and the glueing of Δ , i.e. $\mathcal{E} \downarrow \Delta$ which comes together with the logical functor $\partial_1 : \mathcal{E} \downarrow \Delta \rightarrow \mathbf{Set}$.

Glueing can also be applied for reconstructing the “Joyal topos”. Let T be a topological space. We define \tilde{T} as the space with underlying set $|\tilde{T}| = |T| + |T| = \{0, 1\} \times |T|$ where $U \in \mathcal{O}(\tilde{T})$ iff $U_1 \supseteq U_0 \in \mathcal{O}(T)$. Joyal used $\text{Sh}(\tilde{\mathbb{R}})$ for proving a rule of continuous choice for HAH. Notice that $S = \{1\} \times |T|$ is an open subspace of \tilde{T} and thus its complement $A = \{0\} \times |T|$ is a closed subspace of \tilde{T} . Obviously S is isomorphic to the discrete space with underlying set $|T|$ and A is isomorphic to the space T . The corresponding “fringe” map $f : \mathcal{P}(|T|) \rightarrow \mathcal{O}(T)$ sends X to its interior X° . Obviously, we have $\mathcal{O}(\tilde{T}) \cong \mathcal{O}(T) \downarrow f$. The corresponding “fringe” functor $F : \text{Sh}(|T|) \rightarrow \text{Sh}(T)$ sends

²⁹this may be defined for arbitrary locally small toposes \mathcal{E} since $\Gamma = \mathcal{E}(1, -)$ can be defined without postulating a left adjoint Δ

$X \in \mathbf{Sh}(|T|)$ to the subobject $F(X) \in \mathbf{Sh}(T)$ with $F(X)(U) = \prod_{r \in U} X_r$ and $F(V \subseteq U)(s) = (s_r)_{r \in V}$.³⁰ Thus $\mathbf{Sh}(\tilde{T}) \simeq \mathbf{Sh}(T) \downarrow F$ which comes equipped with the logical functor $\partial_1 : \mathbf{Sh}(T) \downarrow F \rightarrow \mathbf{Sh}(|T|)$.

Benno van den Berg has suggested the following generalisation of the construction of the “Joyal topos”. Consider the map $f : \prod_{r \in |T|} \mathcal{O}(T) \rightarrow \mathcal{O}(T) : (U_r)_{r \in |T|} \mapsto \mathbf{int}\{r \in |T| \mid r \in U_r\}$ which preserves finite meets as one easily checks. Then $\mathcal{O}(T) \downarrow f$ is a locale and one may consider the sheaf topos $\mathbf{Sh}(\mathcal{O}(T) \downarrow f)$ instead of $\mathbf{Sh}(\tilde{T})$.

³⁰Notice that F is the direct image part m_* of the geometric morphism $m : \mathbf{Sh}(|T|) \rightarrow \mathbf{Sh}(T)$ arising from the continuous map $m : |T| \rightarrow T$ with $m(r) = r$ for all $r \in |T|$.

47 Sconing over Arbitrary Bases

If \mathcal{E} is a topos over \mathbf{Set} then $\text{sc}(\mathcal{E}) = \mathbf{Set} \downarrow \Gamma$. Generalizing this to arbitrary geometric morphisms $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ we put $\text{sc}_{\mathcal{S}}(\mathcal{E}) = \mathcal{S} \downarrow U$ which is fibred over \mathcal{E} via $P_U = \partial_1 : \mathcal{S} \downarrow U \rightarrow \mathcal{E}$. This fibration has a right adjoint right inverse $1 : \mathcal{E} \rightarrow \text{sc}_{\mathcal{S}}(\mathcal{E})$ fitting into the commuting triangle

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{1} & \mathcal{S} \downarrow U \\ & \searrow \mathcal{C} & \downarrow \partial_0 \\ & & \mathcal{S} \end{array}$$

All three functors involved are direct image parts of geometric morphism since $F \dashv U$, $P_U \dashv 1$ and $L \dashv \partial_0$. The functor L is given by $LI = \eta_I : I \rightarrow UFI$ since for $u : I \rightarrow J$ and $a : I \rightarrow UA$ we have

$$\begin{array}{ccc} I & \xrightarrow{u} & J \\ \eta_I \downarrow & & \downarrow a \\ UFI & \xrightarrow{U\widehat{au}} & UA \\ & & \downarrow \widehat{au} \\ FI & \dashrightarrow & A \end{array}$$

and obviously L preserves finite limits since F and U do so by assumption.

A fibred version of sconing is given by $\text{sc}_{\mathcal{S}}(\mathcal{E})$ fibred over $\mathcal{E} \downarrow F$ as in

$$\begin{array}{ccccccc} \mathcal{S} \downarrow U & \longrightarrow & \text{sc}_{\mathcal{S}}(\mathcal{E}) & \longrightarrow & \text{Fam}(P_{\mathcal{S}}) & \longrightarrow & \mathcal{S} \downarrow \mathcal{S} \\ \downarrow P_U & & \downarrow & & \downarrow \text{Fam}(P_{\mathcal{S}}) & & \downarrow P_{\mathcal{S}} \\ \mathcal{E} & \hookrightarrow & \mathcal{E} \downarrow F & \xrightarrow{\Gamma} & \mathcal{S} \downarrow \mathcal{S} & \xrightarrow{\partial_0} & \mathcal{S} \\ \downarrow & & \downarrow P_F & & \swarrow P_{\mathcal{S}} & & \\ \mathbf{1} & \hookrightarrow & \mathcal{S} & & & & \end{array}$$

where all squares are pullbacks.

48 Separators vs. Generating Families (J.-L. Moens)

Let $F \dashv U : \mathbb{C} \rightarrow \mathbb{B}$ be a geometric morphism and $P : \mathbb{X} \rightarrow \mathbb{B}$ be the ensuing geometric fibration. Recall that in \mathbb{X} cocartesian arrows and collective epis are stable under arbitrary pullbacks.

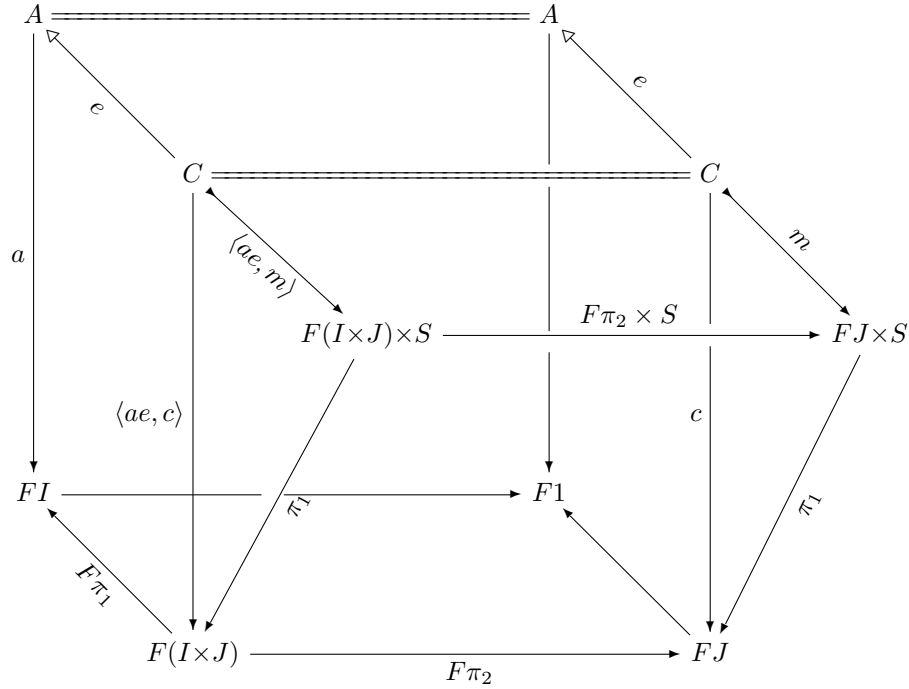
An S over 1 is called a *separator* iff for all A over 1 there is a collective epi $C \rightarrow A$ where C is a vertical subobject of J^*S for some $J \in \mathbb{B}$. Then for X over I we have

$$\begin{array}{ccccc}
 X & \xrightarrow{\psi_0} & \coprod_I X & & \\
 & \text{cocart.} & & & \\
 & \swarrow e & & \nwarrow e_0 & \\
 & & Y & \xrightarrow[\psi]{\text{cocart.}} & Z \\
 & & \downarrow m & & \downarrow m_0 \\
 & & (I \times J)^* S & \xrightarrow[\varphi]{\text{cart.}} & J^* S
 \end{array}$$

where the top square is a pullback and e_0 and e are collective epis. Thus G_S over $\text{Sub}(S)$ is a generating family for P where $m_S : G_S \rightarrow \text{Sub}(S)^*S$ is a generic family of subobjects of S , i.e. m_S is a vertical mono such that for every vertical mono $m : P \rightarrow I^*S$ there is a unique map $p : I \rightarrow \text{Sub}(S)$ with $m \cong p^*m_S$. Actually S is a separator if and only if G_S is a generating family for P .

49 A Nice Picture Illustrating the Argument Showing that Bounded Geometric Morphisms Give Rise to Geometric Fibrations with a Small Generating Family

Suppose $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ is a geometric morphism with bound S . Suppose $a : A \rightarrow FI$ in \mathcal{E} . Then, since S is a bound, there exists a cover $e : C \rightarrow A$ with $m : C \rightarrow FJ \times S$ for some $J \in \mathcal{S}$. Then we have the following diagram

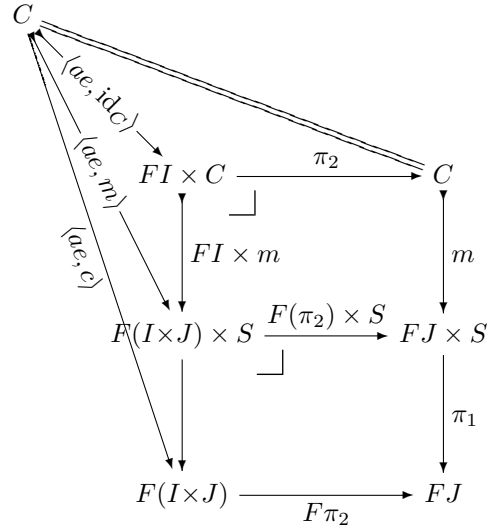


showing that there is a collectively epic map e to a over $\pi_1 : I \times J \rightarrow I$ starting from $\langle ae, c \rangle : C \rightarrow F(I \times J)$ which appears as subobject of $(I \times J)^*S$ via the monomorphism $\langle ae, m \rangle$. Thus, the map $g = \pi_1 \circ m_S : G \rightarrow FUP(S)$ in

$$\begin{array}{ccc}
 G & \xrightarrow{\quad} & \exists_S \\
 m_S \downarrow \lrcorner & & \downarrow \\
 FUP(S) \times S & \xrightarrow{\varepsilon_{\mathcal{P}(S)} \times S} & \mathcal{P}(S) \times S \\
 \pi_1 \downarrow \lrcorner & & \downarrow \pi_1 \\
 FUP(S) & \xrightarrow{\varepsilon_{\mathcal{P}(S)}} & \mathcal{P}(S)
 \end{array}$$

is a generating family for the geometric fibration $P_F = F^*P_{\mathcal{E}}$ over \mathcal{S} .

Notice that (since $F(I \times J) \cong FI \times FJ$) we have



illustrating why the cocartesian arrow from $\langle ae, c \rangle$ to $c = \pi_1 \circ m$ over $\pi_2 : I \times J \rightarrow J$ factors through the cartesian arrow to c over $\pi_2 : I \times J \rightarrow J$ via the monomorphism $\langle ae, id_C \rangle$.

50 For bounded geometric morphisms $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ the fibration P_F has a small generating family

Let $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ be a geometric morphism with bound S . Then for every $a : A \rightarrow FI$ there exist a $J \in \mathcal{S}$, a mono $m : C \rightarrow FJ \times S$ and an epi $e : C \rightarrow A$ as depicted³¹ in

$$\begin{array}{ccccc}
 & & A & \xleftarrow{e} & C \\
 & & \downarrow a & & \downarrow \langle ae, \text{id}_C \rangle \\
 & & & & FI \times C \\
 & & & & \downarrow \langle ae, m \rangle \\
 & & & & F(I \times J) \times S \\
 & & & & \downarrow \langle ae, \pi_1 m \rangle \\
 & & & & F(I \times J) \\
 & & & & \downarrow F\pi_1 \\
 & & & & FI \\
 & & & & \downarrow F\pi_2 \\
 & & & & FJ \\
 & & & & \downarrow \pi_1 \\
 & & & & FJ \times S \\
 & & & & \downarrow m \\
 & & & & C \\
 & & & & \downarrow \pi_2 \\
 & & & & C
 \end{array}$$

$\begin{array}{ccc} \lrcorner & & \lrcorner \\ \lrcorner & & \lrcorner \end{array}$

illustrating also that the cocartesian arrow from $\langle ae, \pi_1 m \rangle$ to $\pi_1 m$ over $\pi_2 : I \times J \rightarrow J$ factors through the cartesian arrow to $\pi_1 m$ over $\pi_2 : I \times J \rightarrow J$ via the monomorphism $\langle ae, \text{id}_C \rangle$.

A small generating family for $P_F : \mathcal{E} \downarrow F \rightarrow \mathcal{S}$ is given by $g : G \rightarrow FUP(S)$ as defined in

$$\begin{array}{ccc}
 G & \xrightarrow{\quad} & \exists_S \\
 \downarrow g & \lrcorner & \downarrow \\
 FUP(S) & \xrightarrow{\quad} & \mathcal{P}(S) \times S \\
 & \varepsilon_{\mathcal{P}(S)} & \downarrow \pi_1 \\
 & & \mathcal{P}(S)
 \end{array}$$

since $\langle ae, \pi_1 m \rangle$ arises as pullback of g along $F(\tilde{u})$ where $\tilde{u} : I \times J \rightarrow UP(S)$ is the transpose of the unique map $u : F(I \times J) \rightarrow \mathcal{P}(S)$ with $\langle ae, m \rangle \cong (u \times S)^* \exists_S$.

³¹where we use that $F(I \times J) \cong FI \times FJ$ since F preserves finite limits

51 Locally Bounded Implies Bounded

(on CATEGORIES 31.7.-3.8.2017)

End of July 2017 Mamuka Jibladze asked whether a geometric morphism $f : \mathcal{F} \rightarrow \mathcal{E}$ is already bounded if it is locally bounded, i.e. $f/X : \mathcal{F}/f^*X \rightarrow \mathcal{E}/X$ is bounded for some $X \in \mathcal{E}$ with global support. A few days later Peter Johnstone came up with the following proof providing a positive answer.

First observe that \mathcal{F}/f^*X is bounded over \mathcal{E}/X iff it is bounded over \mathcal{E} since $\mathcal{E}/X \rightarrow \mathcal{E}$ is bounded. And the following proposition holds.

Proposition If $Y \in \mathcal{F}$ has global support and G is a bound for \mathcal{F}/Y over \mathcal{E} then $\Sigma_Y(G)$ is a bound for f .

Proof: By assumption any object B of \mathcal{F}/Y is a subquotient of $G \times Y^*f^*I$ for some object I of \mathcal{E} . But the Frobenius reciprocity condition $\Sigma_Y(G \times Y^*f^*I) \cong \Sigma_Y(G) \times f^*I$ holds, so $\Sigma_Y(B)$ is a subquotient of $\Sigma_Y(G) \times f^*I$. Finally, since Y has global support, any object A of \mathcal{F} is a quotient of $\Sigma_Y(Y^*A) \cong A \times Y$ and thus a subquotient of $\Sigma_Y(G) \times f^*I$ for some I of \mathcal{E} . \square

52 Counterexamples to Beck-Chevalley Condition for Adjoints to Change of Base

Consider the following pullback in \mathbf{Cat}

$$\begin{array}{ccc}
 \mathbf{0} & \xrightarrow{\tilde{F}} & \mathbf{1} \\
 \tilde{G} \downarrow & \lrcorner & \downarrow G \\
 \mathbf{1} & \xrightarrow{F} & \mathbf{2}
 \end{array}$$

where $F(\mathbf{0}) = \mathbf{0}$ and $G(\mathbf{0}) = \mathbf{1}$ (we write \mathbf{n} for the ordinal n considered as a category).

Let \mathbf{X} be a category considered as a fibration over $\mathbf{1}$. Apparently both $\coprod_G \mathbf{X}$ and $\coprod_F \mathbf{X}$ are isomorphic to the constant fibration over $\mathbf{2}$ with value \mathbf{X} .

We have $\coprod_{\tilde{G}} \tilde{F}^* \mathbf{X} \cong \mathbf{0}$ whereas $F^* \coprod_G \mathbf{X} \cong \mathbf{X}$ which when taking a nonempty \mathbf{X} gives rise to a counterexample showing that in general the Beck-Chevalley condition fails for $\coprod \vdash (-)^*$.

The (single fibre of) $\coprod_{\tilde{F}} \tilde{G}^* \mathbf{X}$ is equivalent to $\mathbf{1}$ whereas $G^* \coprod_F \mathbf{X}$ is equivalent to \mathbf{X} . Taking for \mathbf{X} a category not equivalent to $\mathbf{1}$ gives rise to a counterexample showing that in general the Beck-Chevalley condition fails for $(-)^* \vdash \coprod$.

Consider a pullback

$$\begin{array}{ccc}
 \mathbf{D} & \xrightarrow{\tilde{P}} & \mathbf{A} \\
 \tilde{F} \downarrow & \lrcorner & \downarrow F \\
 \mathbf{C} & \xrightarrow{P} & \mathbf{B}
 \end{array}$$

in \mathbf{Cat} where P (and thus also \tilde{P}) is a fibration. If the Beck-Chevalley condition holds for this pullback then $F^* \prod_P \simeq \prod_{\tilde{P}} \tilde{F}^*$. Thus, for any fibration Q over \mathbf{B} we have

$$F^*[P \rightarrow Q] = F^* \prod_P P^* Q \simeq \prod_{\tilde{P}} \tilde{F}^* P^* Q \simeq \prod_{\tilde{P}} \tilde{P}^* F^* Q = \prod_{F^* P} (F^* P)^* F^* Q = [F^* P \rightarrow F^* Q]$$

i.e. change of base along F preserves exponentiation of fibrations up to equivalence. But this fails in general already for discrete fibrations since $F^* : \mathbf{Set}^{\mathbf{B}^{\text{op}}} \rightarrow \mathbf{Set}^{\mathbf{A}^{\text{op}}}$ need not preserve exponentiation. Consider e.g. the functor $F : \mathbf{1} \rightarrow \mathbf{2} : 0 \mapsto 1$ giving rise to the global sections functor for the Sierpiński topos. That this does not preserve exponentiation of presheaves can be seen in various ways.

Let A and B be objects in $\hat{\mathbf{2}} = \mathbf{Set}^{\mathbf{2}^{\text{op}}}$ with no global elements, i.e. $A(1) = \emptyset = B(1)$. Then $[F^* A \rightarrow F^* B]$ contains precisely one element whereas $F^*[A \rightarrow B]$ contains as many elements as there are functions from $A(0)$ to $B(0)$.

Let A and B be the presheaves over $\mathbf{2}$ corresponding to the functions $2 \rightarrow 1$ and id_2 , respectively. Then $F^*(B^A)$ contains 2 elements (since there are precisely 2 morphisms from A to B) whereas $(F^* B)^{F^* A}$ contains 4 elements (since both $F^* A$ and $F^* B$ contain precisely 2 elements).

53 Essential and local maps of locales

If A is a locale then there is a unique locale morphism $t : A \rightarrow \Omega$ (where $\Omega = \{\perp, \top\}$ is the lattice with 2 elements corresponding to the space with one point). The map t is uniquely determined by the requirement that t^* preserves finite infima and arbitrary suprema, i.e. $t^*(\perp) = 0_A$ and $t^*(\top) = 1_A$. Accordingly, we have $t_*(a) = \top$ iff $\top \leq t_*(a)$ iff $1_A = t^*(\top) \leq a$, i.e. t_* is the characteristic function of $\{1_A\}$.

Notice that t is always *essential*, i.e. t^* has a left adjoint $t_!$.³² This left adjoint $t_!$ is characterized by $t_!(a) \leq \perp$ iff $a \leq t^*\perp = 0_A$, i.e. $t_!$ is the characteristic function of $A \setminus \{0_A\}$. The locale map t is *local* iff t_* has a right adjoint $t^!$. Thus, the map t is local iff t_* preserves arbitrary suprema which is the case iff $A \setminus \{1_A\}$ has a greatest element. In case $A = \mathcal{O}(X)$ for a T_0 space X this amounts to the requirement that there is a $\perp \in X$ whose only open neighbourhood is X itself (such a point is called *focal*).

A point of a locale A is a locale map $p : \Omega \rightarrow A$. A point p is *essential* iff p^* has a left adjoint iff p^* preserves arbitrary infima iff there is a least $a \in A$ with $p^*(a) = \top$. A point p is *local* iff p_* has a right adjoint iff p_* preserves arbitrary suprema iff $p_*(\perp) = 0_A$. If $A = \mathcal{O}(X)$ and $x \in X$ then the corresponding point p is given by $p^*(U) = \top$ iff $x \in U$. Thus $U \subseteq p_*(\perp)$ iff $p^*(U) \leq \perp$ iff $x \notin U$, i.e. $p_*(\perp) = X \setminus \overline{\{x\}}$. Thus, the point x is essential iff there is a least open neighbourhood of x and the point x is local iff $X \setminus \overline{\{x\}} = \emptyset$ iff $X = \overline{\{x\}}$, i.e. iff every nonempty open set U contains x as an element. A T_0 space X has a local point iff it has a greatest element w.r.t. the specialization order \sqsubseteq_X .

54 Preservation properties of inverse image parts of geometric morphisms

If \mathbb{C} is a small category and $\Delta \dashv \Gamma : \widehat{\mathbb{C}} \rightarrow \mathbf{Set}$ is the unique geometric morphism then Δ always preserves the (locally) cartesian closed structure but it does not preserve the subobject classifier unless \mathbb{C} is a groupoid.

Suppose $a \dashv i : \mathcal{F} \hookrightarrow \mathcal{E}$ is an injective geometric morphism then $a : \mathcal{E} \rightarrow \mathcal{F}$ always preserves the subobject classifier and thus is logical iff a preserves the cartesian closed structure. That a need not preserve the cartesian closed structure is shown by the following two counterexamples. Let \mathbb{M} be the monoid $\{1, e\}$ with $e^2 = e$ and $\Gamma \dashv \nabla : \mathbf{Set} \hookrightarrow \widehat{\mathbb{M}}$ be the injective geometric morphism with $\Gamma = \widehat{\mathbb{M}}(1, -)$ which preserves Ω but not exponentials. Let $\Gamma \dashv \nabla : \mathbf{Set} \hookrightarrow \mathbf{Eff}$ be the injective geometric morphism with $\Gamma = \mathbf{Eff}(1, -)$ which preserves Ω but not exponentials.

³²However, there exist spaces X such that the geometric morphism $\mathbf{Sh}(X) \rightarrow \mathbf{Set}$ is not locally connected, e.g. Cantor space $2^{\mathbb{N}}$ or Baire space $\mathbb{N}^{\mathbb{N}}$ (for which $\mathbf{Sh}(X) \rightarrow \mathbf{Set}$ is also not local).

55 Counterexamples based on Sierpiński topos

The Sierpiński topos is defined as $\mathcal{E} = \widehat{\mathbf{2}} = \mathbf{Set}^{\mathbf{2}^{\text{op}}}$ where $\mathbf{2}$ the category $0 \xrightarrow{\alpha} 1$. The global sections functor $\Gamma : \mathcal{E} \rightarrow \mathbf{Set}$ is given by $A \mapsto A(1)$. A left adjoint Δ to Γ is given by $\Delta(S) = \text{id}_S$. A right adjoint ∇ to Γ is given by $\nabla(S) : S \rightarrow 1$. A left adjoint Π to Δ is given by $\Pi(S) = S(0)$. A left adjoint L to Π is given by $L(S) : 0 \rightarrow S$. Summarizing we have

$$L \dashv \Pi \dashv \Delta \dashv \Gamma \dashv \nabla : \mathbf{Set} \rightarrow \mathcal{E}$$

The geometric morphism $\Gamma \dashv \nabla : \mathbf{Set} \rightarrow \mathcal{E}$ is bounded (with bound 1) and essential but not locally connected since for $U = y(0)$ we have

$$\begin{array}{ccc} 0 & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow \\ \Gamma(U) & \longrightarrow & \Gamma(1) \end{array}$$

since $\Gamma(U) \cong 0$ but

$$\begin{array}{ccc} \Delta(0) & \longrightarrow & \Delta(1) \\ \downarrow & & \downarrow \\ U & \longrightarrow & 1 \end{array}$$

is not a pullback since $\Delta(0) \cong 0$ and $\Delta(1) \cong 1$. Moreover, the g.m. $\Gamma \dashv \nabla$ is not connected, i.e. Γ is not full and faithful, although $\Delta(1) \cong 1$.³³

The geometric morphism $\Pi \dashv \Delta : \mathbf{Set} \rightarrow \mathcal{E}$ is bounded (with bound 1) but not local (since Π is not faithful) although Δ has a right adjoint Γ .

However, for the particular case of bounded geometric morphisms $\Delta \dashv \Gamma : \mathbf{Sh}(\mathcal{C}, \mathcal{J}) \rightarrow \mathbf{Set}$ it holds that

- (1) $\Delta \dashv \Gamma$ is locally connected iff Δ has a left adjoint
- (2) $\Delta \dashv \Gamma$ is local iff Γ has a right adjoint.

³³Notice that a locally connected geometric morphism $F \dashv U : \mathcal{F} \rightarrow \mathcal{G}$ is connected iff $L(1) \cong 1$ for $L \dashv F$. This can be seen as follows.

First recall that the functor F is full and faithful iff $\eta : \text{Id} \rightarrow UF$ is an iso iff $\varepsilon : LF \rightarrow \text{Id}$ is an iso. Thus, if $F \dashv U$ is connected then $\varepsilon : LF1 \rightarrow 1$ is an iso and, therefore, $L1 \cong 1$ since $F1 \cong 1$. For the reverse direction suppose $L1 \cong 1$. Since $F \dashv U$ is locally connected

$$\begin{array}{ccc} FX & \longrightarrow & 1 \\ \parallel & \lrcorner & \downarrow \\ FX & \longrightarrow & F1 \end{array} \quad \text{implies} \quad \begin{array}{ccc} LFX & \longrightarrow & L1 \\ \varepsilon_X \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & 1 \end{array}$$

The left square is a pullback since $1 \rightarrow F1$ is an iso. Thus the right square is also a pullback and, therefore, since $L1 \rightarrow 1$ is an iso it follows that ε_X is an iso.

For proofs see the subsequent sections 60 and 61.

Moreover, the geometric morphism $\Pi \dashv \Delta : \mathbf{Set} \rightarrow \mathcal{E}$ is an essential localization since $L \dashv \Pi$. Since one easily checks that

$$\begin{array}{ccc} J & \xrightarrow{u} & I \\ w \downarrow & \lrcorner & \downarrow v \\ \Pi Y & \xrightarrow{\Pi f} & \Pi X \end{array} \quad \text{implies} \quad \begin{array}{ccc} LJ & \xrightarrow{Lu} & LI \\ \hat{w} \downarrow & \lrcorner & \downarrow \hat{v} \\ Y & \xrightarrow{f} & X \end{array}$$

the geometric morphism $\Pi \dashv \Delta$ is even locally connected.

But $L \dashv \Pi$ is not a local geometric morphism because L does not preserve 1 and thus is not even a geometric morphism although Π has a full and faithful right adjoint.

56 An essential hyperconnected geometric morphism which is not locally connected

Let \mathbf{V} be the category with two objects V and E and two morphisms $s, t : V \rightarrow E$. Then the topos $\widehat{\mathbf{V}} = \mathbf{Set}^{\mathbf{V}^{\text{op}}}$ is the topos of graphs. The functor $H : \mathbf{V} \rightarrow \mathbf{2}$ with $H(V) = 0$ and $H(E) = 1$ induces an essential geometric morphism from $\widehat{\mathbf{V}}$ to $\widehat{\mathbf{2}}$ whose inverse image part is given by $F = H^* = \mathbf{Set}^{H^{\text{op}}}$ since $L = \prod_H \dashv F \dashv \prod_H = U$. The left adjoint L to F can be described as follows: it sends a graph $s, t : G_1 \rightarrow G_0$ to $q \circ s = q \circ t : G_1 \rightarrow Q$ where $q : G_0 \rightarrow Q$ is the coequalizer of s and t . The functor F is full and faithful since it includes graphs where all edges are loops into the category of all graphs. Thus $F \dashv U$ is connected. It is even hyperconnected since the image of F is closed under subobjects (if all edges of a graph are loops then this holds also for all its subgraphs). The right adjoint U to F removes from a graph all edges that are not loops. But U does not have a right adjoint since U does not preserve the coequalizer of $s, t : V \rightarrow E$ which is the terminal object whereas the coequalizer of $U(s)$ and $U(t)$ is the graph with one node but no edge.

That $L \dashv F \dashv U$ is not locally connected can be seen by considering the pullback

$$\begin{array}{ccc} X \times G & \longrightarrow & G \\ \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & 1 \end{array}$$

where G is the generic graph $\mathbf{Y}_{\mathbf{V}}(E)$ and X is the discrete graph with 2 nodes and no edge. Obviously $L(G)$ is terminal and thus $X \times L(G) \cong X$. The graph $X \times G$ has 4 nodes but no edges and thus $L(X \times G) \cong X \times G$. Thus $X \times L(G)$ and $L(X \times G)$ are not isomorphic – as would be the case if $F \dashv U$ were locally connected – since X has 2 nodes whereas $X \times G$ has 4 nodes.

There is an even simpler argument. Choose for X the graph with one node and no edge. Then $X \times G$ is the graph with two nodes and no edge. This is not changed when applying L to it. Thus $L(X \times G)$ has two nodes whereas $X \times L(G) \cong X$ has only one node since $L(G)$ is terminal. Thus $L(X \times G)$ and $X \times L(G)$ are not isomorphic as would be the case if $F \dashv U$ were locally connected.

57 A hyperconnected essential geometric morphism from the topos of reflexive graphs to the Dirichlet topos

Let M_n be the monoid with elements e_0, \dots, e_{n-1} besides 1 satisfying the identities $e_i e_j = e_i$ for $i, j < n$. Let $h : M_2 \rightarrow M_1$ be the monoid homomorphism with $h(e_i) = e_0$ for $i < 2$. Then $h^* : \widehat{M}_1 \rightarrow \widehat{M}_2$ is the inverse image part of an essential geometric morphism from the topos of reflexive graphs to the “Dirichlet topos”. The right adjoint h_* to h^* removes from a reflexive graph all edges that are not loops. The geometric morphism $h^* \dashv h_*$ is also hyperconnected since reflexive graphs all whose edges are loops are closed under subobjects. The left adjoint $h_!$ to h^* is the cocontinuous functor from \widehat{M}_2 to \widehat{M}_1 preserving the terminal object and sending the generic edge to the generic loop. However, h_* does not have a right adjoint.

58 \mathcal{S} -essential geometric morphisms

Let \mathcal{S} be a base topos and $e : \mathcal{E} \rightarrow \mathcal{S}$ and $f : \mathcal{F} \rightarrow \mathcal{S}$ be (bounded) geometric morphisms. An \mathcal{S} -essential geometric morphism from f to e is a geometric morphism $\varphi : \mathcal{F} \rightarrow \mathcal{E}$ with $f = e\varphi$, i.e. $f^* = \varphi^*e^*$, such that φ^* has a left adjoint $\varphi_!$ which is \mathcal{S} -fibered in the sense that

$$\begin{array}{ccc}
 B & \xrightarrow{h} & A \\
 \downarrow b & \lrcorner & \downarrow a \\
 f^*J & \xrightarrow{f^*u} & f^*I
 \end{array}
 \quad \text{implies} \quad
 \begin{array}{ccc}
 \varphi_!B & \xrightarrow{\varphi_!h} & \varphi_!A \\
 \downarrow \widehat{b} & \lrcorner & \downarrow \widehat{a} \\
 e^*J & \xrightarrow{e^*u} & e^*I
 \end{array}$$

where \widehat{b} and \widehat{a} are the upper transposes of b and a , respectively, w.r.t. $\varphi_! \dashv \varphi^*$.

Obviously, if e equals $\text{id}_{\mathcal{S}}$ then φ is \mathcal{S} -essential iff φ is molecular (i.e. locally connected).

This terminology was originally introduced by Barr and Paré in their 1980 paper introducing molecular toposes and later taken up in Bunge and Funk’s book.

59 An Open Problem brought up by Lawvere and Menni

Matias Menni has asked whether hyperconnected and local geometric morphisms $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ which are essential are necessarily also locally connected, i.e. whether F preserves dependent products whenever F has a left adjoint. Hemelaer and Rogers have come up with a counterexample in September 2020. Actually they show that the left adjoint L of F does not preserve binary products which by Lemma 3.5 of [Joh11] would be the case if the geometric morphism $F \dashv U$ were locally connected.

But there remains the question by Lawvere and Menni whether for hyperconnected and local geometric morphisms $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ the functor F preserves dependent products, i.e. $F \dashv U$ is locally connected, whenever F preserves ordinary exponentials. This has to be considered as a strengthening of the previous question since as shown by Menni in Cor. 3.8 of his 2017 paper *The construction of π_0 in axiomatic cohesion* a hyperconnected geometric morphism $p : \mathcal{E} \rightarrow \mathcal{S}$ is essential, i.e. p^* has a left adjoint $p_!$, whenever p^* preserves ordinary exponentials. In the subsequent Cor. 3.9 Menni has shown that for a hyperconnected geometric morphism $p : \mathcal{E} \rightarrow \mathcal{S}$ the full subcategory $p^* : \mathcal{S} \rightarrow \mathcal{E}$ is an exponential ideal iff p^* has a left adjoint $p_!$ preserving finite products. In December 2020 R. Garner has come up with an example of an essential and local geometric morphism $p : \mathcal{E} \rightarrow \mathcal{S}$ such that p^* gives rise to an exponential ideal although it does not preserve dependent products. But, alas, this p is not hyperconnected and thus the original question of Lawvere and Menni is still open.

But I expect the answer to be negative since the requirement that a geometric morphism $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ is locally connected / local is known to be equivalent to the requirement that the corresponding cartesian functor $\Delta : P_{\mathcal{S}} \rightarrow P_F / \Gamma : P_F \rightarrow P_{\mathcal{S}}$ has a cartesian left / right adjoint Π / ∇ . This is the correct fibrational generalization to arbitrary base toposes \mathcal{S} of the folklore observations (proved in the next two sections) that in case the base topos \mathcal{S} is **Set** a geometric morphism $\Delta \dashv \Gamma$ is locally connected / local iff $\Delta : \mathbf{Set} \rightarrow \mathcal{E} / \Gamma : \mathcal{E} \rightarrow \mathbf{Set}$ has a left / right adjoint Π / ∇ .

60 Locally Connected Toposes over Set

Suppose $F \dashv U : \mathcal{E} \rightarrow \mathbf{Set}$ is a geometric morphism. Notice that $F^*P_{\mathcal{E}}$ is equivalent to $\text{Fam}_b(\mathcal{E})$, the full subfibration of $\text{Fam}(\mathcal{E})$ over \mathbf{Set} on bounded families (where a family $(A_i)_{i \in I}$ is bounded iff there exists an object A in \mathcal{E} such that all A_i appear as subobjects of A). Notice that the fibred functor $\Delta : P_{\mathbf{Set}} \rightarrow F^*P_{\mathcal{E}}$, i.e. the fibred version of F , is equivalent to $\text{Fam}(F) : \text{Fam}(\mathbf{Set}) \rightarrow \text{Fam}(\mathcal{E})$ whose image stays within $\text{Fam}_b(\mathcal{E})$.

Now, if F has a left adjoint L then $\text{Fam}(L)$ is a fibred left adjoint of $\text{Fam}(F)$. Since $\text{Fam}(F)$ and $\text{Fam}(L)$ restrict to fibred functors between $\text{Fam}_b(\mathcal{E})$ and $\text{Fam}_b(\mathbf{Set}) = \text{Fam}(\mathbf{Set})$ and thus Δ has a fibred left adjoint Λ . Thus, we have shown the following

Theorem

A geometric morphism $F \dashv U : \mathcal{E} \rightarrow \mathbf{Set}$ is locally connected iff it is essential, i.e. F has a left adjoint L .

Notice that for $a = \coprod_{i \in I} !A_i : \prod_{i \in I} A_i \rightarrow \prod_{i \in I} 1_{\mathcal{E}} = \Delta(I)$ we have $\Lambda(a) = \prod_{i \in I} !L(A_i) : \prod_{i \in I} L(A_i) \rightarrow \prod_{i \in I} 1 \cong I$. But $\Lambda(a)$ is isomorphic to $\hat{a} = \varepsilon_I \circ L(a)$ since L (being a left adjoint) preserves sums and thus $L(a)$ is isomorphic to $\prod_{i \in I} L(!A_i) : \prod_{i \in I} L(A_i) \rightarrow \prod_{i \in I} L(1_{\mathcal{E}})$ and $\varepsilon_I : LFI \rightarrow I$ is isomorphic to $\prod_{i \in I} \varepsilon_1$ because L and F (being left adjoints) preserve sums and by naturality of the counit ε we have

$$\begin{array}{ccc} LFI & \xrightarrow{\varepsilon_I} & I \\ LFi \uparrow & & \uparrow i \\ LFI & \xrightarrow{\varepsilon_1} & 1 \end{array}$$

for all (global) elements i of I . But, of course, this could have been concluded also more abstractly since fibred left adjoints are unique up to (fibred natural) isomorphism.

With some hand waving Barr and Paré have stated this in the last section of their 1980 JPAA paper “*Molecular Toposes*” where they also observe that a geometric morphism $F \dashv U : \mathcal{E} \rightarrow \mathbf{Set}$ is locally connected iff F preserves exponentials. This can be seen as follows. Suppose Δ has a fibred left adjoint Λ . Then Δ preserves internal products and thus all $F_{/I} : \mathbf{Set}/I \rightarrow \mathcal{E}/FI$ preserve exponentials. In particular $F_{/1} \cong F$ preserves exponentials. On the other hand if F preserves exponentials then $\text{Fam}(F)$ preserves exponentials and thus also internal products.³⁴ Thus, the equivalent fibred functor Δ preserves internal products from which it follows by the (appropriate) Fibred Adjoint Functor Theorem that Δ has a fibred left adjoint Λ .

³⁴In C3.3.1 of PTJ’s *Elephant* it is shown that a geometric morphism $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ is locally connected, i.e. Δ has a fibred left adjoint Λ , iff F preserves dependent products iff all $F_{/I} : \mathcal{S}/I \rightarrow \mathcal{E}/FI$ preserve exponentials.

61 Local Grothendieck Toposes over Set

Suppose $\mathcal{E} = \text{Sh}(\mathbb{C}, J)$ is a Grothendieck topos where we assume w.l.o.g. that (\mathbb{C}, J) is a subcanonical site, i.e. that all representable objects are J -sheaves. Let $\Delta \dashv \Gamma : \mathcal{E} \rightarrow \mathbf{Set}$ be the corresponding geometric morphism. We will show that if this geometric morphism is local, i.e. Γ has a right adjoint ∇ , then ∇ (and thus also Δ) are full and faithful.

By Yoneda we have

$$\nabla(S)(I) \cong \text{Sh}(\mathbb{C}, J)(y(I), \nabla(S)) \cong \mathbf{Set}(\Gamma(y(I)), S)$$

from which it follows that $\nabla(S) = S^{G(-)}$ where $G = \Gamma \circ y$.

Suppose $G(I)$ were empty for all I in \mathbb{C} . Then $\nabla(2) \cong 1$ which is impossible³⁵ since it entails $2^{\Gamma(X)} \cong \text{Sh}(\mathbb{C}, J)(X, \nabla(2)) \cong 1$ and thus $\Gamma(X) \cong 0$ for all $X \in \text{Sh}(\mathbb{C}, J)$. Thus $G(I)$ is nonempty for some I , i.e. there is a global element $i_0 : 1 \rightarrow y(I_0)$.

From this it follows immediately that $\nabla : S \mapsto S^{G(-)}$ is faithful. For showing fullness of ∇ suppose $\tau : \nabla(S) \rightarrow \nabla(T)$. We have to show that $\tau = \nabla(f)$ for some map $f : S \rightarrow T$. First we show that $\tau_I : S^{G(I)} \rightarrow T^{G(I)}$ preserves constant functions. Let $c_s \in S^{G(I)}$ be the constant function with value s . For $i \in G(I)$ and $J \in \mathbb{C}$ let $u_{J,i}$ be the map $J \rightarrow 1 \xrightarrow{i} I$. Obviously $G(u_{J,i})$ is the constant map from $G(J)$ to $G(I)$ with value i . Since $c_s \circ G(u_{J,i}) = c_s$ it follows from naturality of τ that $\tau_I(c_s) = \tau_I(c_s \circ G(u_{J,i})) = \tau_I(c_s) \circ G(u_{J,i})$ and thus $\tau_I(c_s)$ is constant. Now we define $f(s) = \tau_{I_0}(c_s)(i_0)$ for $s \in S$. For showing that $\tau = \nabla(f)$ suppose $c : G(I) \rightarrow S$. We have to show that $\tau_I(c) = f \circ c$. Suppose $i \in G(I)$. We write u as shorthand for $u_{I_0,i} : I_0 \rightarrow I$ and observe that $G(u)(i_0) = i$. By naturality of τ the diagram

$$\begin{array}{ccc} S^{G(I)} & \xrightarrow{\tau_I} & T^{G(I)} \\ \downarrow S^{G(u)} & & \downarrow T^{G(u)} \\ S^{G(I_0)} & \xrightarrow{\tau_{I_0}} & T^{G(I_0)} \end{array}$$

commutes. For this reason and because $c \circ G(u)$ is constant we have $\tau_I(c)(i) = \tau_I(c)(G(u)(i_0)) = (T^{G(u)} \circ \tau_I)(c)(i_0) = (\tau_{I_0} \circ S^{G(u)})(c)(i_0) = \tau_{I_0}(c \circ G(u))(i_0) = f((c \circ G(u))(i_0)) = f(c(G(u)(i_0))) = f(c(i)) = (f \circ c)(i)$. Since this holds for all $i \in G(I)$ we have shown that $\tau_I(c) = f \circ c = \nabla(f)_I(c)$. Since this holds for all $I \in \mathbb{C}$ and $c \in S^{G(I)}$ we finally have $\tau = \nabla(f)$ as desired.

³⁵Alternatively, one might argue that if $G(I)$ is empty for all $I \in \mathbb{C}$ then since Γ is cocontinuous and $1 \cong \text{colim}_{I \in \mathbb{C}} y(I)$ we have $\Gamma(1)$ is empty which clearly is impossible!

62 Lawvere Comprehension

A fibration $P : \mathbb{X} \rightarrow \mathbb{B}$ has (*Lawvere*) *comprehension* iff $P \dashv 1 \dashv G$ where $P \circ 1 = \text{Id}_{\mathbb{B}}$ (and thus also $G \circ 1 = \text{Id}_{\mathbb{B}}$). This requirement expresses that the fibration P has small global sections.

If P has Lawvere comprehension there is an induced functor $\Gamma : P \rightarrow P_{\mathbb{B}}$ (where $P_{\mathbb{B}} = \partial_1 : \mathbb{B}^2 \rightarrow \mathbb{B}$) sending $f : Y \rightarrow X$ in \mathbb{X} to the square

$$\begin{array}{ccc} GX & \xrightarrow{Gf} & GY \\ P\varepsilon_X \downarrow & \Gamma(f) & \downarrow P\varepsilon_Y \\ PX & \xrightarrow{Pf} & PY \end{array}$$

which is obtained by applying P to the square

$$\begin{array}{ccc} 1_{GX} & \xrightarrow{1_{Gf}} & 1_{GY} \\ \varepsilon_X \downarrow & & \downarrow \varepsilon_Y \\ X & \xrightarrow{f} & Y \end{array}$$

since $G \circ 1 = \text{Id}_{\mathbb{B}}$. Instantiating f by the terminal projection $!_X : X \rightarrow 1_{PX}$ in the fibre over PX we obtain (since one may choose $\varepsilon_{1_{PX}}$ as $\text{id}_{1_{PX}}$)

$$\begin{array}{ccc} 1_{GX} & \xrightarrow{1_{G!_X}} & 1_{PX} \\ \varepsilon_X \downarrow & & \parallel \\ X & \xrightarrow{!_X} & 1_{PX} \end{array}$$

from which it follows that $P\varepsilon_X = G!_X$. Accordingly we have

$$\begin{array}{ccc} GX & \xrightarrow{Gf} & GY \\ G!_X \downarrow & \Gamma(f) & \downarrow G!_Y \\ G1_X & \xrightarrow{G1_{Pf}} & G1_Y \end{array}$$

since $G1_{Pf} = Pf$. From this it follows that $\Gamma(f)$ is a pullback whenever f is cartesian, i.e. that $\Gamma : P \rightarrow P_{\mathbb{B}}$ is a cartesian functor.³⁶

Moreover, one easily sees that Γ is full and faithful as a cartesian functor iff for all objects X in \mathbb{X} the arrow $\varepsilon_X : 1_{GX} \rightarrow X$ is cocartesian.

³⁶Notice that $P_{\mathbb{B}}$ is a fibration iff \mathbb{B} has pullbacks.

63 Comprehension for posetal hyperdoctrines

A fibration $P : \mathbb{X} \rightarrow \mathbb{B}$ has *comprehension* iff $P \dashv 1 \dashv G$ where $P \circ 1 = \text{Id}_{\mathbb{B}}$ (and thus also $G \circ 1 = \text{Id}_{\mathbb{B}}$). If \mathbb{B} has pullbacks and P has internal sums and the fibres have terminal objects stable under reindexing then P has comprehension iff the cartesian functor $\Delta : P_{\mathbb{B}} \rightarrow P$ sending $u : J \rightarrow I$ to $\coprod_u 1_J$ has a cartesian right adjoint Γ (where $\Gamma_I(X) = G(X \rightarrow 1_I) : G(X) \rightarrow G(1_I) = I$). If P is posetal then $\Gamma_I(X)$ is a subobject of I for all $X \in \mathbb{X}_I$.

We will show that for \mathbb{B} with a terminal object Γ_1 need be neither one-to-one nor onto. For that purpose recall that for a geometric morphism $F \dashv U : \mathcal{F} \rightarrow \mathcal{E}$ the fibration $F^*P_{\mathcal{F}}$ has comprehension given by $\Gamma_I(A \xrightarrow{a} FI) = \eta_I^*U(a)$. Thus, also the posetal hyperdoctrine $P = F^*\text{Sub}_{\mathcal{E}}$ has comprehension given by $\Gamma_I(m) = \eta_I^*U(m)$. We consider now the particular case of a geometric morphism $F \dashv U : \text{Sh}(X) \rightarrow \text{Sh}(Y)$ induced by a continuous map $f : X \rightarrow Y$ in which case $\Gamma_1 : \mathcal{O}(X) \rightarrow \mathcal{O}(Y)$ is given by the right adjoint f_* to $f^* = f^{-1} : \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$. Since $f^*f_*f^* = f^*$ and $f_*f^*f_* = f_*$ the map f_* is one-to-one iff $f^*f_* = \text{id}$ iff f^* is onto and f_* is onto iff $f_*f^* = \text{id}$ iff f^* is one-to-one. Thus $\Gamma_1 = f_*$ is neither one-to-one nor onto iff f^* is neither onto nor one-to-one. A typical example of such a map is $f : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto x^2$ because f^* is not one-to-one since $f^*(\emptyset) = \emptyset = f^*((-\infty, 0))$ and f^* is not onto since $(0, \infty)$ is not in the image of f^* (as if $1 \in f^*(U)$ then $f(1) = 1 \in U$ and thus also $f(-1) = 1 \in U$ from which it follows that also $-1 \in f^*(U)$).

Finally notice that posetal hyperdoctrines over a logos \mathbb{B} with one-to-one comprehension correspond to factorization systems $(\mathcal{E}, \mathcal{M})$ on \mathbb{B} where all maps in \mathcal{M} are monos, \mathcal{E} is stable under pullbacks and \mathcal{M} is stable under universal quantification along maps in \mathbb{B} . Typical such examples are j -closed monos in a topos \mathcal{E} where j is a topology on \mathcal{E} (e.g. $\neg\neg$ -closed monos in a realizability topos).

64 f^* has a right adjoint iff f is exponentiable

Let \mathcal{C} be a category with pullbacks and $f : J \rightarrow I$ be a map in \mathcal{C} . Then the pullback functor $f^* : \mathcal{C}/I \rightarrow \mathcal{C}/J$ has a right adjoint $\Pi_f : \mathcal{C}/J \rightarrow \mathcal{C}/I$ if and only if $(-) \times f$ has a right adjoint $(-)^f$ in \mathcal{C}/I .

Suppose $f^* \dashv \Pi_f$. Then $(-) \times f = \Sigma_f \circ f^* \dashv \Pi_f \circ f^* = (-)^f$. On the other hand if $(-) \times f \dashv (-)^f$ then for $g : K \rightarrow J$ we can construct $\Pi_f g$ as

$$\begin{array}{ccc} \Pi_f g & \longrightarrow & (fg)^f \\ \downarrow & \lrcorner & \downarrow g^f \\ 1_I & \xrightarrow{\lrcorner \text{id}_f \lrcorner} & f^f \end{array}$$

where the right vertical arrow is $(-)^f$ applied to $g : fg \rightarrow f$ and $\lrcorner \text{id}_f \lrcorner$ is the “name” of id_f , i.e. the exponential transpose (λ -abstraction) of the projection $1_I \times f \rightarrow f$.

65 Exponentials in \mathbf{Fib}/\mathbb{B}

Let P and Q be fibrations over \mathbb{B} then the exponential Q^P in \mathbf{Fib}/\mathbb{B} is given by the *split* fibration

$$Q^P(I) = \mathbf{Fib}/\mathbb{B}(\underline{I} \times P, Q)$$

where $\underline{I} = P_I = \partial_0 : \mathbb{B}/I \rightarrow \mathbb{B}$. Since $\underline{I} \times P = \Sigma_I I^* P_{\mathbb{B}}$ we have

$$Q^P(I) = \mathbf{Fib}/\mathbb{B}(\Sigma_I I^* P, Q) \cong \mathbf{Fib}/(\mathbb{B}/I)(I^* P, I^* Q)$$

providing an alternative construction of Q^P .

66 Surjective Geometric Morphisms of Toposes

Let $f^* \dashv f_* : \mathcal{F} \rightarrow \mathcal{E}$ be a geometric morphism. The geometric morphism $f = (f^*, f_*)$ is called *surjective* iff one of the following to equivalent condition holds

- (1) f^* reflects isomorphisms
- (2) f^* is faithful.

Suppose f^* reflects isos. Suppose $u, v : A \rightarrow B$ with $f^*u = f^*v$. Let e be an equalizer of u and v . Since $f^*u = f^*v$ and f^* preserves finite limits f^*e is isomorphic because it is the equalizer of the equal arrows f^*u and f^*v . Since by assumption f^* reflects isos it follows that e is an iso and thus $f = g$. Thus we have shown that f^* is faithful.

Suppose f^* is faithful. Suppose $m : P \rightarrow A$ is a mono with f^*m iso. Let $\chi : A \rightarrow \Omega$ be the characteristic map of m . The mono m is the equalizer of χ and $\top_A = \top \circ !_A$. Since f^* preserves finite limits f^*m is the equalizer of $f^*\top_A$ and $f^*\chi$. But as by assumption f^*m is an iso it follows that $f^*\top_A = f^*\chi$. As f^* is assumed as faithful we have $\top_A = \chi$ and thus m is an iso. Now suppose $h : A \rightarrow B$ with f^*h iso. Let $k_1, k_2 : E \rightarrow A$ be a kernel pair of h and $\alpha : A \rightarrow E$ the unique arrow with $k_i \circ \alpha = \text{id}_A$ for $i = 1, 2$. Since f^* preserves finite limits $f^*k_1, f^*k_2 : f^*E \rightarrow f^*A$ is the kernel pair of the iso f^*h and thus $f^*k_1 = f^*k_2$ is an isomorphism. Since $f^*k_i \circ f^*\alpha = \text{id}$ and f^*k_i is an isomorphism it follows that $f^*\alpha$ is an iso as well. Thus, since α is a (split) mono it follows from our previous consideration that α is an iso. Since $k_1 \circ \alpha = \text{id} = k_2 \circ \alpha$ it follows that k_1 and k_2 are equal isos and, therefore, the map h is monic. Since by assumption f^*h is an iso it follows from the previous consideration that h is an iso as desired.

A further equivalent condition is

- (3) $f_A^* : \text{Sub}_{\mathcal{E}}(A) \rightarrow \text{Sub}_{\mathcal{F}}(f^*A)$ is one-to-one for all $A \in \mathcal{E}$.

Suppose (3) holds. We show that this entails (2). Suppose $u, v : A \rightarrow B$ with $f^*u = f^*v$. Let e be an equalizer of u and v . Then f^*e is an equalizer of f^*u and f^*v and thus f^*e is an iso. From assumption (3) it follows that e is itself an iso and thus $u = v$ as desired.

Suppose (1) holds. We show (3). Suppose $m : P \rightarrow A$ and $n : Q \rightarrow A$ with $f^*m \cong f^*n$ as subobjects of f^*A . Consider the pullback

$$\begin{array}{ccc}
 P \cap Q & \xrightarrow{j} & Q \\
 \downarrow i & \lrcorner & \downarrow n \\
 P & \xrightarrow{m} & A
 \end{array}$$

Since f^* preserves pullbacks and $f^*m \cong f^*n$ both f^*i and f^*j are isos. From assumption (1) it follows that both i and j are isos and thus $m \cong n$ as subobjects of A .

67 Indexed Categories

Let \mathbb{B} be a category. A *category indexed over* \mathbb{B} is given by

- (1) a category $P(I)$ for every object I of \mathbb{B}
- (2) a functor $u^* : P(I) \rightarrow P(J)$ for every morphism $u : J \rightarrow I$ in \mathbb{B}
- (3) a natural isomorphism $c_{u,v} : v^*u^* \rightarrow (uv)^*$ for all $u : J \rightarrow I$ and $v : K \rightarrow J$ in \mathbb{B}
- (4) a natural isomorphism $i_I : 1_I^* \rightarrow 1_{P(I)}$ for all objects I in \mathbb{B}

satisfying the coherence conditions

- (i) $c_{uv,w} \circ w^*c_{u,v} = c_{u,vw} \circ c_{v,w}u^*$
- (ii) $c_{1_I,u} = u^*i_I$.

Instantiating v and w by identity maps in (i) we get $1_J^*c_{u,1_J} = c_{1_J,1_J}u^*$. Instantiating u by 1_J in (ii) we get $c_{1_J,1_J} = 1_J^*i_J$. Thus $1_J^*c_{u,1_J} = 1_J^*i_Ju^*$ from which it follows that

- (iii) $c_{u,1_J} = i_Ju^*$

since 1_J^* is full and faithful because it is isomorphic to $1_{P(J)}$.

The conditions expressed in (i) are reminiscent (see Section 189) of the so-called ‘‘cocycle conditions’’ known from *group cohomology* and *descent theory*.

The unpleasant aspect of the above coherence conditions is that the isomorphisms $c_{u,v} : v^*u^* \rightarrow (uv)^*$ and $i_I : 1_I^* \rightarrow 1_{P(I)}$ are in no sense canonical. If $P : \mathbb{X} \rightarrow \mathbb{B}$ is a fibration then a corresponding indexed category is induced by a *choice* of a *cleavage* choosing for every $u : J \rightarrow I$ in \mathbb{B} and $X \in P(I)$ a cartesian morphism $\text{Cart}(u, X) : u^*X \rightarrow X$ over u . For $u : J \rightarrow I$ the induced functor $u^* : P(I) \rightarrow P(J)$ sends a vertical arrow $\alpha : Y \rightarrow X$ to the unique vertical arrow $u^*\alpha : u^*Y \rightarrow u^*X$ making the diagram

$$\begin{array}{ccc}
 u^*Y & \xrightarrow{\text{Cart}(u, Y)} & Y \\
 \downarrow u^*\alpha & & \downarrow \alpha \\
 u^*X & \xrightarrow{\text{Cart}(u, X)} & X
 \end{array}$$

commute. For $u : J \rightarrow I$ and $v : K \rightarrow J$ the component of $c_{u,v}$ at X is the unique vertical arrow making the diagram

$$\begin{array}{ccc}
v^*u^*X & \xrightarrow{\text{Cart}(v, u^*X)} & u^*X \\
\downarrow (c_{u,v})_X & & \searrow \text{Cart}(u, X) \\
(uv)^*X & \xrightarrow{\text{Cart}(uv, X)} & X
\end{array}$$

commute. For $I \in \mathbb{B}$ the component $(i_I)_X$ is given by $\text{Cart}(1_I, X)$. Of course, one may restrict attention to *normalized* cleavages where always $\text{Cart}(1_I, X) = 1_X$ in which case $1_I^* = \text{Id}_{P(I)}$ and i_I is the identity natural transformation on $\text{id}_{P(I)}$ for all $I \in \mathbb{B}$. But in general there do not exist *split* cleavages which besides being normalized satisfy $\text{Cart}(uv, X) = \text{Cart}(u, X) \circ \text{Cart}(v, u^*X)$ which would give rise to $v^*u^* = (uv)^*$ and all $c_{u,v}$ being identities.

Since one may always choose a normalized cleavage it would make a lot of sense to simplify the definition of indexed category by omitting (4) and instead requiring that 1_I^* is always an identity functor. Then besides the “cocycle” condition (1) one simply has to require that $c_{u,v}$ is an identity whenever u or v is an identity.

But experience shows that instead of working with indexed categories one better works with fibrations as was immediately recognised by A. Grothendieck who invented first indexed categories and then immediately replaced it by the superior notion of fibred category. Jean Bénabou has over and over emphasized this and given many examples showing the superiority of fibred over indexed categories and how working with the latter is error prone because the bureaucracy of coherence conditions tempts one to neglect them and assuming that $c_{u,v}$ and i_I are identities leads to wrong conclusions in many cases. A typical example is the **wrong** claim that for an indexed category P the objects form a presheaf over \mathbb{B} . Alas, Jean Bénabou’s long awaited comprehensive book *Des Catégories Fibrées* exposing his “correct” view of fibred categories will never see the light of day because he got lost due to his somewhat exaggerated desire for accuracy and elegance and a tendency to pursue excessive generalizations.

68 Local properties of functors

(Bénabou October 2007 on CATEGORIES list and talk in Paris in June 2013)

Let \mathcal{P} be a property of functors then a functor $F : \mathbb{X} \rightarrow \mathbb{Y}$ has *locally the property \mathcal{P}* iff for every $X \in \mathbb{X}$ the slice $F/X : \mathbb{X}/X \rightarrow \mathbb{Y}/FX$ has property \mathcal{P} . We write $\mathcal{L}(\mathcal{P})$ for the class of functors which locally have the property \mathcal{P} .

We now consider a few examples.

- (1) A functor $P : \mathbb{X} \rightarrow \mathbb{B}$ is locally an isomorphism of categories iff P is a *discrete fibration*.
- (2) A *surjective equivalence* is a functor which is an equivalence and surjective on objects. A functor $P : \mathbb{X} \rightarrow \mathbb{B}$ is a surjective equivalence iff P is a fibration all whose fibres are equivalent to the terminal category $\mathbf{1}$. Thus surjective equivalences are stable under arbitrary pullbacks (whereas equivalences are not).
A functor $P : \mathbb{X} \rightarrow \mathbb{B}$ is locally a surjective equivalence iff P is a *fibration of groupoids*.
- (3) A functor $F : \mathbb{A} \rightarrow \mathbb{B}$ has a right adjoint right inverse (rari) iff F has a right adjoint U such that all counits are identities, i.e. $\varepsilon_B = \text{id}_B$ for all $B \in \mathbb{B}$. Apparently F has a rari iff for every $B \in \mathbb{B}$ there is a $UB \in \mathbb{A}$ such that $FUB = B$ and for all $f : FA \rightarrow B$ there exists a unique $g : A \rightarrow UB$ with $g = Ff$.
A functor $P : \mathbb{X} \rightarrow \mathbb{B}$ is a *fibration* iff P has locally the property of having a rari.³⁷
- (4) Consider the following property of a functor $F : \mathbb{A} \rightarrow \mathbb{B}$, namely that all its fibres have a terminal object. A functor $P : \mathbb{X} \rightarrow \mathbb{B}$ locally has this property iff P is a *prefibration*.
- (5) Consider the following weaker property of a functor $F : \mathbb{A} \rightarrow \mathbb{B}$, namely that all its fibres are non-empty and connected. A functor $P : \mathbb{X} \rightarrow \mathbb{B}$ locally has this property iff P is a *homotopy prefibration*.

³⁷The functor $P/X : \mathbb{X}/X \rightarrow \mathbb{B}/PX$ has a rari iff for all $u : J \rightarrow I = PX$ there exists $\varphi : u^*X \rightarrow X$ such that $P\varphi = u$ and for all $f : Y \rightarrow X$ and $v : PY \rightarrow J$ with $uv = Pf$ there exists a unique $g : Y \rightarrow u^*X$ with $\varphi g = f$ and $Pg = v$.

69 Street's Generalization of Fibrations

J. Gray has characterised (Grothendieck) fibrations as functors $P : \mathbb{X} \rightarrow \mathbb{B}$ such that for every $X \in \mathbb{X}$ the functor $P_{/X} : \mathbb{X}/X \rightarrow \mathbb{B}/PX$ has a right adjoint right inverse C_X . Being a right inverse means that all counits are identities.

A notion of fibration stable under equivalence is obtained by just requiring that all $P_{/X}$ have right adjoints C_X whose counits are isomorphisms. In elementary terms this means that for every $u : J \rightarrow I = PX$ there is a morphism $C_X(u) : u^*X \rightarrow X$ together with an isomorphism $\varepsilon_{X,u} : P(u^*X) \rightarrow J$ satisfying $P(C_X(u)) = u \circ \varepsilon_{X,u}$ such that for all $f : Y \rightarrow X$ and $v : PY \rightarrow J$ with $Pf = uv$ there exists a unique $g : Y \rightarrow u^*X$ with $f = C_X(u) \circ g$ and $v = \varepsilon_{X,u} \circ Pg$. In other words for every $u : J \rightarrow I = PX$ there is a(n ordinary) cartesian arrow $C_X(u) : u^*X \rightarrow X$ such that $P(C_X(u)) = u \circ \varepsilon_{X,u}$ for some isomorphism $\varepsilon_{X,u} : P(u^*X) \rightarrow J$. Such a generalised fibration (in **Cat**) we call a *weak fibration* or a *Street fibration*.³⁸

For Street fibrations $P : \mathbb{X} \rightarrow \mathbb{B}$ and $Q : \mathbb{Y} \rightarrow \mathbb{B}$ a cartesian functor from P to Q is a functor $F : \mathbb{X} \rightarrow \mathbb{Y}$ such that $P = Q \circ F$ and F sends P -cartesian arrows to Q -cartesian arrows.

One can extend the notion of Grothendieck fibration to arbitrary 2-categories \mathcal{K} in the following way. A 1-cell $p : E \rightarrow B$ in \mathcal{K} is a *strict fibration* iff for all X the functor $\mathcal{K}(X, p) : \mathcal{K}(X, E) \rightarrow \mathcal{K}(X, B)$ is a Grothendieck fibration and for all $f : Y \rightarrow X$ the commuting square

$$\begin{array}{ccc} \mathcal{K}(X, E) & \xrightarrow{\mathcal{K}(f, E)} & \mathcal{K}(Y, E) \\ \mathcal{K}(X, p) \downarrow & & \downarrow \mathcal{K}(Y, p) \\ \mathcal{K}(X, B) & \xrightarrow{\mathcal{K}(f, B)} & \mathcal{K}(Y, B) \end{array}$$

is a morphism in **Fib**, i.e. $\mathcal{K}(f, E)$ is cartesian over $\mathcal{K}(f, B)$.

Requiring that all $\mathcal{K}(X, p)$ are just Street fibrations gives rise to a notion of *Street* or *weak fibration* in \mathcal{K} which can be extended even to bicategories (*c.f.* R. Street *Fibrations in Bicategories*).

Although for a weak fibration $P : \mathbb{X} \rightarrow \mathbb{B}$ in **Cat** in general there does not exist for all $u : J \rightarrow I$ in \mathbb{B} a reindexing functor $u^* : P(J) \rightarrow P(I)$ there

³⁸Notice that for a functor $P : \mathbb{X} \rightarrow \mathbb{B}$ and a morphism $\varphi : Y \rightarrow X$ in \mathbb{X} one can formulate cartesianness of φ as

$$\forall \psi : Z \rightarrow X. \forall v : PZ \rightarrow PX. P(\psi) = P(\varphi) \circ v \Rightarrow \exists^1 \theta : Z \rightarrow Y. \psi = \varphi \circ \theta \wedge P(\theta) = v$$

i.e. without reference to equality of objects.

That P is a weak fibration can be formulated as

$$\forall u : J \rightarrow PX. \exists \varphi : Y \rightarrow X. \varphi \text{ cartesian} \wedge \exists \iota : PY \rightarrow J. \iota \text{ iso} \wedge P(\varphi) = u \circ \iota$$

i.e. also without reference to equality of objects.

But this weaker notion of fibration is not appropriate for developing category theory over an arbitrary base (topos) \mathbb{B} because it does not allow one to always transport an X over I to an u^*X over J along a arbitrary $u : J \rightarrow I$ in the base.

does exist a reindexing functor between the *essential fibres*. The essential fibre $P[I]$ of P over I has as objects pair (X, i_X) where X is an object of \mathbb{X} and $i_X : P(X) \rightarrow I$ is an isomorphism. A morphism in $P[I]$ from (Y, i_Y) to (X, i_X) is a morphism $f : Y \rightarrow X$ with $i_X \circ P(f) = i_Y$ (of course, the map $P(f)$ is then an isomorphism). Every morphism $u : J \rightarrow I$ induces a functor $u^* : P[I] \rightarrow P[J]$ as follows

$$\begin{array}{ccc}
 u^*(X, i_X) & \xrightarrow[\text{cart.}]{\varphi} & X \\
 \vdots & & \vdots \\
 P(u^*(X, i_X)) & \xrightarrow{P(\varphi)} & P(X) \\
 \downarrow i_{u^*(X, i_X)} \cong & \nearrow & \downarrow i_X \cong \\
 J & \xrightarrow{u} & I
 \end{array}$$

From the ensuing pseudo-functor $\mathcal{H} : \mathbb{B}^{\text{op}} \rightarrow \mathbf{Cat}$ one obtains by the Grothendieck construction a fibration $P_c : \mathbb{X}_c \rightarrow \mathbb{B}$ such that $P_c \circ E = P$ for some cartesian equivalence $E : \mathbb{X} \rightarrow \mathbb{X}_c$.

One easily shows that all equivalences are weak fibrations and that weak fibrations are closed under composition. Thus weak fibrations over \mathbb{B} are obtained from Grothendieck fibrations by precomposing with equivalences. But since every fibration is equivalent to a split(able) one the weak fibrations over \mathbb{B} are obtained from split(able) fibrations by precomposing with equivalences.

Though there are many natural examples of non-split(able) fibrations there are hardly any natural examples of weak fibrations that are not already Grothendieck fibrations. So one may doubt that anything is gained by generalising fibration to weak fibrations!

Acknowledgement

This has been pointed out to me by M. Shulman around 1. October 2010 during discussions on the CATEGORIES network. A useful but more condensed discussion of Street fibrations influenced by our discussions can be found at <http://ncatlab.org/nlab/show/Street+fibration>.

Genuine Examples

In a mail from 24. Jan. 2011 on CATEGORIES Janelidze, Lack and Street have pointed out some examples of Street fibrations, there called *abstract fibrations*, which are not Grothendieck fibrations.

The first one was mentioned in a paper by Janelidze from 1984 in the context for categorical Galois theory. Let \mathcal{C} be the opposite of the category of commutative rings (with 1) and \mathcal{X} the category of Stone spaces then the functor $P : \mathcal{C} \rightarrow \mathcal{X}$ sending a comm. ring to the Stone space of the boolean algebra of idempotents of the ring.

Another example is the following one: \mathcal{C} is the category of étale spaces over a locally connected space S (i.e. $\mathbf{Sh}(S)$) and \mathcal{X} the category of sets and $P : \mathcal{C} \rightarrow \mathcal{X}$ sends $p : A \rightarrow S$ to the set of connected components of A .

This generalises as follows. Call an object C of a category \mathcal{C} with small sums connected iff $\mathcal{C}(C, -) : \mathcal{C} \rightarrow \mathbf{Set}$ preserves sums. Such a category \mathcal{C} is called locally connected iff every object of \mathcal{C} is isomorphic to a small sum of connected objects. For such a category \mathcal{C} the functor $P : \mathcal{C} \rightarrow \mathbf{Set}$ sending an object to its sets of connected components is an abstract fibration but not a Grothendieck fibration. There are plenty of examples in this vein like the category \mathbf{cat} of small categories or any locally connected Grothendieck topos.

Another class of examples are finite limit preserving functors between finite limit categories which have a full and faithful right adjoint.

70 A recent reformulation of Gray's definition

J. Gray's definition of a cloven Grothendieck fibration is a functor $P : \mathbb{X} \rightarrow \mathbb{B}$ such that for every object $X \in \mathbb{X}$ the functor $P_{/X} : \mathbb{X}/X \rightarrow \mathbb{B}/PX$ has a right adjoint right inverse C_X . One may formulate this more globally as follows. Consider the functor $i : \mathbb{X} \rightarrow \mathbb{B}\downarrow P$ sending $f : Y \rightarrow X$ to

$$\begin{array}{ccc} PY & \xrightarrow{Pf} & PX \\ \parallel & & \parallel \\ PY & \xrightarrow{Pf} & PX \\ \\ Y & \xrightarrow{f} & X \end{array}$$

Notice that i is a functor in \mathbf{Cat}/\mathbb{B} from P to $\partial_0 : \mathbb{B}\downarrow P \rightarrow \mathbb{B}$. Then P is a cloven fibration iff i has a right adjoint r whose counit is a *vertical* natural transformation. More elementarily, this means that for every object $u : J \rightarrow PX$ in $\mathbb{B}\downarrow P$ there is a vertical arrow

$$\begin{array}{ccc} P(u^*X) & \xlongequal{\quad} & J \\ \parallel & & \downarrow u \\ P(u^*X) & \xrightarrow{P(\varphi_{u,X})} & PX \\ \\ u^*X & \xrightarrow{\varphi_{u,X}} & X \end{array}$$

such that for every

$$\begin{array}{ccc}
 PZ & \xrightarrow{v} & J \\
 \parallel & & \downarrow u \\
 P(u^*X) & \xrightarrow{P(\psi)} & PX \\
 & & \\
 Z & \xrightarrow{\psi} & X
 \end{array}$$

there exists a unique $\theta : Z \rightarrow u^*X$ making

$$\begin{array}{ccccc}
 PZ & & & & \\
 \parallel & \searrow^{P(\theta)} & & \searrow^v & \\
 PZ & \xrightarrow{P(\theta)} & P(u^*X) & \xrightarrow{P(\varphi_{u,X})} & PX \\
 & & \parallel & & \downarrow u \\
 & & P(u^*X) & \xrightarrow{P(\varphi_{u,X})} & PX \\
 & & & & \\
 Z & \xrightarrow{\theta} & u^*X & \xrightarrow{\varphi_{u,X}} & X \\
 & \searrow^{\psi} & & \searrow^{\psi} & \\
 & & & & X
 \end{array}$$

commute, i.e. $(v, \psi) = (\text{id}_J, \varphi_{u,X}) \circ i(\theta)$.

M. Shulman and E. Riehl have observed that a weakening can be obtained by requiring that the counit of $i \dashv r$ at (u, X) is just above an iso and not necessarily above an identity.

71 Locally small fibrations

Let P be a fibration over \mathbb{B} . For $X \in P(I)$ and $Y \in P(J)$ let $\text{HOM}(X, Y)$ be the category whose objects are pairs (φ, ψ) of morphisms where $\varphi : Z \rightarrow X$ and $\psi : Z \rightarrow Y$ and φ is cartesian and whose morphisms from (φ', ψ') to (φ, ψ) are morphisms θ with $\varphi\theta = \varphi'$ and $\psi\theta = \psi'$. Notice that θ is necessarily cartesian and uniquely determined by φ, φ' and $P(\theta)$. There is an obvious functor $\text{HOM}(X, Y) \rightarrow \text{Elts}(y(I) \times y(J))$ (sending (φ, ψ) to $(P(\varphi), P(\psi))$) which is an elementary fibration, i.e. a fibration all whose fibres are posetal groupoids.

The fibration P is *locally small* iff for all $X \in P(I)$ and $Y \in P(J)$ the elementary fibration $\text{HOM}(X, Y)$ is representable, i.e. $\text{HOM}(X, Y)$ has a terminal object (φ_0, ψ_0) . The universal property of (φ_0, ψ_0) means that for all morphisms $\varphi : Z \rightarrow X$ and $\psi : Z \rightarrow Y$ with φ cartesian there exists a unique (cartesian) morphism θ with $\varphi_0\theta = \varphi$ and $\psi_0\theta = \psi$.

If \mathbb{C} is a locally small category and $P = \text{Fam}(\mathbb{C})$ over **Set** then for families $X = (X_i)_{i \in I}$ and $Y = (Y_j)_{j \in J}$ a terminal object in $\text{HOM}(X, Y)$ is given by the span

$$X \xleftarrow{\varphi_0} Z \xrightarrow{\psi_0} Y$$

where Z is the family over $\coprod_{i \in I, j \in J} \mathbb{C}(X_i, Y_j)$ with $Z_{(i,j,\alpha)} = X_i$, φ_0 is the canonical cartesian arrow over $\pi_0 : \coprod_{i \in I, j \in J} \mathbb{C}(X_i, Y_j) \rightarrow I$ and $\psi_0 = (\pi_1, f)$ where $\pi_1 : \coprod_{i \in I, j \in J} \mathbb{C}(X_i, Y_j) \rightarrow J$ and $f_{(i,j,\alpha)} = \alpha$.

In Bénabou's original definition he considers only X and Y in the same fibre over I and considers the full subcategory $\text{HOM}_I(X, Y)$ of $\text{HOM}(X, Y)$ on those spans (φ, ψ) where $P(\varphi) = P(\psi)$. This subcategory $\text{HOM}_I(X, Y)$ now gives rise to an elementary fibration over \mathbb{B}/I and he requires all such fibrations to be representable by a morphism $\text{hom}_I(X, Y) \rightarrow I$ (in case of $\text{Fam}(\mathbb{C})$ over **Set** this arrow is the projection of $\coprod_{i \in I} \mathbb{C}(X_i, Y_i)$ to I). In case \mathbb{B} has binary products Bénabou's notion of local smallness is equivalent to the previous one using $\text{HOM}_{I \times J}(\pi_1^*X, \pi_2^*Y)$ for constructing $\text{HOM}(X, Y)$.

72 Internal vs. External Families of Objects

Let \mathcal{E} be a topos and $F : \mathbf{Set} \rightarrow \mathcal{E}$ a functor preserving terminal objects. Let I be a set and $i \in I$ then we write also $i : 1 \rightarrow I$ for the function picking i out of I . The functor $P_F = \partial_1 : \mathcal{E}/F \rightarrow \mathbf{Set}$ is a fibration of toposes called the *fundamental fibration of F* . With every $a : A \rightarrow FI$ in \mathcal{E} we may associate the family $(A_i)_{i \in I}$ where

$$\begin{array}{ccc} A_i & \xrightarrow{\iota_i^{(a)}} & A \\ \downarrow \lrcorner & & \downarrow a \\ F1 & \xrightarrow{Fi} & FI \end{array}$$

for $i \in I$ and with every $f : a \rightarrow b$ in \mathcal{E}/FI we may associate the family $(f_i : A_i \rightarrow B_i)_{i \in I} \in I$ where

$$\begin{array}{ccc} A_i & \xrightarrow{\iota_i^{(a)}} & A \\ \downarrow \lrcorner & & \downarrow f \\ B_i & \xrightarrow{\iota_i^{(b)}} & B \\ \downarrow \lrcorner & & \downarrow b \\ F1 & \xrightarrow{Fi} & FI \end{array}$$

for $i \in I$. This way we obtain a cartesian functor C_F from P_F to $\mathbf{Fam}_b(\mathcal{E})$, the fibration of bounded families in \mathcal{E} , which in general, however, does not reflect isomorphisms as shown by the following two examples.³⁹

³⁹There arises the question to which extent this observation can be reversed, i.e. whether F has a right adjoint whenever C_F reflects isomorphisms. However, the answer to this question is negative as shown by the following counterexample. Let \mathcal{E} be a nontrivial topos and $F : \mathbf{Set} \rightarrow \mathcal{E}$ constant with value $1_{\mathcal{E}}$. Obviously F preserves finite limits but has no right adjoint (since $0 \not\cong 1$ in \mathcal{E}). Nevertheless $C_F : P_F \rightarrow \mathbf{Fam}_b(\mathcal{E})$ does reflect isomorphisms since

$$\begin{array}{ccc} A & \xlongequal{\quad} & A \\ \downarrow \lrcorner & & \downarrow \\ F1 & \xlongequal{F(i)} & FI \end{array}$$

for all $i \in I$.

Let \mathcal{A} be a nontrivial pca and $\mathbf{RT}(\mathcal{A})$ the associated realizability topos. Then the global elements functor $\Gamma : \mathbf{RT}(\mathcal{A}) \rightarrow \mathbf{Set}$ has a full and faithful right adjoint $\nabla : \mathbf{Set} \rightarrow \mathbf{RT}(\mathcal{A})$. Then the cartesian functor $C_\nabla : P_\nabla \rightarrow \mathbf{Fam}_b(\mathbf{RT}(\mathcal{A}))$ does not reflect isomorphisms for the following reason. Both $\text{id}_{\nabla(2)}$ and $\eta_2 : 2 \rightarrow \nabla(2)$ induce the constant family over 2 with value 1 but the two maps are not isomorphic over $\nabla(2)$ since 2 is not isomorphic to $\nabla(2)$ (because otherwise \mathcal{A} were trivial).

Let \mathcal{G} be the local topos of reflexive graphs, i.e. $\Delta \dashv \Gamma \dashv \nabla : \mathbf{Set} \leftrightarrow \mathcal{G}$. Then $C_\nabla : P_\nabla \rightarrow \mathbf{Fam}_b(\mathcal{G})$ does not reflect isomorphisms for the following reason. Let $\eta_I : I \rightarrow \Gamma\Delta(I)$ be the unit of the adjunction $\Delta \dashv \Gamma$ at I . Since Δ is full and faithful η_I is an isomorphism. Let $m_I : \Delta(I) \rightarrow \nabla(I)$ be the transpose of $\eta_I^{-1} : \Gamma\Delta(I) \rightarrow I$. Then both $\text{id}_{\nabla(I)}$ and m_I induce the constant family over I with value 1. But these two maps are isomorphic only if $|I| \leq 1$.

In the general case of a terminal object preserving functor $F : \mathbf{Set} \rightarrow \mathcal{E}$ the fundamental fibration P_F is locally small if and only if F has a right adjoint if and only if \mathcal{E} is locally small as an ordinary category and F preserves small sums, i.e. is isomorphic to $\Delta : \mathbf{Set} \rightarrow \mathcal{E} : I \mapsto \coprod_I 1_{\mathcal{E}}$.

Lemma 1

If $F : \mathbf{Set} \rightarrow \mathcal{E}$ preserves terminal objects and small sums then $C_F : P_F \rightarrow \mathbf{Fam}_b(\mathcal{E})$ is a cartesian weak equivalence.

Proof: Suppose $a : A \rightarrow FI$ is an internal family in \mathcal{E} over I . For $i \in I$ we may consider the pullback

$$\begin{array}{ccc} A_i & \xrightarrow{\iota_i^{(a)}} & A \\ \downarrow t_i^{(a)} & \lrcorner & \downarrow a \\ F1 & \xrightarrow{F_i} & FI \end{array}$$

Since $(F_i)_{i \in I}$ is a coproduct cocone and $a^* : \mathcal{E}/FI \rightarrow \mathcal{E}/A$ preserves sums the family $(\iota_i^{(a)})_{i \in I}$ is a coproduct cone as well. Thus a is isomorphic to $\coprod_{i \in I} t_i^{(a)}$ in \mathcal{E} , i.e. the internal family a is uniquely determined up to isomorphism by its induced external family $(A_i)_{i \in I}$. Thus, for $a : A \rightarrow FI$ and $b : B \rightarrow FI$ in \mathcal{E}/FI for every family $(f_i : A_i \rightarrow B_i)_{i \in I}$ there exists a unique $f : A \rightarrow B$ with $f \circ \iota_i^{(a)} = \iota_i^{(b)} \circ f_i$, i.e.

$$\begin{array}{ccc} A_i & \xrightarrow{\iota_i^{(a)}} & A \\ \downarrow f_i & & \downarrow f \\ B_i & \xrightarrow{\iota_i^{(b)}} & B \end{array}$$

for all $i \in I$. Since for all $i \in I$ the diagram

$$\begin{array}{ccc}
 A_i & \xrightarrow{l_i^{(a)}} & A \\
 f_i \downarrow & & \downarrow f \\
 B_i & \xrightarrow{l_i^{(b)}} & B \\
 \downarrow \lrcorner & & \downarrow b \\
 F1 & \xrightarrow{Fi} & FI
 \end{array}$$

commutes it follows that $b \circ f = a$, i.e. $f : a \rightarrow b$. Since in the previous diagram the lower square and the outer rectangle are pullbacks we have

$$\begin{array}{ccc}
 A_i & \xrightarrow{l_i^{(a)}} & A \\
 f_i \downarrow & \lrcorner & \downarrow f \\
 B_i & \xrightarrow{l_i^{(b)}} & B
 \end{array}$$

for all $i \in I$. Thus we have shown that $C_F(f) = (f_i)_{i \in I}$. If $g : a \rightarrow b$ with $C_F(g) = (f_i)_{i \in I}$ then, in particular, for all $i \in I$ we have

$$\begin{array}{ccc}
 A_i & \xrightarrow{l_i^{(a)}} & A \\
 f_i \downarrow & & \downarrow g \\
 B_i & \xrightarrow{l_i^{(b)}} & B
 \end{array}$$

and thus $f = g$. Thus, we have shown that C_F is full and faithful.

Since \mathcal{E} has copowers of 1 sums of small bounded families exist in \mathcal{E} . Moreover, small sums which exist in \mathcal{E} are extensive, i.e. stable and disjoint. Thus, the cartesian functor C_F is also essentially surjective and thus a cartesian weak equivalence. \square

On the other hand

Lemma 2

If $F : \mathbf{Set} \rightarrow \mathcal{E}$ is a terminal object preserving functor with $C_F : P_F \rightarrow \mathbf{Fam}_i(\mathcal{E})$ full and faithful then F preserves small sums.

Proof: Suppose C_F is full and faithful. We have to show that for an arbitrary set I the cocone $(F_i)_{i \in I}$ is colimiting. For this purpose suppose C is an object of \mathcal{E} and $(f_i : F1 \rightarrow C)_{i \in I}$. Since C_F is full and faithful there exists a unique map g from id_{FI} to $\pi : FI \times C \rightarrow FI$ in \mathcal{E}/FI with $C_F(g) = (f_i)_{i \in I}$, i.e.

$$\begin{array}{ccc}
 F1 & \xrightarrow{F_i} & FI \\
 \downarrow f_i & & \downarrow g \\
 C & \xrightarrow{\quad} & FI \times C \\
 \downarrow \lrcorner & & \downarrow \pi \\
 F1 & \xrightarrow{F_i} & FI
 \end{array}$$

for all $i \in I$. Since g is necessarily of the form $\langle \text{id}_I, f \rangle$ this means that – as desired – there exists a unique $f : FI \rightarrow C$ with $f \circ F_i = f_i$ for all $i \in I$. \square

Thus we have shown that

Theorem 1

A terminal object preserving functor $F : \mathbf{Set} \rightarrow \mathcal{E}$ preserves small sums iff the cartesian functor $C_F : P_F \rightarrow \mathbf{Fam}_b(\mathcal{E})$ is a weak equivalence iff C_F is full and faithful.

Notice that functors $F : \mathbf{Set} \rightarrow \mathcal{E}$ preserving terminal objects and small sums also preserve pullbacks because this is equivalent to P_F having internal sums validating the Chevalley condition which holds for the equivalent fibration $\mathbf{Fam}_b(\mathcal{E})$.

Theorem 2

A terminal object preserving functor $F : \mathbf{Set} \rightarrow \mathcal{E}$ has a right adjoint iff \mathcal{E} is locally small as an ordinary category and the cartesian functor $C_F : P_F \rightarrow \mathbf{Fam}_b(\mathcal{E})$ is a weak equivalence.

Thus, an elementary topos \mathcal{E} is over \mathbf{Set} , i.e. has a unique (up to isomorphism) geometric morphism to \mathbf{Set} , iff \mathcal{E} is locally small and has copowers of 1.

73 Notions of Definable Subfibration

A subfibration of a fibration $P : \mathbb{X} \rightarrow \mathbb{B}$ is given by a subcategory \mathbb{X}' of \mathbb{X} such that

- (1) if $\varphi : Y \rightarrow X$ is cartesian with $X \in \mathbb{X}'$ then $\varphi \in \mathbb{X}'$
- (2) if $\varphi : Y \rightarrow X$ is cartesian, $\alpha : Z \rightarrow Y$ is vertical and $\varphi\alpha \in \mathbb{X}'$ then $\alpha \in \mathbb{X}'$.

Is there a notion of subfibration avoiding the closure under vertical isos implicit in (2)? Well, Grothendieck originally defined a subfibration of a fibration $P : \mathbb{X} \rightarrow \mathbb{B}$ as a subcategory \mathbb{X}' of \mathbb{X} such that $P' = P|_{\mathbb{X}'}$ is a fibration and $I : \mathbb{X}' \hookrightarrow \mathbb{X}$ is a cartesian functor from P' to P . But for this more general notion condition (1) will not hold in general as shown by the following example. Let \mathbb{C} be $\nabla(2)$ and \mathbb{C}' the subcategory $\mathbf{2}$ obtained from \mathbb{C} by deleting the arrow $i^{-1} : 1 \rightarrow 0$ inverse to $i : 0 \rightarrow 1$ in \mathbb{C} . Considering \mathbb{C} and \mathbb{C}' as fibrations over $\mathbf{1}$ we observe that $i^{-1} : 1 \rightarrow 0$ is a cartesian arrow in \mathbb{C} with codomain in \mathbb{C}' but i^{-1} is not in \mathbb{C}' . This holds *a fortiori* for the subfibration $\text{Fam}(\mathbb{C}')$ of $\text{Fam}(\mathbb{C})$. Notice, however, that \mathbb{X}' is a subfibration in our sense iff it is a subfibration in the sense of Grothendieck and satisfies condition (1).

Apparently Bénabou's notion of definability applies only to subfibrations in our sense since for non-split fibrations reindexing is defined only up to isomorphism. However, for split fibrations $P : \mathbb{X} \rightarrow \mathbb{B}$ one may define a subfibration of P as a subcategory \mathbb{X}' of \mathbb{X} such that

- (1') for every $X \in \mathbb{X}'$ and $u : J \rightarrow I = P(X)$ the chosen cartesian arrow $\text{Cart}(u, X) : u^*X \rightarrow X$ is in \mathbb{X}' and
- (2') for every vertical arrow $\alpha : X \rightarrow Y$ in \mathbb{X}' over I and $u : J \rightarrow I$ the unique vertical arrow $\beta = u^*\alpha : u^*X \rightarrow u^*Y$ with $\text{Cart}(u, Y) \circ \beta = \alpha \circ \text{Cart}(u, X)$ is in \mathbb{X}'

and then formulate the notion of definability w.r.t. the reindexing chosen by the splitting.

74 Definable subfibrations of wellpowered fibrations need not be wellpowered

Let \mathbb{C} be the category of all ordinals and monotone maps between them and \mathbb{C}' the (lluf) subcategory of \mathbb{C} consisting of terminal projections and identity morphisms. Notice that every isomorphism in \mathbb{C} is an identity map and thus \mathbb{C}' is closed under composition with isos in \mathbb{C} . Moreover, all morphisms in \mathbb{C}' are monic. Obviously, the category \mathbb{C} is wellpowered but \mathbb{C}' is not since $\mathbf{1}$ has as many subobjects as there are objects in \mathbb{C}' .

Since \mathbb{C}' is closed under isos in \mathbb{C} it follows that $\text{Fam}(\mathbb{C}')$ is a subfibration of $\text{Fam}(\mathbb{C})$. Actually, it is even a definable subfibration. Since $\text{Fam}(\mathbb{C})$ is a

wellpowered fibration whereas $\text{Fam}(\mathbb{C}')$ is not we have shown that definable subfibrations of wellpowered fibrations need not be wellpowered.

Bénabou originally suggested to consider $\mathbb{C} = \mathbf{Set}$ and the (lluf) subcategory \mathbb{C}' consisting of identity morphisms and terminal projections. Again \mathbb{C}' is not wellpowered though $\mathbb{C} = \mathbf{Set}$ is. However, since \mathbb{C}' is not closed under isomorphisms in \mathbb{C} we can't consider $\text{Fam}(\mathbb{C}')$ as a subfibration of $\text{Fam}(\mathbb{C})$.

75 Geometric Morphisms, Logical Morphisms and Cartesian Closedness

In Lemma A.1.5.8 of the *Elephant* it is shown that for an adjunction $L \dashv F : \mathcal{C} \rightarrow \mathcal{D}$ between cartesian closed categories the functor F preserves the cartesian closed structure iff $L \dashv F$ satisfies the *Frobenius reciprocity law*, i.e. the canonical morphism

$$\langle L\pi_1, \varepsilon_A \circ L\pi_2 \rangle : L(B \times FA) \longrightarrow LB \times A$$

is an isomorphism for all $A \in \mathcal{C}$ and $B \in \mathcal{D}$. As Corollary A.1.5.9 one obtains that for adjunctions $L \dashv F$ between cartesian closed categories it holds that

1. if F preserves the cartesian closed structure and L preserves 1 then F is full and faithful
2. if F is full and faithful and L preserves binary products then F preserves the cartesian closed structure.

This has the consequence (stated in Lemma A.4.2.9) that a geometric morphism between toposes is an inclusion iff its direct image part preserves the cartesian closed structure. However, for every nontrivial monoid \mathbb{M} the geometric morphism $\Delta \dashv \Gamma : \widehat{\mathbb{M}} \rightarrow \mathbf{Set}$ is not an inclusion but Γ preserves Ω since $\widehat{\mathbb{M}}$ is 2-valued.

In Proposition A.2.3.8 of the *Elephant* it is shown that for a logical functor $F : \mathcal{E} \rightarrow \mathcal{F}$ between toposes with left adjoint L t.f.a.e.

1. $L \dashv F$ is of the form $\Sigma_B \dashv B^* : \mathcal{E} \rightarrow \mathcal{E}/B$ (where B is $L1$)
2. L is faithful
3. L preserves equalizers
4. L preserves pullbacks.

Thus, as observed in Scholium A.2.3.9 a logical functor between toposes has a finite limit preserving left adjoint iff it is an equivalence.

In Lemma A.2.2.10 of the *Elephant* it is shown that a logical functor F between toposes has a left adjoint L iff it has a right adjoint U . Thus, logical functors F with a faithful left adjoint are inverse image parts of certain geometric morphisms, namely the atomic and localic ones (Lemma C.3.5.4 (iii)).⁴⁰ Such geometric morphisms are called *local homeomorphisms*.

In Proposition A.4.5.1 of the *Elephant* it is shown that for an inclusion of toposes $i : \mathcal{L} \hookrightarrow \mathcal{E}$ t.f.a.e.

1. i is of the form $U^* \dashv \Pi_U : \mathcal{E}/U \rightarrow \mathcal{E}$ for some subterminal object U (i.e. \mathcal{L} is an *open* subtopos of \mathcal{E})

⁴⁰A geometric morphism $f : \mathcal{F} \rightarrow \mathcal{E}$ is called *atomic* iff $f^* : \mathcal{E} \rightarrow \mathcal{F}$ is logical and it is called *localic* iff every $X \in \mathcal{F}$ appears as subquotient of some f^*I .

2. i^* is logical
3. i^* preserves the cartesian closed structure.

Notice, however, that for an inclusion $i : \mathcal{L} \hookrightarrow \mathcal{E}$ the inverse image part $i^* : \mathcal{E} \rightarrow \mathcal{L}$ might well preserve Ω without preserving the cartesian closed structure. Examples are global sections functors $\Gamma : \mathcal{E} \rightarrow \mathbf{Set}$ where \mathcal{E} is the topos of reflexive graphs or the effective topos where $\Gamma \dashv \nabla : \mathcal{E}_{\neg\neg} \hookrightarrow \mathcal{E}$ is the inclusion of $\neg\neg$ -sheaves.

76 A topos with no geometric morphism to any boolean one

can be found in a paper by R. Paré from 1985 with this very title. Let I be the inclusion of the topos \mathbf{Set}_f of finite sets into the topos \mathbf{Set} . Let $\mathcal{E} = \mathbf{Set} \downarrow I$ be the topos obtained by glueing along I . It can be shown easily that there is no geometric morphism from \mathcal{E} to a boolean topos \mathcal{B} .

Suppose $f : \mathcal{E} \rightarrow \mathcal{B}$ is a geometric morphism with \mathcal{B} boolean. Since subtoposes of boolean toposes are boolean we may assume without loss of generality that f is surjective. Since f^* preserves decidability of objects and an object $a : A_0 \rightarrow A_1$ in \mathcal{E} is decidable iff a is monic for all $X, Y \in \mathcal{B}$ the hom-set $\mathcal{E}(f^*X, f^*Y)$ is finite. Since f^* is faithful all hom-sets $\mathcal{B}(X, Y)$ are finite. For an arbitrary set A we have

$$\mathcal{B}(1, f_*(A \rightarrow 1)) \cong \mathcal{E}(f^*1, A \rightarrow 1) \cong \mathcal{E}(1 \rightarrow 1, A \rightarrow 1) \cong A$$

which leads to a contradiction when choosing A to be infinite.

77 Mail to CATEGORIES from 25 June 2002

In my reply to the question about large sites for toposes I made the WRONG claim that if $\mathbf{Psh}(C)$ is a topos then all slices of C are essentially small. As Peter Johnstone pointed out this is a sufficient condition and he asked me for an argument for the reverse direction. Alas, I couldn't find one but instead I came across the following counterexample.

Let G be a large group (e.g. all permutations of the universe whose collection of non-fixpoints is small, see 1.96(10) of "Cats & Alligators") then $\mathbf{Psh}(G)$ is a locally small cocomplete topos although G is not even locally small. However, every slice of G is trivial. But if one takes for C the large group G augmented with a terminal object then $\mathbf{Psh}(C)$ is a locally small cocomplete topos, too, whereas $C/1$ is not essentially small.

Thomas Streicher

78 Local Homeomorphism of Toposes

In P. T. Johnstone's book *Topos Theory* (1977) in Theorem 1.47 (pp. 37-38) one finds the following characterisation of local homeomorphism between toposes, i.e. geometric morphisms of the form $I^* \dashv \Pi_I : \mathcal{E}/I \rightarrow \mathcal{E}$. A geometric morphism $f : \mathcal{F} \rightarrow \mathcal{E}$ is a local homeomorphism of toposes iff $f^* : \mathcal{E} \rightarrow \mathcal{F}$ is logical and has a left adjoint $f_!$ preserving equalizers. As shown in A.2.3.8 of the *Elephant* if f^* is logical then $f_!$ preserves equalizers iff $f_!$ preserves pullbacks iff $f_!$ is faithful.

Obviously, the conditions are necessary. For the reverse direction one first derives from the assumptions that $f_!$ reflects isos and thus is comonadic. Then one shows that the canonical morphism of comonads from $f_!f^*$ to $f_!1 \times (-)$ whose component at X is given by $\langle f_!(!_{f^*X}), \varepsilon_X \rangle : f_!f^*X \rightarrow f_!1 \times X$ is an isomorphism (an instance of the Frobenius reciprocity law for the adjunction $f_! \dashv f^*$).

Thus, alternatively, one could characterize local homeomorphisms of toposes as those essential geometric morphisms $f : \mathcal{F} \rightarrow \mathcal{E}$ such that

- (1) $f_!$ preserves equalizers
- (2) $f_!$ reflects isomorphism
- (3) $\langle f_!(!_{f^*X}), \varepsilon_X \rangle : f_!f^*X \rightarrow f_!1 \times X$ is an isomorphism for all X in \mathcal{E} .

One may replace (1) by the requirement that $f_!$ preserves pullbacks since functors of the form $\Sigma_I \dashv I^*$ do and pullback preserving functors between finite limit categories preserve also equalizers.

79 Ordinary Frobenius Reciprocity

An adjunction $F \dashv U : \mathbb{B} \rightarrow \mathbb{A}$ between categories with finite products satisfies the *Frobenius Reciprocity* law iff for all $A \in \mathbb{A}$ and $B \in \mathbb{B}$ the canonical morphism

$$F(A \times UB) \xrightarrow{\langle F\pi_1, F\pi_2 \rangle} FA \times FUB \xrightarrow{FA \times \varepsilon_B} FA \times B$$

is an isomorphism.⁴¹

This canonical morphism can be described alternatively as the exponential transpose $\hat{\alpha} = \varepsilon_{FA \times B} \circ F\alpha$ of the morphism $\alpha : A \times UB \rightarrow U(FA \times B)$ with

$$\begin{array}{ccc} A \times UB & \xrightarrow{\eta_A \times UB} & UFA \times UB \\ & \searrow \alpha & \uparrow \cong \langle U\pi_1, U\pi_2 \rangle \\ & & U(FA \times B) \end{array}$$

where $\langle U\pi_1, U\pi_2 \rangle$ is an isomorphism since U is a right adjoint. We observe that

$$\begin{aligned} \pi_1 \circ \hat{\alpha} &= \pi_1 \circ \varepsilon_{FA \times B} \circ F\alpha = \varepsilon_{FA} \circ FU(\pi_1) \circ F\alpha = \\ &= \varepsilon_{FA} \circ F(U\pi_1 \circ \alpha) = \varepsilon_{FA} \circ F(\eta_A \circ \pi_1) = \\ &= \varepsilon_{FA} \circ F\eta_A \circ F\pi_1 = F\pi_1 = \\ &= \pi_1 \circ (FA \times \varepsilon_B) \circ \langle F\pi_1, F\pi_2 \rangle \end{aligned}$$

and

$$\begin{aligned} \pi_2 \circ \hat{\alpha} &= \pi_2 \circ \varepsilon_{FA \times B} \circ F\alpha = \varepsilon_B \circ FU(\pi_2) \circ F\alpha = \\ &= \varepsilon_B \circ F(U\pi_2 \circ \alpha) = \varepsilon_B \circ F\pi_2 = \\ &= \pi_2 \circ (FA \times \varepsilon_B) \circ \langle F\pi_1, F\pi_2 \rangle \end{aligned}$$

from which it follows that

$$\hat{\alpha} = (FA \times \varepsilon_B) \circ \langle F\pi_1, F\pi_2 \rangle$$

as claimed.

F. W. Lawvere introduced Frobenius reciprocity in the late 1960s because the existence of a natural isomorphism $F(A \times UB) \cong FA \times B$ is equivalent to preservation of exponentials by U (provided they exist in \mathbb{B}) since we have

$$\frac{A \rightarrow UC^{UB}}{A \times UB \rightarrow UC} = \frac{F(A \times UB) \rightarrow C}{FA \times B \rightarrow C} = \frac{FA \rightarrow C^B}{A \rightarrow U(C^B)}$$

naturally in A , B and C .

⁴¹If U is full and faithful, i.e. ε is a natural isomorphism, Frobenius reciprocity amounts to the requirement that $\langle F\pi_1, F\pi_2 \rangle$ is an isomorphism, i.e. F sends products of A and UB in \mathbb{A} to products in \mathbb{B} .

80 Stably Frobenius

Suppose now that \mathbb{A} and \mathbb{B} have finite limits. Then $L \dashv F : \mathbb{B} \rightarrow \mathbb{A}$ is *stably Frobenius* (this notion is due to Ch. Townsend) iff for all $I \in \mathbb{B}$ the adjunction $L_I \dashv F_I := F_{/I} : \mathbb{B}/I \rightarrow \mathbb{A}/FI$ satisfies the ordinary Frobenius reciprocity law. Notice that $L_I(b : B \rightarrow FI) = \hat{b} = \varepsilon_I \circ Lb$.

We now explicitate this requirement. Suppose $a : A \rightarrow FI$ and $b : B \rightarrow I$ then $a \times F_I b$ is given by

$$\begin{array}{ccc} P & \xrightarrow{q} & FB \\ \downarrow p & \lrcorner & \downarrow Fb \\ A & \xrightarrow{a} & FI \end{array} \quad (1)$$

whose transpose is

$$\begin{array}{ccc} LP & \xrightarrow{\hat{q}} & B \\ \downarrow Lp & & \downarrow b \\ LA & \xrightarrow{\hat{a}} & I \end{array} \quad (2)$$

i.e.

$$\begin{array}{ccccc} LP & \xrightarrow{Lq} & LFB & \xrightarrow{\varepsilon_B} & B \\ \downarrow Lp & & \downarrow LFb & \searrow L_I F_I b & \downarrow b \\ LA & \xrightarrow{La} & LFI & \xrightarrow{\varepsilon_I} & I \end{array}$$

Frobenius reciprocity for $L_I \dashv F_I$ requires the outer rectangle to be a pullback. Thus Frobenius reciprocity for $L_I \dashv F_I$ means that whenever (1) is a pullback then (2) is a pullback.

Thus F has a left adjoint L with $L \dashv F$ stably Frobenius iff $\Delta_F : P_{\mathbb{B}} \rightarrow P_F$ has a fibred left adjoint Π_F (whose fiber over $1_{\mathbb{B}}$ is L). Thus, we may call stably Frobenius adjunctions $L \dashv F$ *co-geometric morphisms* since for finite limit preserving $F : \mathbb{B} \rightarrow \mathbb{A}$ the existence of a right adjoint U is equivalent to Δ_F having a fibred right adjoint Γ_F (given by $\eta_I^* \circ U_{/FI}$).

One easily shows that co-geometric morphisms are closed under composition. Moreover, if $L \dashv F$ is a co-geometric morphism then F is full and faithful iff L preserves terminal objects (i.e. $L1 \cong 1$).

81 Funk's Frobenius Reciprocity

Let \mathbb{C} and \mathbb{B} be categories with finite limits. We will characterize those adjunctions $F \dashv U : \mathbb{C} \rightarrow \mathbb{B}$ which are “local homeomorphisms” in the sense that up to equivalence they are of the form $\Sigma_I \dashv I^* : \mathbb{B} \rightarrow \mathbb{B}/I$ for some $I \in \mathbb{B}$. Obviously, this is the case iff $F_{/1} : \mathbb{C} \rightarrow \mathbb{B}/F1$ is an equivalence (since $F = \Sigma_{F1} \circ F_{/1}$).

Notice that $F_{/1}$ has right adjoint $\eta_1^* \circ U_{/F1}$. Thus $F_{/1}$ is an equivalence iff

- (1) $F_{/1}$ reflects isos
- (2) $F_{/1} \dashv \eta_1^* \circ U_{/F1}$ is a reflection.

Obviously, condition (1) holds iff F reflects isos. Condition (2) holds iff for all $b : B \rightarrow F1$ for the map q in

$$\begin{array}{ccc} P & \xrightarrow{q} & UB \\ \downarrow \lrcorner & & \downarrow Ub \\ 1 & \xrightarrow{\eta_1} & UF1 \end{array}$$

it holds that $\hat{q} = \varepsilon_B \circ Fq$ is an isomorphism.

We will show now that from conditions (1) and (2) it follows that $F \dashv U$ satisfies a condition called *Funk's Frobenius Reciprocity* from which it follows that $F_{/1}$ is an equivalence provided F reflects isomorphisms.

First notice that from (1) and (2) it follows that F preserves pullbacks (since equivalences and functors of the form Σ_I preserve pullbacks) and thus $F_{/1}$ preserves finite limits. If $G \dashv V : \mathbb{Y} \rightarrow \mathbb{X}$ is an adjunction between categories with finite limits and G preserves finite limits then $G \dashv V$ is a reflection iff $G_{/X} \dashv \eta_X^* \circ V_{/GX}$ is a reflection for all $X \in \mathbb{X}$. Thus, since $F_{/1}$ preserves finite limits and $F_{/1} \dashv \eta_1^* \circ U_{/F1}$ is a reflection for all objects C in \mathbb{C} the adjunction $F_{/C} \dashv \eta_C^* \circ U_{/FC}$ is a reflection, i.e. for all $C \in \mathbb{C}$ and $b : B \rightarrow FC$ for the map q in

$$\begin{array}{ccc} P & \xrightarrow{q} & UB \\ p \downarrow \lrcorner & & \downarrow Ub \\ C & \xrightarrow{\eta_C} & UFC \end{array}$$

it holds that $\hat{q} = \varepsilon_B \circ Fq$ is an isomorphism. We will show now that for all $a : A \rightarrow B$ in \mathbb{B} and $c : C \rightarrow UB$

$$\text{from } \begin{array}{ccc} P & \xrightarrow{q} & UA \\ p \downarrow \lrcorner & & \downarrow Ua \\ C & \xrightarrow{c} & UB \end{array} \quad \text{it follows that } \begin{array}{ccc} FP & \xrightarrow{\hat{q}} & A \\ Fp \downarrow \lrcorner & & \downarrow a \\ FC & \xrightarrow{\hat{c}} & B \end{array}$$

which condition is called *Funk's Frobenius Reciprocity* (FFR).⁴² Suppose

$$\begin{array}{ccc} P & \xrightarrow{q} & UA \\ p \downarrow \lrcorner & & \downarrow Ua \\ C & \xrightarrow{c} & UB \end{array} \quad \text{and} \quad \begin{array}{ccc} Q & \xrightarrow{q'} & A \\ p' \downarrow \lrcorner & & \downarrow a \\ FC & \xrightarrow{\widehat{c}} & B \end{array}$$

and consider

$$\begin{array}{ccccc} P & \xrightarrow{\alpha} & UQ & \xrightarrow{Uq'} & UA \\ p \downarrow \lrcorner & & Up' \downarrow \lrcorner & & \downarrow Ua \\ C & \xrightarrow{\eta_C} & UFC & \xrightarrow{U\widehat{c}} & UB \end{array}$$

with $Uq' \circ \alpha = q$. By transposition w.r.t. $F \dashv U$ we obtain

$$\begin{array}{ccccc} FP & \xrightarrow{\widehat{\alpha}} & Q & \xrightarrow{q'} & A \\ Fp \downarrow \cong & & p' \downarrow \lrcorner & & \downarrow a \\ FC & \xrightarrow{=} & FC & \xrightarrow{\widehat{c}} & B \end{array}$$

with $q' \circ \widehat{\alpha} = \widehat{q}$ where $\widehat{\alpha}$ is an isomorphism. Thus

$$\begin{array}{ccc} FP & \xrightarrow{\widehat{q}} & A \\ Fp \downarrow \lrcorner & & \downarrow a \\ FC & \xrightarrow{\widehat{c}} & B \end{array}$$

is a pullback as desired.

Thus, Funk's Frobenius Reciprocity states that the canonical morphism $\widehat{\alpha} : F(C \times_{UB} UA) \rightarrow FC \times_B A$ is an isomorphism (which is ordinary Frobenius Reciprocity for the adjunction $\Sigma_{\varepsilon_B} \circ F_{/UB} \dashv U_{/B}$).

The following theorem can be found in an appendix of J. Funk's Thesis *Descent for Cocomplete Categories* (McGill 1990) where also FFR was introduced.

⁴²Instantiating c by η_C in FFR it follows that \widehat{q} is an isomorphism whenever

$$\begin{array}{ccc} P & \xrightarrow{q} & UB \\ p \downarrow \lrcorner & & \downarrow Ub \\ C & \xrightarrow{\eta_C} & UFC \end{array}$$

since \widehat{c} is the identity on FC .

Theorem 1

For an adjunction $F \dashv U : \mathbb{C} \rightarrow \mathbb{B}$ between categories with finite limits the functor $F_{/1} : \mathbb{C} \rightarrow \mathbb{B}/F1$ is an equivalence iff F reflects isos and the adjunction $F \dashv U$ satisfies Funk's Frobenius Reciprocity.

Proof: The forward direction has just been established. The backwards direction follows by observing that FFR implies condition (2) when instantiating c by η_1 . \square

This theorem together with the simple proof of Moens' Lemma in Section 82 gives rise to the following theorem which also can be found in Funk's Thesis and for which purpose he introduced FFR.

Theorem 2

Let \mathbb{I} be a category with finite limits. Then a fibration $P : \mathbb{X} \rightarrow \mathbb{I}$ is equivalent to one of the form $P_F = \partial_1 : \mathbb{C} \downarrow F \rightarrow \mathbb{I}$ for a finite limit preserving functor $F : \mathbb{I} \rightarrow \mathbb{C}$ where \mathbb{C} has finite limits iff P is a fibration of categories with finite limits and internal sums such that for every object I in \mathbb{I} the functor $\coprod_I : \mathbb{X}_I \rightarrow \mathbb{X}_1$ reflects isos and the adjunction $\coprod_I \dashv I^* : \mathbb{X}_1 \rightarrow \mathbb{X}_I$ satisfies Funk's Frobenius Reciprocity.

Proof: Let \mathbb{I} be a category with finite limits. By Section 82 a fibration $P : \mathbb{X} \rightarrow \mathbb{I}$ is equivalent to one of the form $P_F = \partial_1 : \mathbb{C} \downarrow F \rightarrow \mathbb{I}$ for a finite limit preserving functor $F : \mathbb{I} \rightarrow \mathbb{C}$ where \mathbb{C} has finite limits iff P is a fibration of categories with finite limits and internal sums which is extensive in the sense that a commuting square

$$\begin{array}{ccc}
 V & \xrightarrow{\varphi} & U \\
 v \downarrow & & \downarrow u \\
 1_I & \xrightarrow[\varphi_I]{\text{cocart.}} & \coprod_I 1_I
 \end{array}$$

with u and v vertical is a pullback iff φ is cocartesian.

Thus it remains to show that a fibration $P : \mathbb{X} \rightarrow \mathbb{I}$ of categories with finite limits and internal sums is extensive iff for all $I \in \mathbb{I}$ the functor \coprod_I reflects isos and the adjunction $\coprod_I \dashv I^*$ satisfies FFR. Suppose $P : \mathbb{X} \rightarrow \mathbb{I}$ is a fibration of categories with finite limits and internal sums. By Theorem 1 the requirement on the adjunctions $\coprod_I \dashv I^*$ is equivalent to

- (1) \coprod_I reflects isos
- (2) $\coprod_{I/1_I} \dashv \eta_1^* \circ I^*_{/\coprod_I 1_I}$ is a reflection.

Condition (2) is equivalent to the requirement that pullbacks of $\varphi_I : 1_I \rightarrow \coprod_I 1_I$ along vertical $u : U \rightarrow \coprod_I 1_I$ are cocartesian. This can be seen from

$$\begin{array}{ccccc}
& & & & \coprod_I V \\
& & & \nearrow \text{cocart.} & \downarrow \hat{\alpha} \\
V & \xrightarrow{\quad} & I^*U & \xrightarrow{\quad} & U \\
\downarrow \lrcorner & \alpha & \downarrow \lrcorner & \text{cart.} & \downarrow u \\
1_I & \xrightarrow{\quad} & I^* \coprod_I 1_I & \xrightarrow{\quad \text{cart.}} & \coprod_I 1_I \\
& \eta_1 & & &
\end{array}$$

where the bottom row is φ_I since $\hat{\alpha}$ is an iso iff $u^*\varphi_I$ is cocartesian. Now assuming that $\varphi_I : 1_I \rightarrow \coprod_I 1_I$ satisfies (2) the functor \coprod_I reflects isos iff for every vertical $v : V \rightarrow 1_I$ the square

$$\begin{array}{ccc}
V & \xrightarrow{\quad \varphi \quad} & \coprod_I V \\
\downarrow v & & \downarrow \coprod_I v \\
1_I & \xrightarrow{\quad \varphi_I \quad} & \coprod_I 1_I
\end{array}$$

is a pullback. Suppose all such squares are pullbacks and $\alpha : U \rightarrow V$ is a vertical arrow above I such that $\coprod_I \alpha$ is an isomorphism. Then in

$$\begin{array}{ccc}
U & \xrightarrow{\quad \text{cocart.} \quad} & \coprod_I U \\
\downarrow \alpha & & \cong \downarrow \coprod_I \alpha \\
V & \xrightarrow{\quad \varphi \quad} & \coprod_I V \\
\downarrow v & & \downarrow \coprod_I v \\
1_I & \xrightarrow{\quad \varphi_I \quad} & \coprod_I 1_I
\end{array}$$

the lower square and the outer rectangle are pullbacks. Thus the upper square is a pullback, too, from which it follows that α is an isomorphism. For the reverse direction suppose \coprod_I reflects isos and consider a diagram

$$\begin{array}{ccc}
V & & \\
\downarrow \iota & \searrow \varphi & \\
I^* \coprod_I V & \xrightarrow{\text{cocart.}} & \coprod_I V \\
\downarrow e & \lrcorner & \downarrow \coprod_I v \\
1_I & \xrightarrow[\varphi_I]{\text{cocart.}} & \coprod_I 1_I
\end{array}$$

where the top arrow of the square is cocartesian due to assumption (2) and the already established equivalence. Then $\coprod_I \iota$ is an isomorphism from which it follows that ι is an isomorphism. Thus we have shown that as desired P is extensive iff (1) and (2) hold for all $I \in \mathbb{I}$ iff for all $I \in \mathbb{I}$ the functor \coprod_I reflects isos and $\coprod_I \dashv_I I^*$ satisfies FFR. \square

Essentially the same argument allows one to prove

Theorem 3

Let \mathbb{I} be a category with finite limits. Then a fibration $P : \mathbb{X} \rightarrow \mathbb{I}$ is equivalent to one of the form $P_F = \partial_1 : \mathbb{C} \downarrow F \rightarrow \mathbb{I}$ for a terminal object preserving functor $F : \mathbb{I} \rightarrow \mathbb{C}$ where \mathbb{C} has finite limits iff P is a fibration of categories with finite limits and a cofibration such that for every object I in \mathbb{I} the functor $\coprod_I : \mathbb{X}_I \rightarrow \mathbb{X}_1$ reflects isos and the adjunction $\coprod_I \dashv I^* : \mathbb{X}_1 \rightarrow \mathbb{X}_I$ satisfies Funk's Frobenius Reciprocity.

which provides an alternative characterization of Zawadowski's cartesian bifibrations (see section 99).

However, the characterisations in terms of extensivity as given in subsequent section 82 are certainly much more intuitive.

82 The World's Simplest Proof of Moens' Lemma

Let \mathbb{B} be a category with finite limits and $P : \mathbb{X} \rightarrow \mathbb{B}$ be a fibration of finite limit categories which is also a cofibration. For $I \in \mathbb{B}$ let $\varphi_I : 1_I \rightarrow \Delta(I)$ be a cocartesian arrow from 1_I (terminal object in the fibre \mathbb{X}_I) over the terminal projection $I \rightarrow 1$ in \mathbb{B} . For $u : J \rightarrow I$ in \mathbb{B} let $\Delta(u) : \Delta(J) \rightarrow \Delta(I)$ be the unique vertical arrow making the diagram

$$\begin{array}{ccc} 1_J & \xrightarrow{\varphi_J} & \Delta(J) \\ 1_u \downarrow & (1) & \downarrow \Delta(u) \\ 1_I & \xrightarrow{\varphi_I} & \Delta(I) \end{array}$$

commute. This gives rise to a functor $\Delta : \mathbb{B} \rightarrow \mathbb{X}_1$.

A fibration P as above is called *internally extensive* iff every commuting diagram with α and β vertical

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & U \\ \alpha \downarrow & & \downarrow \beta \\ 1_I & \xrightarrow{\varphi_I} & \Delta(I) \end{array}$$

is a pullback iff φ is cocartesian.⁴³

Obviously, for internally extensive fibrations P the functor $\varphi_I^* : \mathbb{X}_1/\Delta(I) \rightarrow \mathbb{X}_I/1_I \cong \mathbb{X}_I$ is an equivalence and thus, since (1) commutes, the diagram

$$\begin{array}{ccc} \mathbb{X}_J & \xleftarrow[\simeq]{\varphi_J^*} & \mathbb{X}_1/\Delta(J) \\ u^* \uparrow & & \uparrow \Delta(u)^* \\ \mathbb{X}_I & \xleftarrow[\varphi_I^*]{\simeq} & \mathbb{X}_1/\Delta(I) \end{array}$$

commutes up to isomorphism for all $u : J \rightarrow I$ in \mathbb{B} and thus $P \simeq P_\Delta = \Delta^* P_{\mathbb{X}_1}$ (where $P_{\mathbb{X}_1} = \partial_1 : \mathbb{X}_1^2 \rightarrow \mathbb{X}_1$ is the fundamental fibration of \mathbb{X}_1).

Obviously, the functor Δ preserves 1. Recall⁴⁴ that a functor $F : \mathbb{B} \rightarrow \mathbb{C}$ between categories with pullbacks preserves pullbacks iff $P_F = F^* P_{\mathbb{C}}$ has internal sums, i.e. cocartesian arrows are stable under pullbacks along cartesian arrows. Thus, the fibration $P \simeq P_\Delta = \Delta^* P_{\mathbb{X}_1}$ has internal sums iff Δ preserves pullbacks (and thus all finite limits).

⁴³Equivalently one may require that (1) cocartesian arrows are stable under pullbacks along vertical arrows and (2) from φ and $\varphi\alpha$ cocartesian and α vertical it follows that α is an iso, i.e. \prod_u reflects isos for all u in \mathbb{B} .

⁴⁴see Lemma 13.2 (on pp.46-47) of my notes on *Fibred Categories*

83 M. Jibladze's Theorem on Fibered Toposes

Let \mathbb{B} be a category with finite limits. A *topos fibered over* \mathbb{B} is a fibration $P : \mathbb{X} \rightarrow \mathbb{B}$ all whose fibres are toposes and all whose reindexing functors are logical. M. Jibladze has shown that if a fibered topos has internal sums then these sums are necessarily universal (i.e. pullback stable) and disjoint. Thus, by Moens' Theorem it follows that $P \simeq P_\Delta$ where $\Delta : \mathbb{B} \rightarrow \mathbb{E} = P(1) : I \mapsto \coprod_I 1_I$.

As a preparation we need the following results about logical functors $F : \mathcal{E} \rightarrow \mathcal{F}$ between toposes. If $L \dashv F$ then (by A.2.4.8 of the *Elephant*) functor L preserves monos and (by A.2.3.8 of the *Elephant*) the following are equivalent

- (1) $L_{/1} : \mathcal{F}_{/1} \rightarrow \mathcal{E}_{/L1}$ is an equivalence⁴⁵
- (2) L is faithful
- (3) L preserves equalizers
- (4) L preserves pullbacks.

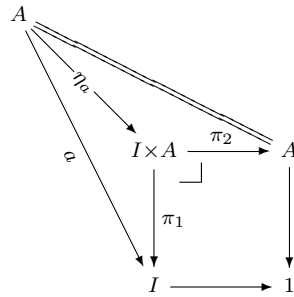
One easily shows⁴⁶ that if L is full and faithful then $L1$ is subterminal from which it follows that all components of the counit ε are monos.

We are now ready to prove Jibladze's Theorem on Fibered Toposes.

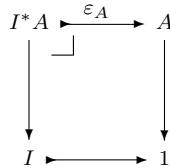
Proof: First observe that for a mono $m : J \rightarrow I$ we have $m^* \prod_m \cong \text{Id}_{P(J)}$ as follows from the Chevalley condition for internal sums at the pullback square

⁴⁵the right adjoint of $L_{/1}$ is given by $\eta_1^* \circ F_{/L1}$

⁴⁶Suppose $\Sigma_I : \mathcal{E}_{/I} \rightarrow \mathcal{E}$ is full and faithful. Then the unit η of $\Sigma_I \dashv I^*$ is an isomorphism. For $a : A \rightarrow I$ we have $\eta_a = \langle a, \text{id}_A \rangle : a \rightarrow I^* \Sigma_I a$ as depicted in



Since η_a is an isomorphism the projection $\pi_2 : I \times A \rightarrow A$ is an isomorphism, too. Thus, for $a' : A \rightarrow I$ we have $\pi_2 \circ \langle a, \text{id}_A \rangle = \text{id}_A = \pi_2 \circ \langle a', \text{id}_A \rangle$ from which it follows that $a = a'$ since π_2 is an isomorphism. Thus I is subterminal. But then the counit ε_A at A is given by



and thus monic.

$$\begin{array}{ccc}
J & \xlongequal{\quad} & J \\
\parallel & \lrcorner & \downarrow m \\
J & \xrightarrow{\quad} & I \\
& & m
\end{array}$$

Since m^* is logical and \coprod_m is full and faithful all components of the counit of $\coprod_m \dashv m^*$ are monic.

Next we show that for all $u : J \rightarrow I$ in \mathbb{B} and $X \in P(J)$ the map $\eta_X : X \rightarrow u^* \coprod_u X$ is monic (where η is the counit of $\coprod_u \dashv u^*$). Recall that η_X is the unique vertical map such that

$$\begin{array}{ccc}
X & \xrightarrow[\text{cocart.}]{\varphi_u(X)} & \coprod_u X \\
\eta_X \downarrow & \nearrow[\text{cart.}]{\theta_u(X)} & \\
u^* \coprod_u X & &
\end{array}$$

Let $k_0, k_1 : K \rightarrow J$ be a kernel pair of u in \mathbb{B} and $d_u : J \rightarrow K$ with $k_0 d_u = \text{id}_J = k_1 d_u$. Consider the diagram

$$\begin{array}{ccccc}
d_u^* k_1^* X & \xrightarrow[\text{cart.}]{\psi} & k_1^* X & \xrightarrow[\text{cart.}]{\theta} & X \\
\parallel & & \downarrow \varphi & \lrcorner & \downarrow \varphi_u(X) \\
X & \xrightarrow[\eta_X]{} & u^* \coprod_u X & \xrightarrow[\theta_u(X)]{\text{cart.}} & \coprod_u X
\end{array}$$

with $\theta \circ \psi = \text{id}_X$. Notice that φ is cocartesian by the Chevalley condition for internal sums. Next consider the diagram

$$\begin{array}{ccc}
d_u^* k_1^* X & & \\
\downarrow \widetilde{\varphi}_1 & \searrow \psi & \\
\coprod_{d_u} d_u^* k_1^* X & \xrightarrow{\varepsilon_{k_1^* X}} & k_1^* X \\
\downarrow \widetilde{\varphi}_2 & & \downarrow \varphi \\
\coprod_{k_0} \coprod_{d_u} d_u^* k_1^* X = X & \xrightarrow{\eta_X} & u^* \coprod_u X
\end{array}$$

where φ and $\widetilde{\varphi}_2$ are cocartesian over k_0 and $\widetilde{\varphi}_1$ is cocartesian over d_u . Since d_u is monic the map $\varepsilon_{k_0^* X}$ is monic. Since \coprod_{k_0} is left adjoint to the logical functor k_0^* it preserves monos from which it follows that η_X is monic.

Now since all components of the counit η of $\coprod_u \dashv u^*$ are monic it follows that \coprod_u is faithful. Since u^* is logical it follows that $\coprod_{u/1_J}$ is an equivalence. Recall that its right adjoint is given by $\eta_{1_J}^* \circ u^* / \coprod_u 1_J$, i.e. pullback along the cocartesian arrow $\varphi_u : 1_J \rightarrow \coprod_u 1_J$.

That the counit of the adjunction $\coprod_{u/1_J} \vdash \varphi_u^*$ is an isomorphism means that for vertical α in the pullback

$$\begin{array}{ccc}
 \varphi_u^* X & \xrightarrow{\alpha^* \varphi_u} & X \\
 \varphi_u^* \alpha \downarrow & \lrcorner & \downarrow \alpha \\
 1_J & \xrightarrow[\varphi_u]{\text{cocart.}} & \coprod_u 1_J
 \end{array}$$

the top arrow $\alpha^* \varphi_u$ is cocartesian. This is sufficient for showing that cocartesian arrows are stable under pullbacks along vertical arrows, i.e. that internal sums are universal (since by the Chevalley condition cocartesian arrows are stable under pullbacks along cartesian arrows anyway).

That the unit of the adjunction $\coprod_{u/1_J} \vdash \varphi_u^*$ is an isomorphism means that

$$\begin{array}{ccc}
 X & \xrightarrow{\text{cocart.}} & \coprod_u X \\
 !X \downarrow & & \downarrow \coprod_u !X \\
 1_J & \xrightarrow[\varphi_u]{\text{cocart.}} & \coprod_u 1_J
 \end{array}$$

is a pullback. From this it follows that all diagrams of the form

$$\begin{array}{ccc}
 X & \xrightarrow{\text{cocart.}} & \coprod_u X \\
 \alpha \downarrow & & \downarrow \coprod_u \alpha \\
 Y & \xrightarrow{\text{cocart.}} & \coprod_u Y
 \end{array}$$

are pullbacks. But (from the proof of Moens' Theorem) this is known to imply disjointness of internal sums.⁴⁷ \square

Though claimed otherwise in the *Elephant* Jibladze's Theorem was not

⁴⁷One can see this more easily as follows. Since $\coprod_{u/1_J}$ is an equivalence it follows that \coprod_u reflects isomorphisms which is known (from the proof of Moens' Theorem) to entail that internal sums are disjoint provided they are universal.

proved in Moens' Thèse from 1982. Johnstone claims that Moens proved in some other way that for a fibered topos internal sums are universal and disjoint. But this is not the case because he considered fibered variants of Giraud's Theorem where internal sums are *assumed* as universal and disjoint. The only known way of showing that for a fibered topos internal sums are universal and disjoint is via Jibladze's Theorem.

However, in Jibladze's original formulation he did not prove universality and disjointness for internal sums in a fibered topos. For him it was sufficient to show that all \coprod_u are faithful because from this it follows that the adjunctions $\coprod_u \dashv u^*$ are (equivalent to ones) of the form $\Sigma_A \dashv A^*$ for some A in \mathbb{X}_1 and this is sufficient for showing that $P \simeq P_\Delta$.

84 Johnstone's Fibred Giraud Theorem

In part B of the *Elephant* Peter Johnstone formulates a fibred version of Giraud's Theorem employing the following notion of an ∞ -pretopos.

Definition

An ∞ -pretopos over a topos \mathcal{S} is a category \mathcal{E} fibred over \mathcal{S} such that

- (1) all fibres \mathcal{E}_I are pretoposes and all reindexing functors u^* are coherent
- (2) in all fibres \mathcal{E}_I the functor $\mathbf{Sub}_{\mathcal{E}_I} : (\mathcal{E}_I)^{\text{op}} \rightarrow \mathbf{Set}$ sends coequalizers in \mathcal{E}_I to equalizers in \mathbf{Set}
- (3) the fibration \mathcal{E} over \mathcal{S} is well-powered
- (4) the fibration \mathcal{E} over \mathcal{S} has stable and disjoint internal sums.

Fibred Giraud's Theorem Let \mathcal{S} be a topos and \mathcal{E} fibred over \mathcal{S} . Then t.f.a.e.

- (1) \mathcal{E} is an ∞ -pretopos over \mathcal{S} admitting a generating family
- (2) \mathcal{E} is a cocomplete locally small topos over \mathcal{S} admitting a generating family
- (3) \mathcal{E} is a fibred Grothendieck topos over \mathcal{S} for a site internal to \mathcal{S} .

Proof Idea. The equivalence of (2) and (3) is a well-known theorem of R. Diaconescu from 1975. The implication (2) to (1) is straightforward. The key idea of the implication (1) to (2) is to show that the fibred functor \mathbf{Sub} from $(\mathcal{E} \rightarrow \mathcal{S})^{\text{op}}$ to $\mathcal{S}^2 \rightarrow \mathcal{S}$ preserves small limits (for preservation of equalizers use condition (2) of the definition of ∞ -pretopos) from which it follows by the fibred special adjoint functor theorem that \mathbf{Sub} has a fibred left adjoint L whose value at 1 gives Ω in \mathcal{E} .

85 Giraud conditions without small generating family don't guarantee Ω (Johnstone CATEGORIES 14/07/2011)

Let \mathcal{E} be the category whose objects are sets X together with a family $(e_\alpha)_{\alpha \in \mathbf{On}}$ of endomaps of X and whose morphisms are equivariant maps, i.e. maps commuting with all the e_α . One easily checks that \mathcal{E} validates all the Giraud conditions but the existence of a small generating family.

For sake of contradiction suppose \mathcal{E} has a subobject classifier Ω . For infinite cardinals κ let X_κ be the object whose underlying set consists of all κ -indexed families of natural numbers with all but finitely many items equal 0 on which for $\alpha < \kappa$ the map e_α acts as the successor map on the α -th item and as identity otherwise. Obviously, the object X_κ has at least κ many distinct subobjects and, accordingly, there are at least κ many distinct morphisms from X_κ to Ω . Since the constant family $(0)_\alpha$ generates X_κ this family will be mapped to at least κ many different elements of Ω . Thus the underlying set of Ω has at least κ many different elements. Thus, for any infinite cardinal κ the underlying set of Ω has at least κ many elements which is impossible.

86 Connectedness relative to a base category

Let $P : \mathbb{X} \rightarrow \mathbb{B}$ be a fibration of categories with terminal object and (internal) sums. Then for every object I in \mathbb{B} there is a functor $\Delta_I : \mathbb{B}/I \rightarrow \mathbb{X}_I$ sending $u : J \rightarrow I$ to $\Delta_I(u) = \coprod_u 1_J$. We write $\varphi_u : 1_J \rightarrow \Delta_I(u)$ for a (chosen) cocartesian arrow over u .

An object $X \in \mathbb{X}_I$ is an I -indexed family of connected objects iff $\mathbb{X}_I(X, \Delta_I(-)) : \mathbb{B}/I \rightarrow \mathbf{Set}$ is represented by id_I , i.e. for all $u : J \rightarrow I$ and vertical $\alpha : X \rightarrow \Delta_I(u)$ there exists a unique $s : I \rightarrow J$ making the diagram

$$\begin{array}{ccc}
 X & \xrightarrow{!_X} & 1_I \\
 \alpha \downarrow & \text{vert.} & \vdots 1_s \\
 \Delta_I(u) & \xleftarrow[\varphi_u]{\text{cocart.}} & 1_J
 \end{array}$$

commute. Notice that by applying P we get $u \circ s = \text{id}_I$. Intuitively $s(i)$ is the unique $j \in J_i$ such that $\alpha_i : X_i \rightarrow \prod_{j \in J_i} 1$ factors through $\text{in}_j : 1 \rightarrow \prod_{j \in J_i} 1$.

In case I is terminal we write Δ for Δ_I . Notice that $\Delta(1)$ is terminal. Then an object $X \in \mathbb{X}_1$ is connected iff for every $\alpha : X \rightarrow \Delta(I)$ there exists a unique $i : 1 \rightarrow I$ making the diagram

$$\begin{array}{ccc}
 X & \xrightarrow{!_X} & \Delta(1) \\
 \alpha \searrow & & \vdots \Delta(i) \\
 & & \Delta(I)
 \end{array}$$

commute.

In case P is a Moens fibration, i.e. $P = F^*P_{\mathbb{C}}$ for some finite limits preserving functor $F : \mathbb{B} \rightarrow \mathbb{C}$ between categories with finite limits we can formulate things as follows. A morphism $x : X \rightarrow FI$ is an I -indexed family of connected objects iff for every $u : J \rightarrow I$ and $\alpha : x \rightarrow F(u)$ there is a unique section s of u such that $\alpha = F(s) \circ x$. If I is terminal then $X \in \mathbb{C}$ is connected iff for every $\alpha : X \rightarrow FI$ there is a unique $i : 1 \rightarrow I$ with $\alpha = F(i) \circ !_X$.

NB Unless the fibration $P : \mathbb{X} \rightarrow \mathbb{B}$ is locally connected, i.e. Δ has a fibred left adjoint Π , families of connected objects are not necessarily closed under reindexing. Thus, for fibrations which are not locally connected one better defines an $X \in \mathbb{X}_I$ to be a family of connected objects iff for every $u : J \rightarrow I$ the presheaf $\mathbb{X}_J(u^*X, \Delta_J(-))$ is isomorphic to $\mathbb{B}/J(\text{id}_J, -)$.

87 Connected Moens Fibrations

Let $P : \mathbb{X} \rightarrow \mathbb{B}$ be a *Moens fibration*, i.e. \mathbb{B} and \mathbb{X} are categories with finite limits and $P : \mathbb{X} \rightarrow \mathbb{B}$ is a fibration preserving finite limits and having stable and disjoint internal sums. The Moens fibration P is called *connected* iff for every cocartesian arrow $\psi : 1_I \rightarrow \coprod_u 1_I$ over $u : I \rightarrow K$ and every morphism $f : 1_J \rightarrow \coprod_u 1_I$ over $v : J \rightarrow K$ there exists a unique morphism $w : J \rightarrow I$ with $f = \psi \circ 1_w$, i.e.

$$\begin{array}{ccc}
 J & & 1_J \\
 \vdots & & \downarrow \\
 \vdots & & 1_w \\
 \vdots & & \downarrow \\
 I & & 1_I \\
 & & \searrow^{\psi} \\
 & & \coprod_u 1_I
 \end{array}$$

For $u : I \rightarrow K$ and $v : J \rightarrow K$ consider the pullback

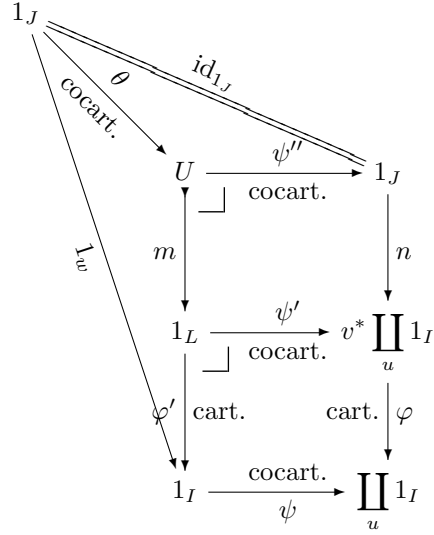
$$\begin{array}{ccc}
 L & \xrightarrow{q} & J \\
 \downarrow p & \lrcorner & \downarrow v \\
 I & \xrightarrow{u} & K
 \end{array}
 \quad (\dagger)$$

in \mathbb{B} . Let $\varphi : v^* \coprod_u 1_I \rightarrow \coprod_u 1_I$ be a cartesian arrow above v . Then arrows $f : 1_J \rightarrow \coprod_u 1_I$ above v are in 1-1-correspondence with vertical arrows $n : 1_J \rightarrow v^* \coprod_u 1_I$ (where $\varphi \circ n = f$). Considering the diagram

$$\begin{array}{ccc}
 U & \xrightarrow{\psi''} & 1_J \\
 \downarrow m & \lrcorner & \downarrow n \\
 1_L & \xrightarrow{\psi'} & v^* \coprod_u 1_I \\
 \downarrow \varphi' & \lrcorner & \downarrow \varphi \\
 1_I & \xrightarrow{\psi} & \coprod_u 1_I
 \end{array}$$

where the lower square lies above the pullback (\dagger) and m is vertical one further observes that vertical arrows $n : 1_J \rightarrow v^* \coprod_u 1_I$ (which are necessarily monic) are in 1-1-correspondence with those vertical subobjects $m : U \rightarrow 1_L$ for which $\coprod_q U \cong 1_J$. From these observations it is immediate that connectedness of P is

equivalent to the requirement that for every such m there exists a unique arrow $\theta : 1_J \rightarrow U$ with $\psi'' \circ \theta = \text{id}_{1_J}$ as illustrated in the diagram



where cocartesianness follows from $\psi'' \circ \theta = \text{id}_{1_J}$ since P is a Moens fibration and both ψ'' and id_{1_J} are cocartesian.

This reformulation of connectedness obviously establishes the connection with the following elementary characterization of connectedness for geometric morphism $F \dashv U : \mathcal{E} \rightarrow \mathbf{Set}$ requiring that for every decomposition $\coprod_{i \in I} U_i \cong 1_{\mathcal{E}}$ there exists a unique $i \in I$ with $U_i \cong 1_{\mathcal{E}}$ (and $U_{i'} = 0_{\mathcal{E}}$ for all $i' \in I$ with $i' \neq i$). In the fibrational generalization of this requirement the unique arrow $w : J \rightarrow I$ plays the role of the unique index i in the elementary characterization.

88 Families of Sheaves

Let $F \dashv U : \mathcal{F} \rightarrow \mathcal{E}$ be an injective geometric morphism. As known from Moens' work this induces a fibred geometric morphism $\Delta \dashv \Gamma : P_{\mathcal{F}} \rightarrow P_{\mathcal{E}}$ whose unit $\tilde{\eta}_a^{(I)}$ (in fibre I) at $a : A \rightarrow I$ is given by

$$\begin{array}{ccc}
 A & & \\
 \searrow^{\tilde{\eta}_a^{(I)}} & & \searrow^{\eta_A} \\
 & \cdot & \longrightarrow UFA \\
 \downarrow^a & \lrcorner & \downarrow^{UFa} \\
 & \Gamma_I \Delta_I(a) & \\
 & \downarrow & \\
 I & \longrightarrow & UFI \\
 & \eta_I &
 \end{array}$$

Since $\Delta_I \dashv \Gamma_I$ is a geometric inclusion (for all $I \in \mathcal{E}$) the family $a : A \rightarrow I$ is a family of sheaves if and only if $\tilde{\eta}_a^{(I)} : a \rightarrow \Gamma_I \Delta_I a$ is an isomorphism, i.e. iff

$$\begin{array}{ccc}
 A & \xrightarrow{\eta_A} & UFA \\
 a \downarrow \lrcorner & & \downarrow^{UFa} \\
 I & \xrightarrow{\eta_I} & UFI
 \end{array}$$

is a pullback. Thus a arises as pullback of some map in \mathcal{F} (which w.l.o.g. we consider as a full subcategory of \mathcal{E}). On the other hand if $g : Y \rightarrow X$ is a map in \mathcal{F} and $f : I \rightarrow X$ a map in \mathcal{E} then for $a = f^*g : A \rightarrow I$ with

$$\begin{array}{ccc}
 A & \xrightarrow{q} & Y \\
 a \downarrow \lrcorner & & \downarrow g \\
 I & \xrightarrow{f} & X
 \end{array}$$

it holds that

$$\begin{array}{ccccc}
 A & \xrightarrow{\eta_A} & UFA & \xrightarrow{UFq} & UFY \\
 a \downarrow \lrcorner & & \downarrow^{UFa} & & \downarrow^{UFg} \\
 I & \xrightarrow{\eta_I} & UFI & \xrightarrow{UFf} & UFX
 \end{array}$$

where the left square is a pullback because the right square is a pullback (since both F and U preserve finite limits) and the outer rectangle is a pullback since

it is equal to the rectangle

$$\begin{array}{ccccc}
 A & \xrightarrow{q} & Y & \xrightarrow[\cong]{\eta_Y} & UFY \\
 a \downarrow & \lrcorner & \downarrow g & & \downarrow UFG \\
 I & \xrightarrow{f} & X & \xrightarrow[\cong]{\eta_X} & UFX
 \end{array}$$

which is a pullback since the left square is a pullback and η_X and η_Y are isomorphisms because by assumption X and Y are in the reflective subcategory \mathcal{F} of \mathcal{E} .

Thus we have shown that $a : A \rightarrow I$ is a family of sheaves iff a arises as pullback in \mathcal{E} of some arrow g in \mathcal{F} .

We now give an alternative proof of this fact. Let $u : J \rightarrow I$ be a map in \mathcal{F} and

$$\begin{array}{ccc}
 B & \xrightarrow{b} & UJ \\
 f \downarrow & \lrcorner & \downarrow Uu \\
 A & \xrightarrow{a} & UI
 \end{array}$$

be a pullback in \mathcal{E} . Then

$$\begin{array}{ccc}
 FB & \xrightarrow{\hat{b}} & J \\
 Ff \downarrow & \lrcorner & \downarrow u \\
 FA & \xrightarrow{\hat{a}} & I
 \end{array}
 \quad \text{since} \quad
 \begin{array}{ccccc}
 FB & \xrightarrow{Fb} & FUJ & \xrightarrow[\cong]{\varepsilon_J} & J \\
 Ff \downarrow & \lrcorner & \downarrow FUu & & \downarrow u \\
 FA & \xrightarrow{Fa} & FUI & \xrightarrow[\cong]{\varepsilon_I} & I
 \end{array}$$

because F preserves pullbacks and ε is a natural isomorphism. Then in

$$\begin{array}{ccccc}
 B & \xrightarrow{\eta_B} & UFB & \xrightarrow{U\hat{b}} & UJ \\
 f \downarrow & & \downarrow Uf & & \downarrow Uu \\
 A & \xrightarrow{\eta_A} & UFA & \xrightarrow{U\hat{a}} & UI
 \end{array}$$

the right square is a pullback since U preserves pullbacks and the outer rectangle is a pullback by assumption from which it follows that

$$\begin{array}{ccc}
 B & \xrightarrow{\eta_B} & UFB \\
 f \downarrow & \lrcorner & \downarrow Uf \\
 A & \xrightarrow{\eta_A} & UFA
 \end{array}$$

is a pullback as desired.

89 Fibred Enriched Categories

In November 1976 Gouzou and Grunig have published their work on *Fibrations Relatives* as an internal report of the Seminaire B enabou. I briefly recall their main definitions.

Let \mathbb{B} be a category with finite limits (or at least pullbacks). A *fibration of monoidal categories over \mathbb{B}* (discussed in Chapter I) is a fibration $V : \mathbb{V} \rightarrow \mathbb{B}$ together with cartesian functors $\otimes : V \times_{\mathbb{B}} V \rightarrow V$ and $I : \text{Id}_{\mathbb{B}} \rightarrow V$ endowing P with the structure of a monoid in $\mathbf{Fib}(\mathbb{B})$ (up to isomorphism). One easily sees that this is equivalent to all fibres being monoidal categories and this structure being preserved by reindexing.

Gouzou and Grunig defined a *V-enriched category over \mathbb{B}* as given by

- (1) an object $C_0 \in \mathbb{B}$
- (2) an objects $C_1 \in V(C_0 \times C_0)$
- (3) a vertical morphism $\eta : I_{C_0} \rightarrow \delta_{C_0}^* C_1$
- (4) a vertical morphism $\mu : \pi_3^* C_1 \otimes \pi_1^* C_1 \rightarrow \pi_2^* C_1$

(where δ_{C_0} is the diagonal on C_0 and π_i projects C_0^3 on those factors whose index is different from i) satisfying some obvious conditions expressing that η gives identity morphisms and μ is composition.

This notion (discussed in Chapter II) just captures **small** V -enriched categories and the associated ordinary fibration over \mathbb{B} is necessarily split and small. To overcome these limitations in Chapter II they define a *V-enriched fibration over \mathbb{B}* as given by the following data

- (1) a $V(I)$ -enriched category \mathbb{C}_I for all $I \in \mathbb{B}$
- (2) a mapping $u^* : \text{Ob}(\mathbb{C}_I) \rightarrow \text{Ob}(\mathbb{C}_J)$ for all $u : J \rightarrow I$ in \mathbb{B}
- (3) an isomorphism $u^* X, Y : u^* \mathbb{C}_I(X, Y) \xrightarrow{\cong} \mathbb{C}_J(X, Y)$ for all $u : J \rightarrow I$ in \mathbb{B} and $X, Y \in \text{Ob}(\mathbb{C}_I)$
- (4) for all $u : J \rightarrow I$ and $v : K \rightarrow J$ in \mathbb{B} and $X \in \text{Ob}(\mathbb{C}_I)$ an isomorphism $\alpha_{u,v,X} : v^* u^* X \xrightarrow{\cong} (uv)^* X$ in the underlying category of \mathbb{C}_K
- (5) for all I in \mathbb{B} and $X \in \text{Ob}(\mathbb{C}_I)$ an isomorphism $\alpha_{I,X} : \text{id}_I^* X \xrightarrow{\cong} X$ in the underlying category of \mathbb{C}_I

satisfying some “obvious” coherence conditions.

This latter definition does not look very “fibrational”. It is rather in the style of indexed categories. But presumably one can’t do better.

Recently (see CATEGORIES 6. Sept. 07) M. Shulman and M. Przybyłek (writing a Master’s Thesis supervised by A. Tarlecki) announced to have found a concept unifying internal and enriched categories. Przybyłek’s notion coincides with Gouzou and Grunig’s notion of V -enriched category over \mathbb{B} and Shulman’s notion is slightly more general⁴⁸ and presented in a way technically very different from Gouzou and Grunig’s account. Both Shulman and Przybyłek observe that one obtains internal categories in \mathbb{B} when instantiating V with the fundamental fibration $P_{\mathbb{B}} = \partial_1 : \mathbb{B}^2 \rightarrow \mathbb{B}$ and one obtains \mathcal{V} -enriched categories when instantiating V with the family fibration $\text{Fam}(\mathcal{V})$ over \mathbf{Set} . But that does not go beyond Gouzou and Grunig’s work.

Possibly a notion of split but not small fibration of enriched categories can be obtained by extending the Australian school’s work on “bicategory enriched categories” to fibrations.⁴⁹ As I understand it a fibred enriched category is necessarily locally small in the sense that for $(Y_i)_{i \in I}$ and $(X_i)_{i \in I}$ there is an I -indexed family $(\text{hom}(Y_j, X_i))_{i \in I, j \in J}$ of objects in the enriching category \mathcal{V} . Penon’s “locally internal categories” (see Appendix of Johnstone’s 1977 book “Topos Theory”) do this for the case $V = P_{\mathbb{B}}$. One might try to generalize his approach to arbitrary fibrations V of monoidal categories and this way obtain a notion of fibrations of enriched categories which is not necessarily split or even small. But I think this way one just again arrives at Gouzou and Grunig’s notion of V -enriched fibration over \mathbb{B} from 1976.

See M. Shulman’s recent paper *Enriched Indexed Categories* (Dec. 2012) section 6 on \mathcal{V} -fibrations for a tentative definition based on the more general notion of “large \mathcal{V} -category”.

⁴⁸In Shulman’s paper *Framed Bicategories and Monoidal Fibrations* (arXiv:0706.1286v1) is based on the concept of a “monoidal fibration”, i.e. a fibration $P : \mathbb{X} \rightarrow \mathbb{B}$ where \mathbb{X} and \mathbb{B} are monoidal categories, P is a monoidal functor and cartesian arrows are closed under \otimes . If the monoidal structure on \mathbb{B} is cartesian, i.e. \otimes is \times , then Shulman’s notion is equivalent to Gouzou and Grunig’s notion of a fibration of monoidal categories.

⁴⁹Actually, in Betti, Carboni, Street and Walters *Variation through Enrichment* they observed that every fibration $P : \mathbb{X} \rightarrow \mathbb{B}$ may be understood as enriched in $\mathcal{W}(\mathbb{B})$, the full subcategory of $\mathbf{Span}(\widehat{\mathbb{B}})$ on representable presheaves. The key point is that for $X \in P(I)$ and $Y \in P(J)$ there is a presheaf $\text{HOM}(X, Y)$ over $y(I) \times y(J)$ which is representable by an object $\text{hom}(X, Y)$ in \mathbb{B} for all $X, Y \in \mathbb{X}$ if and only if the fibration P is locally small.

90 Shulman's Enriched Indexed Categories (2012)

Let \mathbf{S} be a category with finite products and \mathcal{V} a symmetric monoidal category indexed over \mathbf{S} . A large \mathcal{V} -category \mathcal{A} consists of

- i) A collection $\text{Ob}(\mathcal{A})$ of objects x, y, z, \dots
- ii) For each object x , an object $\epsilon x \in \mathbf{S}$, called its *extent*.
- iii) For each objects $x, y \in \text{Ob}(\mathcal{A})$, an object $\mathcal{V}^{\epsilon y \times \epsilon x}$.
- iv) For each $x \in \text{Ob}(\mathcal{A})$ a morphism

$$\begin{array}{ccc} \mathbb{I}_{\epsilon x} & \xrightarrow{\text{id}} & \underline{\mathcal{A}}(x, x) \\ \vdots & & \vdots \\ \epsilon x & \xrightarrow{\delta} & \epsilon x \times \epsilon x \end{array}$$

in $\int \mathcal{V}$.

- v) For each $x, y, z \in \text{Ob}(\mathcal{A})$ a morphism

$$\begin{array}{ccc} \underline{\mathcal{A}}(y, z) \otimes_{\epsilon y} \underline{\mathcal{A}}(x, y) & \xrightarrow{\text{comp}} & \underline{\mathcal{A}}(x, z) \\ \vdots & & \vdots \\ \epsilon z \times \epsilon y \times \epsilon x & \xrightarrow{\pi_{\epsilon y}} & \epsilon x \times \epsilon x \end{array}$$

in $\int \mathcal{V}$.⁵⁰

such that the obvious associative and unit laws hold.

For X in \mathbf{S} a \mathcal{V} -functor $f : \delta X \rightarrow \mathcal{A}$ is given by an object $a \in \text{Ob}(\mathcal{A})$ together with a morphism $\epsilon f_a : X \rightarrow \epsilon a$. If $g : \delta X \rightarrow \mathcal{A}$ is a \mathcal{V} -functor given by b and $\epsilon g_b : X \rightarrow \epsilon b$ then a \mathcal{V} -natural transformation $\alpha : f \rightarrow g$ is given by a morphism $\alpha : \mathbb{I}_X \rightarrow \underline{\mathcal{A}}(a, b)$ in $\int \mathcal{V}$ above $\langle \epsilon g_b, \epsilon f_a \rangle$.

If x is an object of \mathcal{A} we write x also for the \mathcal{V} -functor $\delta(\epsilon x) \rightarrow \mathcal{A}$ as given by x and $\text{id}_{\epsilon x}$. For $x \in \text{Ob}(\mathcal{A})$ and $f : Y \rightarrow \epsilon x$ in \mathbf{S} a *restriction of x along f* is an object f^*x of \mathcal{A} such that $\epsilon(f^*x) = Y$ together with a \mathcal{V} -natural isomorphism between

$$\delta Y \xrightarrow{f^*x} \mathcal{A} \quad \text{and} \quad \delta Y \xrightarrow{\delta f} \delta X \xrightarrow{x} \mathcal{A}$$

where the second \mathcal{V} -functor is given by x and f .

A **\mathcal{V} -fibration** is a large \mathcal{V} -category such that for each object x and each $f : Y \rightarrow \epsilon x$ in \mathbf{S} there exists a restriction f^*x .

⁵⁰in the index of π we list those components which are projected away

91 A left adjoint to $U : \mathbf{Sp}(\mathbb{B}) \rightarrow \mathbf{Fib}(\mathbb{B})$

Let $Y_{\mathbb{B}} : \mathbb{B} \rightarrow \widehat{\mathbb{B}}$ be the Yoneda functor. According to work by Bénabou the fibred Yoneda lemma can be reformulated as $U : \mathbf{Sp}(\mathbb{B}) \rightarrow \mathbf{Fib}(\mathbb{B})$ having a right adjoint $Sp = Y_{\mathbb{B}}^* \prod_{Y_{\mathbb{B}}}$. M. Abbott and M. Warren independently observed that a left adjoint to $U : \mathbf{Sp}(\mathbb{B}) \rightarrow \mathbf{Fib}(\mathbb{B})$ is given by $Y_{\mathbb{B}}^* \coprod_{Y_{\mathbb{B}}}$. Both constructions, however, can be found already in I.2.4 of J. Giraud's 1971 book *Cohomologie non abélienne*.

Explicitating the construction of $\coprod_F \dashv F^*$ as given by Bénabou (*cf.* my lecture notes on Fibred Categories) one sees that this “left adjoint splitting” looks as follows. The fibre over I is the category whose objects are pairs (u, X) with $u : I \rightarrow P(X)$ and morphisms from (u, X) to (v, Y) are spans $(\varphi : Z \rightarrow X, f : Z \rightarrow Y)$ such that $P(\varphi) = u$, $P(f) = v$ and φ is cartesian. Of course, morphisms (φ, f) and (φ', f') have to be considered as equal iff there is a (uniquely determined) vertical iso ι (over I) with $\varphi\iota = \varphi'$ and $f\iota = f'$. For $(\varphi, f) : (u, X) \rightarrow (v, Y)$ and $(\psi, g) : (v, Y) \rightarrow (w, Z)$ their composition in the fibre over I is given by $(\varphi, g\alpha)$ where α is the unique vertical arrow with $\psi\alpha = f$. Reindexing along $u : J \rightarrow I$ is given by precomposition with appropriate cartesian arrows over u . The ensuing split fibration over \mathbb{B} we call $L(P)$.

There is a cartesian functor $H_P : P \rightarrow U(L(P))$ sending X over I to (id_I, X) which serves as unit of the adjunction $L \dashv U$. For a cartesian functor $F : P \rightarrow U(S)$ there is a unique (up to isomorphism) functor $\widehat{F} : L(P) \rightarrow S$ with $U(\widehat{F})H_P = F$ which is constructed as follows using a (normalized) cleavage Cart_P of P . An object (u, X) is sent to $S(u)(F(X))$ and a morphism $(\varphi, f) : (u, X) \rightarrow (v, Y)$ in the fibre over I is sent to the unique vertical arrow $\alpha : u^*F(X) \rightarrow v^*F(Y)$ such that $\text{Cart}_P(v, F(f)) \circ \alpha \circ \iota = F(f)$ where ι is the unique vertical arrow (necessarily an iso!) with $\text{Cart}_P(u, F(X)) \circ \iota = F(\varphi)$.

As emphasized by P. Lumsdaine and M. Warren the “left adjoint splitting” $L(P)$ of a fibrator $P : \mathbb{X} \rightarrow \mathbb{B}$ is isomorphic to the following split fibration over \mathbb{B} described in terms of a (normalized) cleavage Cart_P for P . The objects of the total category are again pairs $(u : I \rightarrow I_0, X)$ with $P(X) = I_0$. But morphisms from $(u : I \rightarrow I_0, X)$ to $(v : J \rightarrow J_0, Y)$ are just morphism $f : u^*X \rightarrow v^*Y$ and their composition is inherited from \mathbb{X} . The functor constituting the fibration sends $(u : I \rightarrow I_0, X)$ to I and $f : (u : I \rightarrow I_0, X) \rightarrow (v : J \rightarrow J_0, Y)$ to $P(f)$. The ensuing fibration can be endowed with the following canonical splitting: for $(u : I \rightarrow I_0, X)$ and $v : J \rightarrow I$ we put $v^*(u : I \rightarrow I_0, X) = (uv, X)$ and let $\text{Cart}(v, (u : I \rightarrow I_0, X)) : (uv)^*X \rightarrow u^*X$ be the unique arrow ψ over v with

$$\text{Cart}_P(u, X) \circ \psi = \text{Cart}_P(uv, X)$$

This new split fibration is isomorphic to $L(P)$ via the split cartesian functor which is the identity on objects and sends a morphism $f : (u, X) \rightarrow (v, Y)$ to the equivalence class of $(\text{Cart}_P(u, X), \text{Cart}_P(v, Y) \circ f)$.

Though this alternative construction of the left adjoint splitting of P relies heavily on global choice, i.e. the axiom of choice for classes, it is in a sense

“more explicit”. Moreover, as shown by Lumsdaine and Warren (2014) this more explicit splitting construction can be used for splitting identity types in non-split models of identity types as arising from e.g. Cisinski model structures.

92 Bénabou’s $\mathbf{Set}(\mathbb{B})$ and $\mathbf{set}(\mathbb{B})$

For a category \mathbb{B} we may consider the split fibration $\mathbf{Set}(\mathbb{B})$ over \mathbb{B} with

$$\mathbf{Set}(\mathbb{B})(I) = \widehat{\mathbb{B}/I} = \mathbf{Set}^{(\mathbb{B}/I)^{\text{op}}} \quad \mathbf{Set}(\mathbb{B})(J \xrightarrow{u} I) = (-) \circ \Sigma_u^{\text{op}}$$

where $\Sigma_u : \mathbb{B}/J \rightarrow \mathbb{B}/I$ is given by postcomposition with u . Apparently, the split fibration $\mathbf{Set}(\mathbb{B})$ is equivalent to the non-split fibration $\mathbf{Y}_{\mathbb{B}}^* P_{\mathbb{B}}$ (where $P_{\mathbb{B}}$ is the fundamental fibration of $\widehat{\mathbb{B}}$).

Now in case \mathbb{B} has pullbacks we may consider the full subfibration $\mathbf{set}(\mathbb{B})$ of $\mathbf{Set}(\mathbb{B})$ where $\mathbf{set}(\mathbb{B})(I)$ is the full subcategory of $\widehat{\mathbb{B}/I}$ on representable presheaves over \mathbb{B}/I .⁵¹ Obviously, the split fibration $\mathbf{set}(\mathbb{B})$ is equivalent to the fundamental fibration $P_{\mathbb{B}}$ of \mathbb{B} .

For $X \in \widehat{\mathbb{B}}$ split cartesian functors from X to $\mathbf{Set}(\mathbb{B})$ are in natural 1-1-correspondence with presheaves over the category $\mathbf{Eelts}(X) = \mathbf{Y}_{\mathbb{B}} \downarrow X$ of elements of X and those which factor through $\mathbf{set}(\mathbb{B})$ correspond to the representable morphisms to X .⁵²

93 Representable Morphisms

Let \mathbb{B} be a category with pullbacks. A morphism $f : Y \rightarrow X$ in $\widehat{\mathbb{B}}$ is called *representable* iff for every $x : y(I) \rightarrow X$ there is a pullback

$$\begin{array}{ccc} y(J) & \longrightarrow & Y \\ y(u) \downarrow & \lrcorner & \downarrow f \\ y(I) & \xrightarrow{x} & X \end{array}$$

with $u : J \rightarrow I$ in \mathbb{B} . Since y preserves pullbacks and \mathbb{B} has pullbacks a morphism $f : X \rightarrow y(I)$ is representable iff it is isomorphic in $\widehat{\mathbb{B}}/y(I)$ to $y(u)$ for some $u : J \rightarrow I$ in \mathbb{B} . Via the equivalence $\widehat{\mathbb{B}}/y(I) \simeq \widehat{\mathbb{B}/I}$ the representable morphism $y(u)$ corresponds to the presheaf $A_u : (\mathbb{B}/I)^{\text{op}} \rightarrow \mathbf{Set}$ with $A_u(v : K \rightarrow I) = \{s : K \rightarrow J \mid us = v\}$ and $A_u(w : vw \rightarrow v)(s) = sw$.

⁵¹Notice that $\mathbf{Set}(\mathbb{B})(u) : \widehat{\mathbb{B}/I} \rightarrow \widehat{\mathbb{B}/J}$ preserves representability of presheaves if and only if \mathbb{B} has pullbacks along $u : J \rightarrow I$. Thus $\mathbf{set}(\mathbb{B})$ is a full subfibration of $\mathbf{Set}(\mathbb{B})$ if and only if \mathbb{B} has all pullbacks.

⁵²The first case was considered already by J. Giraud in his book *Cohomologie non abélienne* from 1971 where he considered the following splitting of $P_{\mathbb{B}}$. The fibre over $X \in \widehat{\mathbb{B}}$ is $\mathbf{Set}^{\mathbf{Eelts}(X)^{\text{op}}}$, the category of presheaves over $\mathbf{Eelts}(X)$, and for $u : Y \rightarrow X$ in $\widehat{\mathbb{B}}$ reindexing along u is given by $\mathbf{Set}^{\mathbf{Eelts}(u)^{\text{op}}}$ where $\mathbf{Eelts}(u) = \mathbf{Y}_{\mathbb{B}} \downarrow u$, i.e. postcomposition with u .

Thus in case $\mathbb{B} = \mathbf{Set}$ a family $(A_i)_{i \in I}$ (thought of as a map $A \rightarrow I$) gives rise to the presheaf $A : (\mathbf{Set}/I)^{\text{op}} \rightarrow \mathbf{Set}$ sending $u : J \rightarrow I$ to $A(u) = \prod_{j \in J} A_{u(j)}$ and $v : uv \rightarrow u$ to $A(v) : s \mapsto s \circ v$.

This explains why a(n ordinary) distributor $\phi : \mathbb{A} \times \mathbb{B}^{\text{op}} \rightarrow \mathbf{Set}$ gives rise to the split cartesian functor $Fam(\phi) : Fam(\mathbb{A}) \times Fam(\mathbb{B})^{\text{op}} \rightarrow \mathbf{set}(\mathbf{Set})$ given by

$$Fam(\phi)_I(A, B)(u : J \rightarrow I) = \prod_{j \in J} \phi(A_{u(j)}, B_{u(j)})$$

and

$$Fam(\phi)_I(f, g)(u : J \rightarrow I)(s) = (\phi(f_{u(j)}, g_{u(j)})(s_j))_{j \in J}$$

However, I have constructed a counterexample showing that in general $Fam(\psi\phi)$ and $Fam(\psi)Fam(\phi)$ are different, i.e. not isomorphic.

94 The split fibration \mathcal{S}^X induced by $X \in \mathcal{S}^{\mathbb{B}^{\text{op}}}$

Let \mathbb{B} be a small category and X an object of $\mathcal{S}^{\mathbb{B}^{\text{op}}}$, the category of presheaves over \mathbb{B} . Let $\int X : \mathbb{X} \rightarrow \mathbb{B}$ be the discrete fibration obtained from X by the Grothendieck construction. Then we may consider the split fibration \mathcal{S}^X over \mathbb{B} whose fibre over $I \in \mathbb{B}$ is given by $\mathcal{S}^{(\mathbb{B}/I) \times \mathbb{X}}^{\text{op}}$ and whose reindexing along $u : J \rightarrow I$ in \mathbb{B} is given by precomposition with the functor $(\Sigma_u \times \mathbb{X})^{\text{op}}$, i.e. $\mathcal{S}^X(u)(A) = A \circ (\Sigma_u \times \mathbb{X})^{\text{op}}$.⁵³

This construction is mentioned in R. Street's paper *The petit topos of globular sets*. But it occurs also implicitly and in a more general form in Bénabou and Streicher's draft *Distributors between Fibrations*, namely as $\mathbf{Set}(\mathbb{B})^X$ where $\mathbf{Set}(\mathbb{B})$ is the split fibration over \mathbb{B} with $\mathbf{Set}(\mathbb{B})(I) = \mathbf{Set}^{(\mathbb{B}/I)^{\text{op}}}$ and $\mathbf{Set}(\mathbb{B})(u) = (-) \circ (\Sigma_u)^{\text{op}}$ (notice that $\mathbf{Set}(\mathbb{B})$ is isomorphic to \mathcal{S}^1 where 1 is terminal in $\mathcal{S}^{\mathbb{B}^{\text{op}}}$).

⁵³One may understand $\mathcal{S}^X(I)$ as the category of " I -indexed families of objects in $\mathcal{S}^{\mathbb{B}^{\text{op}}}/X$ ".

95 Families of Separated Objects

Let \mathbb{B} be a category with finite limits and $\mathbf{Rep}(\mathbb{B})$ be the class of representable morphisms in $\widehat{\mathbb{B}} = \mathbf{Set}^{\mathbb{B}^{\text{op}}}$. The class $\mathbf{Rep}(\mathbb{B})$ contains all isos, is closed under composition and pullbacks along arbitrary morphisms and from f and fg representable it follows that g is representable, too. This latter condition is equivalent (in presence of the other closure conditions) to δ_f being representable whenever f is representable.

A map f in $\widehat{\mathbb{B}}$ is (a family of) *separated* (objects) iff δ_f is representable.⁵⁴ One can show that

if $f : Y \rightarrow X$ and $g : Z \rightarrow Y$ are separated then fg is separated, too.

This follows from the following diagram

$$\begin{array}{ccccc}
 Z & & & & \\
 \delta_g \searrow & & \delta_{fg} \searrow & & \\
 & Z \times_Y Z & \longrightarrow & Z \times_X Z & \\
 \downarrow & \lrcorner & & \downarrow g \times_X g & \\
 & & & & \\
 & Y & \xrightarrow{\delta_f} & Y \times_X Y & \\
 & & & &
 \end{array}$$

because δ_g is representable (since g is separated), the pullback of the representable map δ_f along $g \times_X g$ is representable and δ_{fg} is the composition of these two representable maps.

⁵⁴As observed by Bénabou a presheaf $X \in \widehat{\mathbb{B}}$ is separated, i.e. $\delta_X = \langle \text{id}_X, \text{id}_X \rangle$ is a representable morphism, if and only if for X considered as a discrete fibration over \mathbb{B} equality of objects is definable, i.e. for $x, y \in X(I)$ there is a mono $m : I_0 \rightarrow I$ such that $u : J \rightarrow I$ factors through m iff $u^*x = u^*y$.

96 Anafunctors : ordinary, fibred and internal

Let \mathcal{C} and \mathcal{D} be categories. An *anafunctor* from \mathcal{C} to \mathcal{D} is a pair (P_F, F) where

(A1) $P_F : \mathcal{S}_F \rightarrow \mathcal{C}$ is an elementary fibration (i.e. all fibres of P_F are posetal groupoids) with all fibres of P_F non-empty and connected and

(A2) $F : \mathcal{S}_F \rightarrow \mathcal{D}$ is a(n ordinary) functor.

Obviously (A1) is equivalent to the requirement that P_F is full, faithful and surjective on objects.⁵⁵

If (P_F, F) and (P_G, G) are anafunctors from \mathcal{C} to \mathcal{D} and from \mathcal{D} to \mathcal{E} , respectively, then their composition is given by $(P_F \circ F^*P_G, G \circ P_G^*F)$ where

$$\begin{array}{ccccc}
 \mathcal{S}_H & \xrightarrow{P_G^*F} & \mathcal{S}_G & \xrightarrow{G} & \mathcal{E} \\
 F^*P_G \downarrow & \lrcorner & \downarrow P_G & & \\
 \mathcal{S}_F & \xrightarrow{F} & \mathcal{D} & & \\
 P_F \downarrow & & & & \\
 \mathcal{C} & & & &
 \end{array}$$

and one easily observes that $P_F \circ F^*P_G$ is an elementary fibration with non-empty connected fibres. The identity anafunctor from \mathcal{C} to \mathcal{C} is given by $(\text{Id}_{\mathcal{C}}, \text{Id}_{\mathcal{C}})$.

The notion of anafunctor can be easily extended to fibrations as follows. Let $C : \mathcal{C} \rightarrow \mathcal{B}$ and $D : \mathcal{D} \rightarrow \mathcal{B}$ be fibrations over \mathcal{B} . Then a *cartesian anafunctor* from C to D (over \mathcal{B}) is a pair (P_F, F) where $P_F : \mathcal{S}_F \rightarrow \mathcal{C}$ is an elementary fibration with non-empty connected fibres and F is a cartesian functor from $C \circ P_F$ to D (over \mathcal{B}).

Now if C and D are categories internal to \mathcal{B} (with pullbacks) then an *internal anafunctor* from C to D is a cartesian anafunctor (P_F, F) from \underline{C} to \underline{D} such that $\mathcal{S}_F \xrightarrow{P_F} \mathcal{B}(C) \xrightarrow{C} \mathcal{B}$ is a small fibration. (Here $\underline{C} : \mathcal{B}(C) \rightarrow \mathcal{B}$ denotes the “externalisation” of C as a fibration over \mathcal{B}).

This latter definition may be reformulated within the internal language of \mathcal{B} as follows (where one has to assume that \mathcal{B} is regular). An *internal anafunctor* from C to D is a pair of internal functors $P_F : \mathcal{S}_F \rightarrow C$ and $F : \mathcal{S}_F \rightarrow D$ such that P_F is full and faithful and surjective on objects (it is for expressing fullness and surjectivity on objects that we need \mathcal{B} to be regular).

Anafunctors as Representable Distributors

In January 2011 Bénabou observed on the CATEGORIES list that anafunctors from \mathcal{A} to \mathcal{B} correspond to *representable* distributors from \mathcal{A} to \mathcal{B} , i.e.

⁵⁵It is *not* sufficient to require essential surjectivity!

distributors $\phi : \mathcal{A} \dashrightarrow \mathcal{B}$ such that for all $A \in \mathcal{A}$ the presheaf $\phi(A)$ over \mathcal{B} is representable.

Given such a representable distributor ϕ let \mathcal{S}_ϕ be the category whose objects are $f \in \phi(B, A)$ which represent the presheaf $\phi(A) = \phi(-, A)$ over \mathcal{B} and whose morphisms from f to f' are pairs (β, α) with $\phi(B, \alpha)(f) = \phi(\beta, A')(f')$ for which we write

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ \beta \downarrow & & \downarrow \alpha \\ B' & \xrightarrow{f'} & A' \end{array}$$

Let $P_\phi : \mathcal{S}_\phi \rightarrow \mathcal{A}$ and $F_\phi : \mathcal{S}_\phi \rightarrow \mathcal{B}$ be the obvious projection functors. Then (P_ϕ, F_ϕ) is the anafunctor from \mathcal{A} to \mathcal{B} corresponding to ϕ . Conversely, given an anafunctor (P, F) from \mathcal{A} to \mathcal{B} the associated representable fibration is given by $\phi_F * \phi^P : \mathcal{A} \dashrightarrow \mathcal{B}$.

It is known that distributors correspond to arbitrary functors $P : \mathcal{C} \rightarrow \mathbf{2}$ (so-called “barrels”). This correspondence restricts to one between representable distributors and fibrations over $\mathbf{2}$. Given a fibration $P : \mathcal{C} \rightarrow \mathbf{2}$ consider the category \mathcal{S} whose objects are cartesian arrows over $0 \rightarrow 1$ in $\mathbf{2}$ and whose morphism from φ to φ' are squares

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & A \\ \beta \downarrow & & \downarrow \alpha \\ B' & \xrightarrow{\varphi'} & A' \end{array}$$

with α and β vertical. Let $P_F : \mathcal{S} \rightarrow \mathcal{A}$ and $F : \mathcal{S} \rightarrow \mathcal{B}$ be the obvious projection functors where \mathcal{A} and \mathcal{B} are the fibres of P over 1 and 0, respectively. Then (P_F, F) is the anafunctor from \mathcal{A} to \mathcal{B} associated with P . Conversely, given an anafunctor from \mathcal{A} to \mathcal{B} one first constructs the associated distributor and from this the corresponding fibration over $\mathbf{2}$.

Using the axiom of choice (for classes) for every anafunctor (P_F, F) from \mathcal{A} to \mathcal{B} there is an equivalent anafunctor $(\text{Id}_{\mathcal{A}}, F \circ E)$ where $E : \mathcal{A} \rightarrow \mathcal{S}_F$ is an equivalence with $P_F \circ E = \text{Id}_{\mathcal{A}}$. But notice that there may be anafunctors from \mathcal{A} to \mathcal{B} which are not isomorphic to any anafunctor induced by a representable distributor but every anafunctor is equivalent to an anafunctor induced by a representable distributor.

97 Spreads at 1 need not be spreads

Let \mathbb{M} be the monoid with underlying set $\{1, a, b\}$ where 1 is the neutral element and $xa = a$ and $xb = b$ for all $x \in \{1, a, b\}$. We write \mathbb{M}_0 for the submonoid of \mathbb{M} on $\{1, a\}$. Let $\mathcal{E} = \widehat{\mathbb{M}}$, $\mathcal{F} = \widehat{\mathbb{M}_0}$ and $\psi : \mathcal{F} \rightarrow \mathcal{E}$ the geometric morphism where $\psi^* : \mathcal{E} \rightarrow \mathcal{F}$ restricts right \mathbb{M} -actions to right \mathbb{M}_0 -actions.

Evidently, the functor ψ^* is surjective on objects and, therefore, the geometric morphism ψ is not only localic but also a spread at 1 (because identities are **Set**-definable subobjects).

Nevertheless $\psi : \mathcal{F} \rightarrow \mathcal{E}$ is **not** a spread over **Set** which can be seen as follows. Let A be the representable object of \mathcal{E} , i.e. \mathbb{M} acted on by itself from the right. Let $P \subseteq \psi^*A$ be the subobject with $P = \{1, a\}$. For showing that ψ is not a (**Set**-)spread suppose that for some morphism $u : B \rightarrow A$ and **Set**-definable, i.e. decidable, subobject $Q \subseteq \psi^*B$ the restriction of ψ^*u to Q factors through P via an epimorphism $e : Q \rightarrow P$. Then there exists an $x \in Q$ with $u(x) = e(x) = a$. As $x \cdot b \cdot a = x \cdot a$ and $Q \subseteq \psi^*B$ is decidable it follows that $y = x \cdot b \in Q$. But then $P \ni e(y) = u(x \cdot b) = u(x)b = ab = b$ in contradiction with the definition of P .

This counterexample shows that families generating at 1 need not be generating⁵⁶: take the pullback $g : G \rightarrow \psi^*\psi_*2$ of $true : 1 \rightarrow 2$ along $\varepsilon_2 : \psi^*\psi_*2 \rightarrow 2$ (the counit of $\psi^* \dashv \psi_*$ at 2) then g is generating at 1 but not generating for the fibration $\partial_1 : \mathcal{F} \downarrow \psi^* \rightarrow \mathcal{E}$ as we have seen above. But this “failure” shouldn’t be considered as such as universally quantified statements need not hold even if they hold for all instances by global elements!

The reason for Marta and Jonathon’s confusion (who claimed that “generating at 1” entails already “generating”) was a remark in the article by Paré and Schumacher (pp.98-99) claiming that an object $G \in P(I)$ is generating for $P : \mathbb{X} \rightarrow \mathbb{B}$ if and only if for all $J \in \mathbb{B}$ the object $\pi_2^*G \in P(J \times I)$ (where $\pi_2 : J \times I \rightarrow I$ is second projection) is generating at 1 for the fibration $P_{/J} = \Sigma_J^*P$. That this claim holds can be seen from the following diagram

$$\begin{array}{ccccc}
 X & \xleftarrow{\psi} & u^*G & \xrightarrow{\text{cart.}} & \pi_2^*G & \xrightarrow{\text{cart.}} & G \\
 & & & & & & \\
 J & \xleftarrow{v} & K & \xrightarrow{\langle v, u \rangle} & J \times I & \xrightarrow{\pi_2} & I \\
 & \searrow & \downarrow v & \swarrow \pi_1 & & & \\
 & & J & & & &
 \end{array}$$

noticing that $u = \pi_2 \circ \langle v, u \rangle$.

⁵⁶For a fibration $P : \mathbb{X} \rightarrow \mathbb{B}$ an object $G \in P(I)$ is *generating* iff for all $\alpha, \beta : X \rightarrow Y$ in $P(J)$ with $\alpha \neq \beta$ there exist a cartesian arrow $\varphi : Z \rightarrow G$ and an arrow $\psi : Z \rightarrow X$ such that $\alpha\psi \neq \beta\psi$. If \mathbb{B} has a terminal object 1 then $G \in P(I)$ is *generating at 1* iff for all $\alpha, \beta : X \rightarrow Y$ in $P(1)$ with $\alpha \neq \beta$ there exist a cartesian arrow $\varphi : Z \rightarrow G$ and an arrow $\psi : Z \rightarrow X$ such that $\alpha\psi \neq \beta\psi$,

98 A Fibrational Account of Measures on Toposes

Suppose \mathbb{B} and \mathbb{C} are categories with finite limits and $F : \mathbb{B} \rightarrow \mathbb{C}$. Since $P_{\mathbb{C}}$ is a fibration of categories with finite limits and change of base (along F) preserves this property $F^*P_{\mathbb{C}}$ is also a fibration of categories with finite limits.

Already back in 1974 Jean Bénabou proved that $P_F = F^*P_{\mathbb{C}} = \partial_1 : \mathbb{C} \downarrow F \rightarrow \mathbb{B}$ has (Lawvere) comprehension iff F has a right adjoint. Since cofibrations are stable under change of base (along F) and $P_{\mathbb{C}}$ is a bifibration it follows that $F^*P_{\mathbb{C}}$ is also a bifibration (where cocartesian arrows are stable under pullbacks along vertical arrows). Also already in 1974 Bénabou showed that the bifibration $F^*P_{\mathbb{C}}$ has internal sums, i.e. satisfies the (Beck-)Chevalley condition (i.e. cocartesian arrows are stable under pullbacks along cartesian arrows) iff F preserves pullbacks.

For functors $F : \mathbb{B} \rightarrow \mathbb{C}$ the bifibration $F^*P_{\mathbb{C}}$ satisfies the following *extensivity* property (c.f. section 82): for every commuting square

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \alpha \downarrow & & \downarrow \beta \\ 1_I & \xrightarrow{\varphi_I} & \coprod_I 1_I \end{array}$$

with φ_I cocartesian over $I \rightarrow 1$ and α and β vertical the square is a pullback iff φ is cocartesian. This extensivity property amounts to claiming that the fibre \mathbb{X}_I over I is isomorphic to $\mathbb{X}_1 \downarrow \Delta(I)$ where $\Delta : \mathbb{B} \rightarrow \mathbb{X}_1$ sends $u : J \rightarrow I$ in \mathbb{B} to the unique vertical arrow $\Delta(u)$ making the diagram

$$\begin{array}{ccc} 1_J & \xrightarrow{\varphi_J} & \Delta(J) \\ 1_u \downarrow & & \downarrow \Delta(u) \\ 1_I & \xrightarrow{\varphi_I} & \Delta(I) \end{array}$$

commute. Thus fibrations $P : \mathbb{X} \rightarrow \mathbb{B}$ of finite limit categories which are also bifibrations satisfying the above extensivity property are equivalent to fibrations of the form $F^*P_{\mathbb{C}}$ for some functor $F : \mathbb{B} \rightarrow \mathbb{C}$ where \mathbb{C} has finite limits (take for \mathbb{C} the fibre over 1 and for F the functor Δ as described above).

Thus functors $F : \mathbb{B} \rightarrow \mathbb{C}$ having a right adjoint and preserving terminal objects are in 1-1-correspondence with fibrations $P : \mathbb{X} \rightarrow \mathbb{B}$ of finite limits categories having (Lawvere) comprehension and which are also bifibrations satisfying the above extensivity property. It is a consequence that cocartesian arrows are stable under pullbacks along vertical arrows.

If \mathbb{B} and \mathbb{C} are toposes (\mathbb{C} loc. cart. closed suffices!) then $F^*P_{\mathbb{C}}$ having Lawvere comprehension is equivalent to $F^*P_{\mathbb{C}}$ being locally small (since $P_{\mathbb{C}}$ is locally small iff \mathbb{C} is locally cartesian closed).

Since functors $F : \mathbb{B} \rightarrow \mathbb{C}$ having a right adjoint and preserving terminal objects may be thought of as normalized \mathbb{C} -valued measures on \mathbb{B} there arises

the question whether this fibrational characterisation of normalized measures is of any use?

99 Zawadowski's Cartesian Bifibrations

If \mathbb{C} is a category with finite limits then $P_{\mathbb{C}} = \partial_1 : \mathbb{C} \downarrow \mathbb{C} \rightarrow \mathbb{C}$ is a bifibration such that

- (1) $P_{\mathbb{C}}$ is a fibration of finite limit categories
- (2) for all $u : J \rightarrow I$ the functor $u_!$ (left adjoint to the reindexing functor u^*) preserves pullbacks
- (3) for all $u : J \rightarrow I$ the unit and counit of the adjunction $u_! \dashv u^*$ are cartesian⁵⁷.

If $F : \mathbb{B} \rightarrow \mathbb{C}$ is a functor between categories with finite limits and F preserves terminal objects then $F^*P_{\mathbb{C}}$ is a bifibration over \mathbb{B} still satisfying the conditions (1)-(3) above. We call such fibrations *cartesian bifibrations*.⁵⁸

We now show that every cartesian bifibration $P : \mathbb{X} \rightarrow \mathbb{B}$ is equivalent to one of the form $F^*P_{\mathbb{C}}$ for some terminal object preserving functor $F : \mathbb{B} \rightarrow \mathbb{C}$ between finite limit categories. For this purpose the following Lemma will be crucial.

Lemma If $P : \mathbb{X} \rightarrow \mathbb{B}$ is a cartesian bifibration and

$$\begin{array}{ccc}
 Y & \xrightarrow{\psi} & Z \\
 \alpha \downarrow & (*) & \downarrow \beta \\
 X & \xrightarrow[\varphi]{\text{cocart.}} & u_!X
 \end{array}$$

is a commuting square in \mathbb{X} with α and β vertical and φ cocartesian then this square is a pullback if and only if ψ is cocartesian.

Proof. Suppose ψ is cocartesian. Consider the diagram

$$\begin{array}{ccccc}
 Y & \xrightarrow{\eta_Y} & u^*u_!Y & \xrightarrow{\text{cart.}} & u_!Y \\
 \alpha \downarrow & \lrcorner & \downarrow u^*u_!\alpha & \lrcorner & \downarrow \beta \\
 X & \xrightarrow{\eta_X} & u^*u_!X & \xrightarrow{\text{cart.}} & u_!X
 \end{array}$$

⁵⁷a natural transformation $\tau : F \Rightarrow G$ is cartesian iff for all $\alpha : X \rightarrow Y$ the square

$$\begin{array}{ccc}
 FX & \xrightarrow{\tau_X} & GX \\
 F\alpha \downarrow & & \downarrow G\alpha \\
 FY & \xrightarrow{\tau_Y} & GY
 \end{array}$$

is cartesian, i.e. a pullback

⁵⁸They were introduced by M. Zawadowski in his paper *Lax Monoidal Fibrations* (2009).

where the bottom and top arrow are φ and ψ , respectively. Since both squares are pullbacks so is the rectangle.

Now suppose that $(*)$ is a pullback. Consider the diagram

$$\begin{array}{ccccc}
 u_1 Y & \xrightarrow{u_1 \gamma} & u_1 u^* Z & & \\
 \text{cocart.} \uparrow & & \text{cocart.} \uparrow & \searrow \varepsilon_Z & \\
 Y & \xrightarrow{\gamma} & u^* Z & \xrightarrow{\text{cart.}} & Z \\
 \downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow \beta \\
 \alpha \downarrow & & u^* \beta \downarrow & & \\
 X & \xrightarrow{\eta_X} & u^* u_1 X & \xrightarrow{\text{cart.}} & u_1 X
 \end{array}$$

where the bottom and top arrow of the rectangle are φ and ψ , respectively. Thus, for showing that ψ is cocartesian it suffices to show that $\varepsilon_Z \circ u_1 \gamma$ is an isomorphism. For showing the latter consider the diagram

$$\begin{array}{ccccc}
 u_1 Y & \xrightarrow{u_1 \gamma} & u_1 u^* Z & \xrightarrow{\varepsilon_Z} & Z \\
 u_1 \alpha \downarrow \lrcorner & & u_1 u^* \beta \downarrow \lrcorner & & \downarrow \beta \\
 u_1 X & \xrightarrow{u_1 \eta_X} & u_1 u^* u_1 X & \xrightarrow{\varepsilon_{u_1 X}} & u_1 X
 \end{array}$$

where the left square is a pullback because u_1 preserves pullbacks and the right square is a pullback since ε is cartesian. Thus, the rectangle is a pullback, too. Since the bottom arrow of the rectangle is an identity arrow it follows that the top arrow of the rectangle is an isomorphism as claimed. \square

From this Lemma it follows that P is equivalent to $\Delta^* P_{\mathbb{X}_1}$ where $\Delta : \mathbb{B} \rightarrow \mathbb{X}_1$ is given by

$$\begin{array}{ccc}
 1_J & \xrightarrow{\varphi_J} & \Delta(J) \\
 1_u \downarrow & & \downarrow \Delta(u) \\
 1_I & \xrightarrow{\varphi_I} & \Delta(I)
 \end{array}$$

where φ_I and φ_J are cocartesian over terminal projections.

Notice that Δ preserves pullbacks iff cocartesian arrows are stable under pullbacks along cartesian arrows, i.e. P has sums.

100 Complete Spreads

Let $\varphi : Z \rightarrow X$ be a continuous map with Z locally connected. Then $G = \pi_0 \cdot \varphi^{-1} : \mathcal{O}(X) \rightarrow \mathbf{Set}$ is a cosheaf. For $x \in X$ the set of *cogerm*s at x is given by

$$\psi^{-1}(x) = \text{Lim}_{x \in U} G(U)$$

i.e. the set of all consistent selections $\xi_U \in G(U)$ as U varies over open neighbourhoods U of x . Let Y be the space of all cogerm's of G whose topology is generated by the basic open sets

$$(U, \alpha) = \{ \text{cogerm } \xi \mid \psi(\xi) \in U \text{ and } \xi_U = \alpha \}$$

where U is an open set of X and $\alpha \in G(U)$. The space Y is locally connected and the projection $\psi : Y \rightarrow X$ is continuous. Let $\rho : Z \rightarrow Y$ be the map sending z to the cogerm $\rho(z)$ at $\varphi(x)$ with $\rho(z)_U \ni z$ (i.e. the connected component of $\varphi^{-1}(U)$ containing z as an element). Notice that ρ is continuous and $\psi \cdot \rho = \varphi$. The map φ is called a *complete spread* iff $\rho : Z \rightarrow Y$ is a homeomorphism.

This construction is analogous to the following one from sheaf theory. Let $\varphi : Z \rightarrow X$ be a continuous map and $F : \mathcal{O}(X)^{\text{op}} \rightarrow \mathbf{Set}$ the sheaf of continuous sections of φ . For every $x \in X$ the set of *germ*s at x is given by

$$\psi^{-1}(x) = \text{Colim}_{x \in U} F(U)$$

i.e. the set $\bigcup_{x \in U} F(U)$ modulo the equivalence relation \sim_x where $F(U) \ni s \sim_x s' \in F(U')$ iff $s_{U \cap U'} = s'_{U \cap U'}$. Let Y be the space of all germ's of F whose topology is generated by the basic open sets

$$(U, s) = \{ \text{germ } \xi \mid \psi(\xi) \in U \text{ and } s \in \xi \}$$

where U is an open set of X and $s \in F(U)$. Let $\sigma : Y \rightarrow Z$ be the map sending $[s] \in \psi^{-1}(x)$ to $s(x)$. The map ρ is continuous and satisfies $\varphi \cdot \rho = \psi$. The map φ is called an *etale map* iff σ is a homeomorphism.

101 Geometric morphisms between sheaf toposes

A morphism of locales $f : A \rightarrow B$ induces a geometric morphism $F \dashv U : \mathbf{Sh}(A) \rightarrow \mathbf{Sh}(B)$ where U is given by reindexing along $f^* : \mathcal{O}(B) \rightarrow \mathcal{O}(A)$ and F is given by

$$\begin{array}{ccc} \mathbf{Set}^{\mathcal{O}(B)^{\text{op}}} & \xrightarrow{\coprod_{f^*}} & \mathbf{Set}^{\mathcal{O}(A)^{\text{op}}} \\ \uparrow & & \downarrow \mathbf{a} \\ \mathbf{Sh}(B) & \xrightarrow{F} & \mathbf{Sh}(A) \end{array}$$

where \coprod_{f^*} is left adjoint to reindexing along f^* and \mathbf{a} is the sheafification functor.

Unravelling the construction⁵⁹ of $\coprod_{f^*} : \mathbf{Set}^{\mathcal{O}(B)^{\text{op}}} \rightarrow \mathbf{Set}^{\mathcal{O}(A)^{\text{op}}}$ one observes that $\coprod_{f^*}(Y)(U)$ is the quotient of

$$\coprod_{U \leq f^* V} Y(V)$$

modulo \sim_U where $(V_1, y_1) \sim_U (V_2, y_2)$ iff there exists $V_3 \leq V_1, V_2$ with $U \leq f^* V_3$ and $y_1|_{V_3} = y_2|_{V_3}$.

In part C.2 of the *Elephant* this has been extended to morphisms between sites. If $(\mathbb{C}, \mathcal{J})$ and $(\mathbb{D}, \mathcal{K})$ are sites where \mathbb{C} and \mathbb{D} have finite limits then a morphism from $(\mathbb{C}, \mathcal{J})$ to $(\mathbb{D}, \mathcal{K})$ is a functor $F : \mathbb{C} \rightarrow \mathbb{D}$ preserving finite limits and covers. Then the change of base functor $F^* : \mathbf{Set}^{\mathbb{D}^{\text{op}}} \rightarrow \mathbf{Set}^{\mathbb{C}^{\text{op}}}$ restricts to a functor $F^* : \mathbf{Sh}(\mathbb{D}, \mathcal{K}) \rightarrow \mathbf{Sh}(\mathbb{C}, \mathcal{J})$ which has a left adjoint $F_!$ given by

$$\begin{array}{ccc} \mathbf{Set}^{\mathbb{C}^{\text{op}}} & \xrightarrow{\coprod_F} & \mathbf{Set}^{\mathbb{D}^{\text{op}}} \\ \uparrow & & \downarrow \mathbf{a} \\ \mathbf{Sh}(\mathbb{C}, \mathcal{J}) & \xrightarrow{F_!} & \mathbf{Sh}(\mathbb{D}, \mathcal{K}) \end{array}$$

where $\coprod_F \dashv F^* : \mathbf{Set}^{\mathbb{D}^{\text{op}}} \rightarrow \mathbf{Set}^{\mathbb{C}^{\text{op}}}$ and \mathbf{a} is the sheafification functor.

⁵⁹Recall that $\coprod_{f^*}(Y)(U)$ is obtained by inverting all arrows in $U \downarrow f^* \times_{\mathcal{O}(B)} Y$ where

$$\begin{array}{ccc} U \downarrow f^* \times_{\mathcal{O}(B)} Y & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow \\ U \downarrow f^* & \xrightarrow{\partial_1} & \mathcal{O}(B) \end{array}$$

considering Y as a discrete fibration over $\mathcal{O}(B)$.

102 Axiom of Replacement in Categorical Logic

In E. Zermelo's axiomatisation Z of set theory he forgot to postulate the so-called *replacement* scheme as pointed out by A. Fraenkel. The system ZF of Zermelo's set theory augmented with Fraenkel's replacement scheme is nowadays used as the axiomatic basis of modern mathematics. Often one also adds the axiom of choice to ZF giving rise to the system ZFC.

Informally speaking the replacement scheme says that for a set a and a class function f its image $f[a]$ is also a set where a "class function" is a functional relation on the universe (class) V of all sets. At the time of Zermelo and Fraenkel it was not quite clear what is meant by a class function since the logical basis of set theory was not yet quite clear. But rather quickly people agreed on first order logic as the logical framework relative to which the axioms of ZF(C) have to be understood. Accordingly, a class function was understood as (given by) a first order formula $\varphi(x, y)$ (possibly with further free variables z_1, \dots, z_n) for which it holds that $\forall x \exists^1 \varphi(x, y)$.

This restriction to first order logic forces one to formulate replacement as a *scheme*. A bit later von Neumann, Bernays and Gödel introduced the equivalent system NBG of class theory where (in modern formulation) the logical basis is second order logic where (unary) predicates on the universe of sets are thought of as classes. Second order logic allows one to quantify over classes. But how many classes are forced to exist is regulated by how strong a *comprehension* principle one postulates for defining predicates on classes. Unrestricted comprehension $\exists X \forall x (x \in X \leftrightarrow \varphi(x))$ gives rise to so-called Morse-Kelley set theory whereas in NBG the comprehension principle is restricted to those φ which are "predicative" in the sense that they do not contain any second order quantifiers. Of course, one would get an even stronger class theory if one used *higher order logic* as a logical framework for set theory. A. Grothendieck in the 1950s went even further using ZF(C) as logical framework for set theory thus arriving at the notion of a *Grothendieck universe*, i.e. a transitive set U containing ω , closed under cartesian products and powersets and satisfying a *replacement axiom* saying that $\forall a \in U \forall f \in U^a f[a] \in U$ where $f[a] = \{f(x) \mid x \in a\}$. Notice that this replacement axiom is (much) stronger than the usual *replacement scheme* since f may be instantiated by functions from a to U which are not first order definable in the language of set theory. A formal system for set theory sufficiently strong for almost all purposes of category theory is ZFC together with the axiom that for all sets a there exists a Grothendieck universe U with $a \in U$.

From the above discussion we learn two things, namely that (1) for formulating replacement we need to quantify over a universe (class) of all sets and (2) the strength of replacement depends on the (strength of the) ambient logic.

There is also a pragmatic side, namely the question What is replacement needed for? Experience tells us that one uses replacement for constructing recursively defined families of sets like $(\mathcal{P}^\beta(\omega))_{\beta \in \alpha}$ where $\mathcal{P}^\beta(\omega) = \bigcup_{\gamma \in \beta} \mathcal{P}^\gamma(\omega)$. Donald Martin has used the family $(\mathcal{P}^\alpha(\omega))_{\alpha \in \omega_1}$ in his celebrated proof of Borel

determinacy and a bit later H. Friedman has shown that this uncountable iteration of powerset is indispensable for proving Borel determinacy, i.e. that it cannot be derived in Zermelo's set theory Z. Logicians who indulge in measuring the strength of formal systems and witnessing their different strengths by mathematically natural statements provable in one but not in the other system tend to the opinion that the crucial step is from 2nd Order Arithmetic to ZF and that the step from 2nd Order to Higher Order Arithmetic is not witnessed by any mathematically natural statements provable in higher order but not in 2nd order arithmetic.

But there are other issues than measuring strength when analysing formal systems. From this point of view higher order arithmetic appears quite naturally since it is objectified in a syntax-free way by the notion of an *elementary topos* with a *natural numbers object* N which has been forcefully suggested as an alternative (and better) foundation for mathematics by categorical logicians since end of 1960s. An impressive manifestation of this view can be found in Lawvere and Rosebrugh's book *Sets for Mathematics*. The superiority of categorical logic or equivalently type theory over set theory is analogous to the superiority of a high level programming language over machine language since it is not very user friendly to reduce everything to the relation \in of elementhood just as it is not very user friendly to reduce everything to bit strings. But more important (for me) is that the categorical account of logic opens up the possibility to study models for intuitionistic formal systems and actually provides us with a lot of such models arising in a natural way. (One should also notice that the forcing models used in classical set theory are just sheaf models in disguise!)

However, the notion of elementary topos with natural numbers object does lack the logical strength provided by replacement. Therefore, now and then categorical logicians have paid attention to replacement. Already in Lawvere's 1964 paper *An Elementary Theory of the Category of Sets* (ETCS) one finds an axiomatisation of replacement for wellpointed toposes \mathcal{E} with nno satisfying AC (i.e. every epimorphism splits). His form of replacement claims that for any object A of \mathcal{E} and any $\mathcal{E}(1, A)$ -indexed family S of objects of \mathcal{E} there exists a morphism $X \rightarrow A$ such that $S_a \cong a^*X$ for all $a : 1 \rightarrow A$. But Lawvere requires this only for families S which are *syntactically definable in the language of ETCS*. This axiom makes sense only because for a wellpointed topos \mathcal{E} satisfying AC one can show that objects $X \rightarrow A$ and $Y \rightarrow A$ in \mathcal{E}/A are already isomorphic if $a^*X \cong a^*Y$ for all $a \in \mathcal{E}(1, A)$. Lawvere argues that models of ETCS satisfying his form of replacement are equiconsistent with models of ZFC. Thus, his account is sort of convincing if one wants to axiomatize "the" category of sets. But it is not clear how to extend his approach to toposes which are not wellpointed because external and internal notions of families of objects fall apart: e.g. if A has no global elements then for all internal families $X \rightarrow A$ and $Y \rightarrow A$ it (vacuously) holds that $a^*X \cong a^*Y$ for all $a : 1 \rightarrow A$. Thus, there are many nonisomorphic internal families giving rise to the same external (empty) family. In my opinion it is sheer luck that syntactically definable in the language of ETCS and syntactically definable in the language of set theory turn out as sufficiently equivalent.

Moreover, Lawvere’s formulation of replacement is not in coincidence with set theory since for him families of sets are not given by a functional relation on some universe U of sets *where U itself is an object of the category*. But if – following Quine – existence means to be the range of a quantifier then the class of sets over which the axioms of ZF(C) quantify as a completed totality does exist and, therefore, should be represented by an object of a categorical model. This was the starting point of Ch. Maurer’s paper *Universes in Topoi* from 1975 where (in a topos) he postulates a universe $U \mapsto \mathcal{P}(U)$ satisfying a few well chosen axioms including a strong form of replacement claiming that for every $a \in U$ and $f \in U^a$ the image of f is in U . Maurer’s axiom of replacement (like the one of Grothendieck universes) is stronger than the set theorist’s one because it considers families $a \rightarrow U$ which are not necessarily given by a syntactically definable relation. Moreover, in Maurer’s account the ambient logic is given by a topos, i.e. higher order and not first order logic.

This has been “corrected” in Algebraic Set Theory (AST) originated by A. Joyal and I. Moerdijk where the ambient category is assumed only to be a Heyting category (or Heyting pretopos) where one also postulates a class \mathcal{S} of small maps providing a notion of “size” (c.f. Jean Bénabou’s notion of “calibration” from the 1970s). The class \mathcal{S} is required to be pullback stable and to satisfy quite a few more closure properties depending on how strong a set theory one wants to axiomatize. The requirement for \mathcal{S} corresponding to replacement is that $f \in \mathcal{S}$ whenever e is an epimorphism with $fe \in \mathcal{S}$ since it says that (in every context) the image of a small object is small. Depending on the class \mathcal{S} of “small maps” in AST one postulates a “small powerset” functor $\mathcal{P}_{\mathcal{S}}$ sending X to the unique object $\mathcal{P}_{\mathcal{S}}(X)$ such that morphisms $I \rightarrow \mathcal{P}_{\mathcal{S}}(X)$ are in natural 1-1-correspondence with subobjects $m : R \rightarrow I \times X$ such that $\pi \circ m \in \mathcal{S}$ (i.e. I -indexed families of small subobjects of X). Of course, the object $\mathcal{P}_{\mathcal{S}}(X)$ is determined uniquely up to isomorphism by this requirement. Now every morphism $f : X \rightarrow Y$ gives rise to $\mathcal{P}_{\mathcal{S}}(f) : \mathcal{P}_{\mathcal{S}}(X) \rightarrow \mathcal{P}_{\mathcal{S}}(Y)$ by sending $R \in \mathcal{P}_{\mathcal{S}}(X)(I)$ to the image R' of R under $I \times f$ which is in $\mathcal{P}_{\mathcal{S}}(Y)$ because we have

$$\begin{array}{ccc}
 R & \xrightarrow{e} & R' \\
 \downarrow m & & \downarrow m' \\
 I \times X & \xrightarrow{I \times f} & I \times Y \\
 \searrow \pi & & \swarrow \pi \\
 & I &
 \end{array}$$

and $\pi \circ m$ is in \mathcal{S} and e is epic. In AST one constructs or postulates an initial fixpoint V of $\mathcal{P}_{\mathcal{S}}$ which in presence of the other axioms gives rise to a model of set theory whose class of sets is given by V and whose elementhood predicate is given by $\in \mapsto V \times \mathcal{P}_{\mathcal{S}}(V) \cong V \times V$.

One should notice here that weaker requirements on \mathcal{S} give rise to weaker set theories. For example requiring only regular monomorphisms to be in \mathcal{S}

gives rise to a set theory BIST which as argued by Awodey et.al. in their 2007 Bull. Symb. Logic paper is conservative over higher order arithmetic since every topos with nno appears as the full subcategory of sets of a model of BIST. The theory BIST validates replacement but only bounded separation which demonstrates that the strength of replacement strongly depends on the remaining axioms and also on the ambient logic (since BIST with classical logic is as strong as ZF). The same remarks apply to the even weaker “predicative” set theory CZF of P. Aczel.

If one does not insist on mimicking set theoretic replacement but rather its desired consequence that certain recursively defined families of types do exist an attractive alternative are *universes à la Martin-Löf*. Categorical semantics of type theory was developed in the 1980s with the aim of providing (a notion of) models for Martin-Löf type theory and its “impredicative” relatives like the (Extended) Calculus of Constructions. In this endeavor one considers pullback stable classes \mathcal{S} of “display” maps validating various closure properties and which are generated by a (strongly) generic family $E \rightarrow U$ in \mathcal{S} where “strongly generic” means that every map $X \rightarrow I$ in \mathcal{S} can be obtained as pullback of $E \rightarrow U$ along some (generally not unique) map $I \rightarrow U$. These assumptions guarantee the existence of an exponentiation operation $\rightarrow : U \times U \rightarrow U$ such that $E(a \rightarrow b) \cong E_b^{E_a}$ and if N is in U (i.e. there is a $n : 1 \rightarrow U$ with $E_n \cong N$) then by recursion over the nno N one may define a map $f : N \rightarrow U$ with $f(0) = N$ and $f(n+1) = N^{f(n)}$. If the ambient category is a topos and the subobject classifier $\Omega \in U$ (i.e. $\Omega \cong E_\omega$ for some $\omega : 1 \rightarrow U$) then by primitive recursion one gets a map $g : N \rightarrow U$ with $g(0) = N$ and $g(n+1) = \Omega^{g(n)} = \mathcal{P}(g(n))$ and the pullback of $E \rightarrow U$ along g gives rise to the N -indexed family whose n -th item is $\mathcal{P}^n(N)$.

The axioms for such type-theoretic universes are in one respect stronger than those for \mathcal{S} in AST which postulates just a “weakly generic family” $\pi : E \rightarrow U$ (for every $a : A \rightarrow I$ in \mathcal{S} there exist an epimorphism $e : J \twoheadrightarrow I$ and a map $f : J \rightarrow U$ such that $e^*a \cong f^*\pi$) but weaker in the respect that they don’t postulate a “type-theoretic collection axiom” which is needed for the construction of initial fixpoints of $\mathcal{P}_{\mathcal{S}}$.⁶⁰

But even from a type-theoretic point of view one does not need type-theoretic universes to guarantee the existence of certain recursively defined families. An alternative are so-called “large elimination” schemes which allow one to construct (e.g. N -indexed) families of types by (primitive) recursion. A categorical account of large elimination has been given by P. Taylor in his book *Practical Foundations* (1999). His account essentially amounts to requiring the existence of initial fixpoints for indexed/fibred endofunctors. However, besides such requirements being open ended there arises the question which of them can be assumed consistently. Such problems of this “external” approach do not show up in the “internal” approaches based on universes where the axiomatics is “fixed” by the assumptions on the class \mathcal{S} of small maps. In Streicher’s *Universes in*

⁶⁰The type-theoretic collection axiom says that $\mathcal{P}_{\mathcal{S}}$ preserves epimorphisms (see Joyal and Moerdijk’s book *Algebraic Set Theory* from 1995 for details).

Toposes (2005) it has been shown that in all Grothendieck and all realizability toposes suitable such universes \mathcal{S} do exist when assuming suitable Grothendieck universes on the meta-level.

Summarizing the discussion above I conclude with the following “lessons”

- Replacement as a scheme and replacement as an axiom are different. The latter is more natural and the former just enforced by the restriction to first order logic.
- The strength of replacement strongly depends on the ambient logic and on which other axioms are postulated.
- There are internal and external approaches where the former objectify collections of small types as big types whereas the latter require fixpoints of fibred/indexed endofunctors in analogy with “large elimination” schemes familiar from type theory.

103 Spector's Functional Interpretation of Classical Analysis with Choice

Classical Analysis can be formulated as **EL** + *AC* with classical logic where the Axiom of Choice (*AC*) is given by

$$(AC) \quad \forall x \exists y P(x, y) \rightarrow \exists f \forall x P(x, f(x))$$

as comprehension follows from *AC* and *tertium non datur*. Gödel's functional interpretation for Heyting (and thus also Peano) Arithmetic from 1958 was extended by C. Spector to Classical Analysis (as described above) in a (posthumously published) paper from 1962.

There is no problem to extend Gödel's functional interpretation to (classical) **EL**. The problem rather is to find a functional interpretation for the double negation translation of *AC* which looks as follows

$$(AC^G) \quad \forall x \neg \neg \exists y P^G(x, y) \rightarrow \neg \neg \exists f \forall x P^G(x, f(x))$$

This looks rather hopeless because by double contraposition we only get

$$(\widehat{AC}) \quad \neg \neg \forall x \exists y P^G(x, y) \rightarrow \neg \neg \exists f \forall x P^G(x, f(x))$$

from $\forall x \exists y P^G(x, y) \rightarrow \exists f \forall x P^G(x, f(x))$ which is an instance of *AC*. However, we easily get *AC* from \widehat{AC} using the principle

$$(DNS) \quad \forall x \neg \neg A \rightarrow \neg \neg \forall x A(x)$$

called *Double Negation Shift* which, however, is not derivable in **EL** + *AC* as it contradicts Church's Thesis.

Spector's ingenious idea now was to give a functional interpretation for **EL** + *AC* + *DNS* in $T + \mathbf{B}$, i.e. Gödel's *T* extended by *bar recursion*, thus providing (sort of)⁶¹ a constructive explanation of classical analysis (via Gödel-Gentzen translation of Classical Analysis to **EL** + *AC* + *DNS*).

Before we can explain how to provide a functional interpretation for *DNS* à la Spector, of course, we have to explain what bar recursion actually is. Suppose we are given functionals *Y*, *G* and *H* (of appropriate⁶² types) then bar recursion allows us to define a functional $F = \mathbf{BYGH}$ of type $\sigma^* \rightarrow \tau$ satisfying

$$F(\alpha) = \begin{cases} G(\alpha) & \text{if } Y([\alpha]) < |\alpha| \\ H(\lambda x. F(\alpha * x))\alpha & \text{otherwise} \end{cases}$$

where α ranges over finite lists (of type σ), $|\alpha|$ stands for the length of α and $[\alpha] : 0 \rightarrow \sigma$ is the function with $[\alpha](n) = \alpha_n$ for $n < |\alpha|$ and $[\alpha](n) = 0^\sigma$ for

⁶¹Up to now it is not clarified whether bar recursion is "constructively sound". Independent of this somewhat ideological debate one may consider bar recursion as a concept allowing one to pinpoint the algorithmic content of Classical Analysis. A less problematic(?) way for doing this is via Girard's system *F* from 1972.

⁶²Namely $Y : (0 \rightarrow \sigma) \rightarrow 0$, $G : \sigma^* \rightarrow \tau$ and $H : (\sigma \rightarrow \tau) \rightarrow \sigma^* \rightarrow \tau$ where 0 stands for the type of natural numbers and σ^* for the type of finite lists over elements of type σ .

$n \geq |\alpha|$. Notice that for termination of *BYGH* it is required that for every $f : 0 \rightarrow \sigma$ there exists an $n \in \mathbb{N}$ such that $Y([\bar{f}(n)]) < n$.⁶³ From this point of view bar recursion may be understood as *transfinite induction over wellfounded trees* (of finite lists of elements of type σ).

We now describe how to construct a functional interpretation of *DNS* using bar recursion. The functional interpretation of *DNS* requires one to exhibit constructive Skolem functions for

$$(1) \quad \forall A, Y, D \exists x, Z, C A_D(x, AxZ, Z(AxZ)) \rightarrow A_D(YC, C(YC), DC)$$

where $A(x)^D \equiv \exists u \forall v A_D(x, u, v)$ is the functional interpretation of $A(x)$. A sufficient condition for (1) is

$$(2) \quad \forall A, Y, D \exists x, Z, C x = YC \wedge AxZ = C(YC) \wedge Z(AxZ) = DC$$

which can be reduced to

$$(3) \quad \forall A, Y, D \exists C \forall m \leq YC \exists Z_m AmZ_m = Cm \wedge Z_m(Cm) = DC$$

as (2) follows from (3) putting $x = m = YC$ and $Z = Z_x$.

In the following we employ the abbreviation

$$Q(m, C) \equiv m \leq YC \rightarrow \exists Z_m AmZ_m = Cm \wedge Z_m(Cm) = DC.$$

Using bar recursion we now construct a functional Φ extending every finite list $\langle C_0, \dots, C_{x-1} \rangle$ to a functional $C = \Phi\langle C_0, \dots, C_{x-1} \rangle$ satisfying $\forall m \geq x Q(m, C)$. Then for $C = \Phi\langle \rangle$ we have $\forall m \leq YC \exists Z_m AmZ_m = Cm \wedge Z_m(Cm) = DC$, i.e. (3) as desired.

Assume as induction hypothesis that

$$(IH) \quad \forall X \forall m \geq x+1 Q(m, \Phi\langle C_0, \dots, C_{x-1}, X \rangle)$$

is already known. We are looking for a definition of $\Phi\langle C_0, \dots, C_{x-1} \rangle$. If $Y[\langle C_0, \dots, C_{x-1} \rangle] < x$ then we put $\Phi\langle C_0, \dots, C_{x-1} \rangle = [\langle C_0, \dots, C_{x-1} \rangle]$ which works as for $C = [\langle C_0, \dots, C_{x-1} \rangle]$ we have

$$\forall m \geq x (m \leq YC \rightarrow \exists Z_m AmZ_m = Cm \wedge Z_m(Cm) = DC)$$

because $x \leq m \leq YC$ is impossible due to our assumption $Y(C) < x$. Now suppose that $Y[\langle C_0, \dots, C_{x-1} \rangle] \geq x$. Due to our induction hypothesis it suffices to exhibit an appropriate X_0 with $Q(x, \Phi[\langle C_0, \dots, C_{x-1}, X_0 \rangle])$. Obviously, for this purpose it suffices to exhibit an X_0 with

$$(4) \quad \exists Z_x AxZ_x = Cx \wedge Z_x(Cx) = DC$$

for $C = \Phi[\langle C_0, \dots, C_{x-1}, X_0 \rangle]$. Now putting

⁶³This property is ensured e.g. by continuity of Y because then $Y(f)$ is determined by an initial segment of its argument f . Thus bar recursion can be interpreted in the continuous functionals à la Kleene/Kreisel. An alternative model are the strongly majorizable functionals (due to M. Bezem) containing also discontinuous functionals.

$$X_0 = AxZ_x \quad \text{with} \quad Z_x = \lambda X. D(\Phi(C_0, \dots, C_{x-1}, X))$$

for $C = \Phi[(C_0, \dots, C_{x-1}, X_0)]$ we have

$$Cx = \Phi[(C_0, \dots, C_{x-1}, X_0)]x = X_0 = AxZ_x$$

and, therefore, also

$$Z_x(Cx) = D(\Phi(C_0, \dots, C_{x-1}, Cx)) = DC$$

establishing (4).

Summarising the construction of Φ we notice that Φ is defined recursively as

$$\Phi(C_0, \dots, C_{x-1}) = \begin{cases} [(C_0, \dots, C_{x-1})] & \text{if } Y[(C_0, \dots, C_{x-1})] < x \\ \Phi(C_0, \dots, C_{x-1}, Ax(\lambda X. D(\Phi(C_0, \dots, C_{x-1}, X)))) & \text{else} \end{cases}$$

and, therefore, an instance of the scheme of bar recursion.

Obviously, Spector's functional interpretation of AC^G , the double negation translation of AC , is somewhat involved. A bit later Howard gave a more perspicuous account showing that $HA_\omega + AC + BI$ allows one to prove AC^G where BI is the principle of *bar induction*. Then it is more or less routine to show that $HA_\omega + AC + BI$ can be given a functional interpretation in $T + BR$.

104 Girard's Functional Interpretation for HAS

For this purpose Girard invented his system F , a polymorphic extension of Gödel's system T . With every set variable Z he associates type variables α, β and a variable Z^* of type $\alpha \rightarrow \beta \rightarrow \iota \rightarrow \iota$. Then $t \in Z$ is translated to $\exists x^\alpha. \forall y^\beta. Z^*(x, y, t) = 0$.

Suppose A has been translated as $A^G \equiv \exists x^{\sigma[\alpha, \beta]}. \forall y^{\tau[\alpha, \beta]}. A^*[x, y, Z^*] = 0$. From $\forall \alpha, \beta. \forall Z^*. A^G$ one gets by straightforward quantifier manipulation the formula $\exists X. \forall \alpha, \beta. \forall Z^*. \forall y. A^*(X(\alpha)(\beta)Z^*, y, Z^*)$ which may be reformulated as

$$\exists X. \forall Y. \mathbf{unpack} Y \mathbf{as} (\alpha, \beta, u) \mathbf{in} A^*[X\alpha\beta(\pi_1(u)), \pi_2(u), \pi_1(u)] = 0$$

with $X : \forall \alpha, \beta. (\alpha \rightarrow \beta \rightarrow \iota \rightarrow \iota) \rightarrow \sigma[\alpha, \beta]$ and $Y : \exists \alpha, \beta. (\alpha \rightarrow \beta \rightarrow \iota \rightarrow \iota) \times \tau[\alpha, \beta]$. This serves as interpretation of $(\forall Z. A)^G$.

The formula $\exists \alpha, \beta. \exists Z^*. A^G$, i.e. $\exists \alpha, \beta. \exists Z^*. \exists x^{\sigma[\alpha, \beta]}. \forall y^{\tau[\alpha, \beta]}. A^*[x, y, Z^*] = 0$, may be reformulated as

$$\exists X. \forall Y. \mathbf{unpack} X \mathbf{as} (\alpha, \beta, u) \mathbf{in} A^*[\pi_2(u), Y\alpha\beta u, \pi_1(u)] = 0$$

with $X : \exists \alpha, \beta. (\alpha \rightarrow \beta \rightarrow \iota \rightarrow \iota) \times \sigma[\alpha, \beta]$ and $Y : \forall \alpha, \beta. ((\alpha \rightarrow \beta \rightarrow \iota \rightarrow \iota) \times \sigma[\alpha, \beta]) \rightarrow \tau[\alpha, \beta]$. This serves as interpretation of $(\exists Z. A)^G$.

This is a solution to exercise 7.B.16 in Girard's *Proof Theory and Logical Complexity*. For more details see 3.5.21 of Troelstra's SLNM 344.

105 Unrestricted Bar Induction entails PEM for Π_1^0 -sentences (S. C. Kleene)

In Kleene and Vesley's *Foundations of Constructive Mathematics* (1965) one finds an argument showing that unrestricted bar induction allows one to prove $(\forall\alpha. \alpha(n) = 0) \vee \neg(\forall\alpha. \alpha(n) = 0)$.

Let $R(\alpha, \langle \rangle) \equiv \neg\forall\alpha. \alpha(n) = 0$, $R(\alpha, \langle n \rangle) \equiv \alpha(n) = 0$ and $\neg R(\alpha, s)$ for $|s| \geq 2$. For the predicate

$$A(\alpha, s) \equiv R(\alpha, s) \vee \forall n. R(\alpha, s*\langle n \rangle)$$

one can show that

- (1) $R(\alpha, s) \rightarrow A(\alpha, s)$
- (2) $\forall\beta\exists n R(\alpha, \bar{\beta}(n))$
- (3) $\forall n A(\alpha, s*\langle n \rangle) \rightarrow A(\alpha, s)$.

Claim (1) is obvious. For (2) if $\alpha(\beta(0)) = 0$ put $n = 1$ and otherwise put $n = 0$. For showing (3) suppose $\forall n A(\alpha, s*\langle n \rangle)$, i.e. $\forall n (R(\alpha, s*\langle n \rangle) \vee \forall m R(\alpha, s*\langle n \rangle*\langle m \rangle))$. Since $R(\alpha, s*\langle n \rangle*\langle m \rangle)$ never holds it follows that $\forall n R(\alpha, s*\langle n \rangle)$, i.e. $A(\alpha, s)$ as claimed.

By unrestricted bar induction it follows from (1)-(3) that $A(\alpha, \langle \rangle)$, i.e. $\neg\forall n R(\alpha, n) \vee \forall n R(\alpha, n)$ which does not hold constructively since it contradicts WC-N.

Constructively correct versions of bar induction

Kleene has restricted bar induction to decidable or monotone R . W. Veldman has suggested the following formulation coming closer to Brouwer's original intention. For $B \subseteq \mathbb{N}^*$ let $Sec(B)$ be the least subset C of \mathbb{N}^* with $C \supseteq B$ and

$$s \in C \quad \text{iff} \quad \forall n s*\langle n \rangle \in C$$

Now *Brouwer's Thesis on Bars* says that

$$\text{if } B \subseteq \mathbb{N}^* \text{ is a bar, i.e. } \forall\alpha\exists n\bar{\alpha}(n) \in B, \text{ then } \langle \rangle \in Sec(B).$$

Brouwer mistakenly thought that the above Thesis on Bars entails that from (1)-(3) it follows that $A(\langle \rangle)$. Well, this true classically but it took Kleene's example to show that it contradicts WC-N.

106 Two Applications of Bar Recursion (by P. Oliva)

106.1 Type 2 functionals cannot be 1-1

One can prove classically that for every type 2 functional Ψ there exist different α and β of type 1 with $\Psi(\alpha) = \Psi(\beta)$. Such α and β can be constructed from Ψ using bar recursion.

If s is a sequence of objects of type ρ we write \widehat{s} for the function of type $0 \rightarrow \rho$ with $\widehat{s}(i) = s_i$ for $i < |s|$ and $\widehat{s}(i) = 0^\rho$ for $i \geq |s|$. Let $\delta_g = \lambda k.g(k)(k) + 1$ for $g \in 0 \rightarrow 0 \rightarrow 0$.

Consider the functional B defined by bar recursion as follows

$$B(s) = \begin{cases} \langle \rangle & \text{if } \Psi\delta_{\widehat{s}} < |s| \\ \delta_{\widehat{r}} * B(s * \delta_{\widehat{r}}) & \text{if } \Psi\delta_{\widehat{r}} = |s| \\ 0^1 * B(s * 0^1) & \text{if } \Psi\delta_{\widehat{r}} > |s| \end{cases}$$

where $r = s * 0^1 * B(s * 0^1)$.

Let $t = B(\langle \rangle)$. One shows by induction on $k < |t|$ that $t = \bar{t}(k) * B(\bar{t}(k))$. For $k < |t|$ let $r_k = \bar{t}(k) * 0^1 * B(\bar{t}(k) * 0^1)$. Obviously, for $k < |t|$ we have

- (1) if $\Psi\delta_{\widehat{r}_k} = k$ then $t_k = \delta_{\widehat{r}_k}$ and thus $\Psi t_k = k$
- (2) if $\Psi\delta_{\widehat{r}_k} > k$ then $t = r_k$ and thus $\Psi\delta_{\widehat{t}} = \Psi\delta_{\widehat{r}_k} > k$.

Since $B(t) = \langle \rangle$ we have $n := \Psi\delta_{\widehat{t}} < |t|$. Suppose $\Psi\delta_{\widehat{r}_n} > n$. Then from (2) it follows that $n = \Psi\delta_{\widehat{t}} = \Psi\delta_{\widehat{r}_n} > n$ which clearly is impossible. Thus, we have $\Psi\delta_{\widehat{r}_n} = n$ from which it follows by (1) that $\Psi t_n = n$ and $t_n = \delta_{\widehat{r}_n}$. Thus for $f = \bar{t}$ we have

$$\Psi\delta_f = n = \Psi t_n = \Psi(fn) = \Psi(f(\Psi\delta_f))$$

but $\delta_f \neq f(\Psi\delta_f)$ since $\delta_f(n) = f(n)(n) + 1 \neq f(n)(n) = f(\Psi\delta_{\widehat{t}})(n) = f(\Psi\delta_f)(n)$. Thus, for $\alpha = \delta_f$ and $\beta = f(\Psi\delta_f)$ we have $\Psi\alpha = \Psi\beta$ but $\alpha \neq \beta$ as desired.

106.2 Fixpoints of Update Procedure

In the following let σ and τ range over finite partial functions from \mathbb{N} to \mathbb{N} . Finite partial functions from \mathbb{N} to \mathbb{N} may be coded as finite sequences of pairs of natural numbers. We write $\langle \rangle$ for the everywhere undefined partial function and $\langle k, n \rangle$ for the partial function which is defined only at k with value n . If σ is a partial function and $k, n \in \mathbb{N}$ then $\sigma \oplus \langle k, n \rangle$ is defined as

$$(\sigma \oplus \langle k, n \rangle)(i) = \begin{cases} n & \text{if } i = k \\ \sigma(i) & \text{if } i \neq k \wedge i \in \text{dom}(\sigma) \\ \uparrow & \text{otherwise} \end{cases}$$

If σ is a partial function from \mathbb{N} to \mathbb{N} then we write $\widehat{\sigma}$ for the total function from \mathbb{N} to \mathbb{N} which behaves like σ on $\text{dom}(\sigma)$ and $\widehat{\sigma}(k) = 0$ for $k \notin \text{dom}(\sigma)$.

Let $\Psi : (\mathbb{N} \rightarrow \mathbb{N}) \rightarrow \mathbb{N}$ be a continuous functional on Baire space and $\Phi : (\mathbb{N} \rightarrow \mathbb{N}) \rightarrow \mathbb{N}$. We say that the pair (Ψ, Φ) forms a (unary) *update procedure* if whenever τ extends $\sigma \oplus \langle \Psi\widehat{\sigma}, \Phi\widehat{\sigma} \rangle$ and $\Psi\widehat{\sigma} = \Psi\widehat{\tau}$ then $\Phi\widehat{\sigma} = \Phi\widehat{\tau}$.

Lemma

There exists a σ with $\sigma = \sigma \oplus \langle \Psi\widehat{\sigma}, \Phi\widehat{\sigma} \rangle$.

Proof: We define a sequence $\sigma_{(0)}, \sigma_{(1)}, \dots$ recursively as

$$\sigma_{(0)} = \langle \rangle \quad \sigma_{(i+1)} = \sigma_{(i)} \oplus \langle \Psi\widehat{\sigma_{(i)}}, \Phi\widehat{\sigma_{(i)}} \rangle$$

which is increasing since (Ψ, Φ) is an update procedure. Let $g = \bigcup_{i \in \mathbb{N}} \sigma_{(i)}$. By continuity of Ψ there is an $i \in \mathbb{N}$ with $\Psi\widehat{g} = \Psi\widehat{\sigma_{(i)}} = \Psi\widehat{\sigma_{(i+1)}} = \dots$. Since (ψ, Φ) is an update procedure we have

$$\sigma_{(i+1)} = \sigma_{(i)} \oplus \langle \Psi\widehat{\sigma_{(i)}}, \Phi\widehat{\sigma_{(i)}} \rangle = \sigma_{(i+1)} \oplus \langle \Psi\widehat{\sigma_{(i+1)}}, \Phi\widehat{\sigma_{(i+1)}} \rangle$$

and thus $\sigma_{(i+1)}$ is the desired fixpoint. \square

Alas, the construction of \widehat{g} (as opposed to that of g) is not effective since one has to decide whether $k \in \text{dom}(g)$. It could be done effectively if we could find a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$(\dagger) \quad \forall k (\exists i (k \in \text{dom}(\sigma_{(i)})) \rightarrow k \in \text{dom}(\sigma_{f(k)}))$$

because then we could produce \widehat{g} as

$$g_f(k) = \begin{cases} \sigma_{f(k)}(k) & \text{if } k \in \text{dom}(\sigma_{f(k)}) \\ 0 & \text{otherwise} \end{cases}$$

since the oracle f tells us which i to choose. Let us write ω_Ψ for the function which assigns to every f the number $n = \omega_\Psi(f)$ such that $\Psi(f) = \Psi(f')$ for all f' with $f(i) = f'(i)$ for all $i < n$. (The functional ω_Ψ is part of our assumption that (Ψ, Φ) is an update function!) Now if f satisfied (\dagger) then the value $\Psi(\widehat{g}) = \Psi(g_f)$ is determined already by the first $\omega_\Psi(g_f)$ values of g_f . Thus the fixpoint is attained at stage $n = \max\{f(k) \mid k < m\}$, i.e. $\Psi(g_f) = \Psi(\widehat{\sigma_{(n)}})$.

Although we cannot effectively construct an f satisfying (\dagger) we can construct a sufficiently good approximation as follows. Define (uniformly in the parameters ψ, ϕ) the following bar recursive functional

$$B_{\psi, \phi}(s) = \begin{cases} \langle \rangle & \text{if } \psi(\widehat{s}) < |s| \\ \phi(\widehat{r}) * B_{\psi, \phi}(s * \phi(\widehat{r})) & \text{if } |s| \in \text{dom}(\sigma_{(\phi(\widehat{r}))}) \\ 0 * B_{\psi, \phi}(s * 0) & \text{otherwise} \end{cases}$$

where $r = s * 0 * B_{\psi, \phi}(s * 0)$. Let $t = B_{\psi, \phi}(\langle \rangle)$. One can show by induction on $k < |t|$ that $t = \bar{t}(k) * B_{\psi, \phi}(\bar{t}(k))$. For $k < |t|$ let $r_k = \bar{t}(k) * 0 * B_{\psi, \phi}(\bar{t}(k) * 0)$. Obviously, for $k < |t|$ we have

$$(1) \text{ if } k \in \text{dom}(\sigma_{(\phi(\widehat{r}_k))}) \text{ then } t_k = \phi(\widehat{r}_k) \text{ and thus } k \in \text{dom}(\sigma_{(t_k)})$$

(2) if $k \notin \text{dom}(\sigma_{(\phi(\widehat{r}_k))})$ then $t = r_k$ and thus $\phi(\widehat{t}) = \phi(\widehat{r}_k)$.

Now instantiate the parameters ϕ and ψ as follows

$$\psi(f) := \omega_\Psi(g_f) \quad \phi(f) := \max\{f(k) \mid k \leq \psi(f)\} + 1 .$$

Let $t := B_{\psi, \phi}(\langle \rangle)$ and $n := \phi(\widehat{t})$. Since $B(t) = \langle \rangle$ we have $m := \psi(\widehat{t}) < |t|$. Suppose $k \leq m$ and $k \in \text{dom}(\sigma_{(n)})$. Suppose $k \notin \text{dom}(\sigma_{(\phi(\widehat{r}_k))})$. Then from (2) it follows that $t = r_k$ and thus $n = \phi(\widehat{t}) = \phi(\widehat{r}_k)$. Thus $k \notin \text{dom}(\sigma_{(n)})$ contradicting the assumption $k \in \text{dom}(\sigma_{(n)})$. Thus we have shown that $k \in \text{dom}(\sigma_{(\phi(\widehat{r}_k))})$ from which it follows by (1) that $t_k = \phi(\widehat{r}_k)$ and thus also $k \in \text{dom}(\sigma_{(t_k)})$. Thus we have shown

$$(\ddagger) \quad \forall k \leq \omega_\Psi(g_{\widehat{t}}) (k \in \text{dom}(\sigma_{(n)}) \rightarrow k \in \text{dom}(\sigma_{(t_k)})) .$$

From the definition of ϕ it follows that $n = \phi(\widehat{t}) > t_k$ for all $k \leq \omega_\Psi(\widehat{t})$. Thus all positions in $\sigma_{(n)}$ have been defined before n and, therefore, we have $\sigma_{(n)} = \sigma_{(n-1)}$. Thus, since $\sigma_{(n)} = \sigma_{(n-1)} \oplus \langle \Psi \widehat{\sigma_{(n-1)}}, \Psi \widehat{\sigma_{(n-1)}} \rangle$ it follows that $\sigma_{(n-1)} = \sigma_{(n-1)} \oplus \langle \Psi \widehat{\sigma_{(n-1)}}, \Psi \widehat{\sigma_{(n-1)}} \rangle$, i.e. that $\sigma_{(n-1)}$ is the desired fixpoint.

107 Bar Induction vs. Bar Recursion

In ZF one can prove transfinite induction over well-founded trees and thus, in particular, bar induction. However, ZF proves that bar recursion does not hold since there are type 2 functionals whose associated stopping condition is not a bar. Thus, Krivine's classical realizability for ZF validates bar induction but not bar recursion.

However, from bar recursion one can derive that stopping conditions associated with type 2 functionals give rise to bars (see Lemma 3C of W. A. Howard's *Functional interpretation of bar induction by bar recursion* Compositio Math. 20 1968). Thus, there cannot be models of bar recursion where the stopping conditions don't give rise to bars.

108 Modified Bar Recursion for Modified Realizability of Classical Analysis

However, Gödel's functional interpretation has the disadvantage that it gets complicated rather quickly. Kreisel's Modified Realizability introduced a bit later is much more convenient for the task of extracting algorithms from proofs. Thus, it appears naturally to ask how one can give a modified realizability interpretation for AC^G .

For this purpose U. Berger and P. Oliva recently introduced a so-called *modified bar recursor*

$$\mathbf{B}_m : (\sigma^\omega \rightarrow o) \rightarrow (\sigma^* \rightarrow (\sigma \rightarrow o) \rightarrow \sigma^\omega) \rightarrow \sigma^* \rightarrow o$$

where $\Phi = \mathbf{B}_m Y H$ satisfies the defining equation

$$\Phi(s) = Y(s * H(s, \lambda x. \Phi(s * x)))$$

given $Y : \sigma^\omega \rightarrow o$ and $H : \sigma^* \rightarrow (\sigma \rightarrow o) \rightarrow \sigma^\omega$.⁶⁴

Actually, for interpreting the double negation translation of *dependent choice*

$$(DC) \quad \forall n \forall x \exists y A(n, x, y) \rightarrow \forall x \exists f (f(0) = x \wedge \forall n A(n, f(n), f(n+1)))$$

it suffices to have *weak modified bar recursion* associating a functional $\Phi : \sigma^* \rightarrow o$ with

$$\Phi(s) = Y(s * \lambda k. H(s, \lambda x. \Phi(s * x)))$$

to all $Y : \sigma^\omega \rightarrow o$ and $H : \sigma^* \rightarrow (\sigma \rightarrow o) \rightarrow \sigma$. For showing the modified realizability of DC^G besides $\text{HA}_\omega + DC$ we need also the following axioms

Sequential Continuity

$$\forall F^{\sigma^\omega \rightarrow o}, \alpha \exists n \forall \beta (\forall i < n \alpha(i) = \beta(i)) \rightarrow F(\alpha) = F(\beta)$$

and

Relativized quantifierfree Bar Induction

$$(\forall \alpha \in S \exists n P(\bar{\alpha}(n))) \rightarrow (\forall s \in S [\forall x (S(s*x) \rightarrow P(s*x))] \rightarrow P(s)) \rightarrow S(\langle \rangle) \rightarrow P(\langle \rangle)$$

where S is an arbitrary and P is a quantifierfree predicate on σ^* (we employ the notation $s \in S$ and $\alpha \in S$ as abbreviations for $S(s)$ and $\forall n S(\bar{\alpha}(n))$, respectively). It is a straightforward exercise to derive from the latter principle the principle of

Relativized quantifierfree pointwise Bar Induction

$$(\forall \alpha \in S \exists n P(\bar{\alpha}(n))) \rightarrow (\forall s \in S [\forall x (S(x, |s|) \rightarrow P(s*x))] \rightarrow P(s)) \rightarrow P(\langle \rangle)$$

⁶⁴We write σ^ω as an abbreviation for $\iota \rightarrow \sigma$ where ι is the type of natural numbers.

where $S(x, n)$ is arbitrary and $P(s)$ is quantifierfree and $s \in S$ and $\alpha \in S$ stand as abbreviations for $\forall i < |s| S(s_i, i)$ and $\forall n S(\alpha(n), n)$.

For the purpose of these notes we content ourselves with showing the modified realizability of AC^G . Following Spector we know that it suffices to show the modified realizability of $\forall n \neg \neg B(n) \rightarrow \neg \neg \forall n B(n)$ for $B(n) \equiv \exists y A(n, y)$.

We assume that

$$Y^{\sigma^\omega \rightarrow o} \text{mr } \forall n B(n) \rightarrow \perp \quad \text{and}$$

$$G^{o \rightarrow (\sigma \rightarrow o) \rightarrow o} \text{mr } \forall n \neg \neg B(n)$$

and try to build a realizer for \perp from these. Let $H^{\sigma \rightarrow \sigma}$ be a closed term such that $\forall k H \text{mr } \perp \rightarrow B(k)$ is provable. Using weak bar recursion we get a $\Psi : \sigma^* \rightarrow o$ satisfying

$$\Psi(s) = Y(s * \lambda k. H(G(|s|, \lambda x^\sigma. \Psi(s * x))))$$

for all $s \in \sigma^\omega$. We employ the abbreviations

$$S(x, n) \equiv x \text{mr } B(n) \quad \text{and} \quad P(s) \equiv \Psi(s) \text{mr } \perp$$

respectively. Now using relativized quantifierfree pointwise bar induction we will prove $P(\cdot)$, i.e. $\Psi(\cdot) \text{mr } \perp$ as desired.

i) $\forall \alpha \in S \exists n P(\bar{\alpha}(n))$ is verified as follows:

Suppose $\alpha \in S$. Then by Sequential Continuity there is an n such that $Y(\alpha)$ is determined by $\bar{\alpha}(n)$. By assumptions on Y and α we have $\forall \beta Y(\bar{\alpha}(n) * \beta) \text{mr } \perp$ and thus $\Psi(\bar{\alpha}(n)) \text{mr } \perp$.

ii) $\forall s \in S [\forall x (S(x, |s|) \rightarrow P(s * x))] \rightarrow P(s)$ is verified as follows:

Suppose $s \in S$ and $\forall x (S(x, |s|) \rightarrow P(s * x))$. More explicitly, the latter means $\forall x (x \text{mr } B(|s|) \rightarrow \Psi(s * x) \text{mr } \perp)$ and, therefore, we get $\lambda x^\sigma. \Psi(s * x) \text{mr } \neg B(|s|)$. Using the assumption on G we obtain $G(|s|, \lambda x^\sigma. \Psi(s * x)) \text{mr } \perp$. Thus, for all k we have $H(G(|s|, \lambda x^\sigma. \Psi(s * x))) \text{mr } B(k)$. As $s \in S$, i.e. $\forall i < |s| s_i \text{mr } B(i)$, we get $s * \lambda k. H(G(|s|, \lambda x^\sigma. \Psi(s * x))) \text{mr } \forall k B(k)$. Thus, it follows from the assumption on Y that $\Psi(s) = Y(s * \lambda k. H(G(|s|, \lambda x^\sigma. \Psi(s * x)))) \text{mr } \perp$, i.e. $P(\Psi(s))$, as desired.

109 Double Negation Shift in Type Theory

We work in Martin-Löf type theory with a universe U containing the types N , N_0 and N_1 as elements. We write 0 for the zero element of type N of natural numbers and denote the successor operation on B by S . The single element of N_1 is also denoted by 0 . If $n \leq m$ then $(n \leq m) = N_1$ and $(n \leq m) = N_0$ otherwise. There is a function $\text{less} : \Pi n, m : N. (Sm \leq n) + (n \leq m)$. We write exit for the canonical function from N_0 to A for arbitrary types A and $\neg A$ as an abbreviation for $A \rightarrow N_0$.

If $B : N \rightarrow U$ then $\text{vec}(B) : N \rightarrow U$ is the family with $\text{vec}(B, 0) = N_1$ and $\text{vec}(B, Sn) = \text{vec}(B, n) \times B(n)$. Let $\text{get} : \Pi n, x : N. (Sx \leq n) \rightarrow \text{vec}(B) \rightarrow B(x)$ be the function projecting on the x -th component.

For $B : N \rightarrow U$ there exist functions

$$\begin{aligned} \Phi &: (\Pi n : N. \neg \neg B(n)) \rightarrow \neg(\Pi n : N. B(n)) \rightarrow \Pi n : N. \neg \text{vec}(B, n) \\ \Psi &: (\Pi n : N. \neg \neg B(n)) \rightarrow \neg(\Pi n : N. B(n)) \rightarrow \\ &\quad \Pi n : N. \Pi v : \text{vec}(B, n). \Pi x : N. (Sx \leq n) + (N \leq x) \rightarrow B(x) \end{aligned}$$

which are specified by the equations

$$\begin{aligned} \Phi H K n v &= K(\lambda x. \Psi H K n v x (\text{less } x n)) \\ \Psi H K n v x (\text{inl } p) &= \text{get } n x p v \\ \Psi H K n v x (\text{inr } q) &= \text{exit}(H n (\lambda u. \Phi H K (Sn) (v, u))) \end{aligned}$$

Now using Φ and Ψ double negation shift (for B) is realized by

$$\lambda H. \lambda K. \Phi H K 0 0 : (\Pi n : N. \neg \neg B(n)) \rightarrow \neg \neg(\Pi n : N. B(n))$$

This little note is a correction of a (faulty) suggestion by Th. Coquand and A. Spiwack in a preprint *A Proof of Strong Normalisation using Domain Theory* from 2007.

In older work by Coquand and Spiwack they showed that $\lambda h. \lambda p. \Phi p h \langle \rangle$ realizes $(\Pi n : N. \neg \neg B(n)) \rightarrow \neg \neg(\Pi n : N. B(n))$ in the sense of modified realizability where Φ is defined recursively as

$$\Phi p h \ell = p(\lambda x. \text{get } x \ell (\text{exit}(h x (\lambda y. \Phi p h ((x, y) : \ell))))))$$

where $\text{get } x \ell a = y$ if (x, y) is the first element of ℓ whose first component is x and $\text{get } x \ell a = a$ if such an element does not exist in ℓ .

Alas, the program for Φ is only weakly normalising because one may expand Φ *ad libitum* and so avoid doing any “real computation”. But one may apply “Vogel’s trick”, i.e. introduce an additional case analysis on $x \in \ell$ (meaning that (x, y) occurs in ℓ for some y) thus obtaining

$$\begin{aligned} \Phi p h \ell &= p(\lambda x. \Psi p h \ell x (x \in \ell)) \\ \Psi p h \ell x \text{ true} &= \ell.x \\ \Psi p h \ell x \text{ false} &= \text{exit}(h x (\lambda y. \Psi p h ((x, y) : \ell))) \end{aligned}$$

where $\ell.x = y$ if (x, y) is the first pair in ℓ whose first component is x . This program Φ is very similar to the one above suggested an an extension of type theory.

110 The Inherently Parallel Nature of Exact Real Number Computation

This is a somewhat subjective summary of discussions among A. Bauer, M. Escardó and T. Streicher during September '02 in Birmingham.

In [EHS] it has been shown that the parallel join operation $\vee : \Sigma \times \Sigma \rightarrow \Sigma$ on the Sierpinski domain $\Sigma = \{\perp, \top\}$ can be defined from a few basic undoubtedly sequential operations on the interval domain \mathcal{I} and an arbitrary continuous function $\oplus : \mathcal{I} \times \mathcal{I} \rightarrow \mathcal{I}$ satisfying $x \oplus y = \frac{x+y}{2}$ for all $x, y \in \mathbb{R}$. However, this result doesn't tell us anything yet about the existence of some *sequential model of extensional exact real number computation*. There are essentially two parameters in this question that may vary. The first one is the notion of domain we are using and the second one is the choice of a subspace embedding of \mathbb{R} into a domain \mathcal{R} playing the role of (*formal*) *partial reals*.

The partial reals are inherently parallel

For the first parameter it appears as natural (at least from the point of view of SDT) that a notion of domain is given by some pca A together with a dominance Σ in $\text{RT}(A)$ satisfying the axioms of SDT. In any topos one may define the type \mathcal{R} of partial reals in a purely logical way just simply dropping one condition in the definition of Dedekind reals (namely that for all $n \in \mathbb{N}$ there are $p \in L$ and $q \in U$ with $q < p + 2^{-n}$). One readily checks that all operations of Escardó's Real PCF can be defined in the internal language of the topos including the parallel if-then-else called *pif*. Now if \mathcal{R} is a Σ -domain recursion over \mathcal{R} is available and \mathcal{R} provides a model of Real PCF (together with the lifting of booleans and natural numbers). However, in Real PCF one readily defines $\vee : \Sigma \times \Sigma \rightarrow \Sigma$ as follows: embed Σ into \mathcal{R} via sending \perp to $\iota(\perp) = [0, 1]$ and \top to $\iota(\top) = 1 = [1, 1]$ and then define join on Σ as

$$u \vee v = \left(\frac{1}{4} < (\iota(u) + \iota(v)) \right)$$

where $x + y = [\underline{x} + \underline{y}, \bar{x} + \bar{y}]$ for $x, y \in \mathcal{R}$. Thus, if \mathcal{R} is a Σ -domain then Σ is closed under disjunction. One may also show that closure of Σ under disjunctions allows one to verify the \mathcal{R} is actually a Σ -domain.

Thus, summarizing the discussion of this paragraph we have shown that

\mathcal{R} is a Σ -domain if and only if Σ is closed under disjunction

and, therefore, Escardó's model of Real exists only in SDT models (\mathcal{E}, Σ) where Σ is inherently parallel in the sense that Σ is closed under existential quantification over \mathbb{N} (as $\Sigma^{\mathbb{N}}$ is a Σ -domains and, therefore, existential quantification over \mathbb{N} may be defined recursively as $\exists_{\mathbb{N}}(p) = p(0) \vee \exists_{\mathbb{N}}(\lambda n. p(n+1))$).

Embedding \mathbb{R} into domains

In the previous paragraph we have shown that exact real number computation à la Escardó via the domain of partial reals only works in a world where parallel computation in the form of *extensional dovetailing* is available. However, we may also vary the second parameter, namely the domain \mathcal{R} into which we embed \mathbb{R} as a subspace.

It is not at all evident which axioms this embedding of \mathbb{R} into \mathcal{R} should satisfy simply because Escardó's Real PCF is an implementation of an *unknown specification*. However, a minimal requirement for the subspace embedding $\mathbb{R} \subseteq \mathcal{R}$ seems to be the following *extension property* requiring that every continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ extends to a continuous function $\bar{f} : \mathcal{R} \rightarrow \mathcal{R}$.

It seems to be a very difficult question to characterise those SDT models (\mathcal{E}, Σ) where such an embedding of \mathbb{R} into an appropriate Σ -domain \mathcal{R} can be found. Already for particular sequential pca's such as \mathcal{B} (of Longley and van Oosten), L_U (of Marz, Rohr and Streicher) or $\mathcal{A}_{wb,eff}$ (of Abramsky and Longley) it is unknown whether for the natural choice of Σ there do exist subspace embeddings of \mathbb{R} into a Σ -domain \mathcal{R} satisfying the above extension property. We would rather claim that this is not the case, however, haven't got the faintest idea how to prove it. Of course, the problem is that one hasn't any control over the collection of all subspace embeddings of \mathbb{R} into Σ -domains.

The inherently parallel nature of the canonical embedding of \mathbb{R} into \mathbb{T}_ω

Instead of considering arbitrary subspace embeddings of \mathbb{R} into Σ -domains in arbitrary models of SDT (which seems sort of hopeless) one might ask the following more conservative question staying within the realm of Scott domains.

Is there a subspace embedding of \mathbb{R} into \mathbb{T}_ω such that every continuous function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ extends to a sequential continuous function $\bar{f} : \mathbb{T}_\omega^n \rightarrow \mathbb{T}_\omega$?

Actually, we don't have a negative answer to that question in general. However, for the canonical inclusion

$$\iota : \mathbb{R} \rightarrow \mathbb{T}_\omega : x \mapsto \langle \{n \in \mathbb{N} \mid p_n < x\}, \{n \in \mathbb{N} \mid x < p_n\} \rangle$$

(where $p_0, p_1, \dots, p_n, \dots$ is some enumeration of the set \mathbb{Q} of rational numbers) we can show by a fairly simple argument that almost no continuous function on \mathbb{R} extends to a sequential continuous map on \mathbb{T}_ω .

First we show the following lemma.

Lemma *Every sequentially realizable one-to-one function $f : \mathbb{R} \rightarrow \mathbb{R}$ must send irrational numbers to irrational numbers.*

Proof : Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous one-to-one function that is sequentially realizable, *i.e.* there is a sequential continuous $\Phi : \mathbb{T}_\omega \rightarrow \mathbb{T}_\omega$ with $\iota \circ f = \Phi \circ \iota$.

Now for the sake of contradiction suppose that there is an irrational number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ such that $f(\alpha) = p_n \in \mathbb{Q}$. As Φ was supposed to be sequential the functional $\phi_n := \lambda h: \mathbb{T}_\omega. \phi(h)(n) : \mathbb{T}_\omega \rightarrow \mathbb{B}_\perp$ is sequential, too. Notice that ϕ_n is not constant as we have

$$\phi_n(\iota(\alpha)) = \Phi(\iota(\alpha))(n) = \iota(f(\alpha))(n) = \iota(p_n)(n) = \perp$$

whereas

$$\phi_n(\iota(p_n)) = \Phi(\iota(p_n))(n) = \iota(f(p_n))(n) \neq \perp$$

as $p_n = f(\alpha) \neq f(p_n)$ due to the assumption that f is one-to-one. Thus, as ϕ_n is not constant there is a natural number k at which the argument of ϕ_n has to be evaluated first.

As α is assumed as irrational we have $\iota(\alpha)(k) \neq \perp$ as either $p_k < \alpha$ or $\alpha < p_k$. But if already $\phi_n(\iota(\alpha))$ diverges although $\iota(\alpha)(k)$ terminates then $\phi_n(\iota(p_k))$ must diverge, too, as $\iota(p_k)(k)$ diverges (because $p_k \not\leq p_k$). As α is irrational it is different from p_k and thus $f(\alpha) \neq f(p_k)$ and, therefore, also $\Phi(\iota(p_k)) \neq \Phi(\iota(\alpha)) = \iota(p_n)$ from which it follows that $\phi_n(\iota(p_k)) = \Phi(\iota(p_k))(n) \neq \perp$ contradicting the divergence of $\phi_n(\iota(p_k))$ derived above. \square

Using this Lemma we can show that all basic functions on reals are not sequentially realizable.

Theorem Addition, multiplication and exponentiation on \mathbb{R} do not extend to sequential continuous functions on \mathbb{T}_ω .

Proof : If $+$: $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ were sequentially realizable then so would be $f(x) = x + \sqrt{2}$ contradicting our Lemma as f sends the irrational number $1 - \sqrt{2}$ to the rational number 1.

If \times : $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ were sequentially realizable then so would be $f(x) = x \times \sqrt{2}$ contradicting our Lemma as f sends the irrational number $\sqrt{2}$ to the rational number 2.

If exponentiation were sequentially realizable the so would be $f(x) = 2^x$ contradicting our Lemma as f sends the irrational number $\lg 3$ to the rational number 3. \square

Actually, this way sequential realizability of most other basic functions on \mathbb{R} can be refuted.

Embeddings of \mathbb{R} into dI-domains

Suppose \mathbb{R} appears as subspace of some countably based dI-domain A (as *e.g.* \mathbb{T}_ω) endowed with its Scott topology. Then not for every open subset U of \mathbb{R} there exists a stable continuous map $p : A \rightarrow \Sigma$ with $U = \mathbb{R} \cap p^{-1}(\{\top\})$. This can be seen as follows.

First recall that for a stable $p : A \rightarrow \Sigma$ we have that $p^{-1}(\{\top\}) = \bigcup_{b \in \text{tr}(p)} \uparrow b$ and that the open sets of the family $(\uparrow b)_{b \in \text{tr}(p)}$ are pairwise disjoint. Now suppose that for a connected open subset U of \mathbb{R} we have $U = \mathbb{R} \cap p^{-1}(\{\top\})$ then

there exists a unique $b \in \text{tr}(p)$ with $U \cap \uparrow b \neq \emptyset$ as otherwise U would not be connected. Thus, as by assumption A has a countable basis

only countably many connected opens of \mathbb{R} are realizable by a stable map from A to Σ .

As there are already uncountably many open intervals of the form $(x, -\infty)$ in \mathbb{R} not all of them can be realized by a stable map from A to Σ . Thus, in particular, already

the partial predicate $<: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{B}_\perp$ is not realizable by a stable map from $A \times A$ to Σ

for arbitrary subspace embeddings of \mathbb{R} into a dI-domain A . Thus, it cannot happen for such an embedding that the partial predicate $0 < (-)$ and addition are both stably realizable. Thus, no subspace embedding of \mathbb{R} into a dI-domain gives rise to an even moderately expressive implementation of \mathbb{R} . Moreover, for embeddings into \mathbb{T}_ω our argument does not depend on choosing some sort of (strong) stability as our model of sequentiality because all n -ary PCF definable maps on \mathbb{T}_ω are (strongly) stable (which is not the case anymore if instead of \mathbb{T}_ω we choose for A some higher type).

References

[EHS] M. Escardó, M. Hofmann, T. Streicher *Mediation is inherently parallel* submitted to MSCS (2002).

111 \mathbb{C}^* -algebra approach to quantum mechanics

(à la Segal, Emch, Brattelli & Robinson *et.al.*)

It is known that every \mathbb{C}^* -algebra \mathfrak{A} appears as a closed subalgebra of $\mathcal{L}(\mathcal{H})$ for some Hilbert space \mathcal{H} . Thus, we may define *algebras of observables* (on a quantum space⁶⁵) as \mathbb{C}^* -algebras with a unit I .

A *positive* element of \mathfrak{A} is a selfadjoint $A \in \mathfrak{A}$ whose spectrum is ≥ 0 . One may show that for every positive $A \in \mathfrak{A}$ there exists a unique positive $B \in \mathfrak{A}$ with $A = B^2$. Moreover, for every $R \in \mathfrak{A}$ the element R^*R is positive in \mathfrak{A} . Thus, the positive elements of \mathfrak{A} are those of the form R^*R for some $R \in \mathfrak{A}$.

A *state* of \mathfrak{A} is a (continuous) linear functional $\phi : \mathfrak{A} \rightarrow \mathbb{C}$ with $\phi(I) = 1$ and $\phi(R^*R) \geq 0$ for all $R \in \mathfrak{A}$. The latter condition says that positive observables are sent to positive numbers. This has to be seen in analogy with integration theory à la Daniel-Stone where measures on a (sufficiently reasonable) space X are identified with the *positive linear continuous functionals on $C(X)$* . The requirement $\phi(I) = 1$ just says that the measure of the whole space is 1, i.e. that we are dealing with *probability measures* on the (non-existing) quantum space.

Thus, the intuition is that a state ϕ sends an observable A (i.e. $A^* = A$) to its *expectation value* $\phi(A)$ relative to the state ϕ . As usual the *variance* of A w.r.t. ϕ is defined as $\phi(A^2) - \phi(A)^2$ since from linearity of ϕ it follows that $\phi((A - \phi(A) \cdot I)^2) = \phi(A^2) - 2\phi(\phi(A) \cdot A) + \phi(A)^2 = \phi(A^2) - 2\phi(A)^2 + \phi(A)^2 = \phi(A^2) - \phi(A)^2$. Thus, the variance of A w.r.t. ϕ vanishes, i.e. ϕ is *dispersion-free* w.r.t. A , iff $\phi(A^2) = \phi(A)^2$.

This approach has first been presented by I. E. Segal in 1947.

There arises the question in which sense elements of a \mathbb{C}^* -algebra \mathfrak{A} can be thought of as observables. Well, if $\mathfrak{A} = \mathfrak{B}(\mathcal{H})$, the \mathbb{C}^* -algebra of continuous operators on Hilbert space \mathcal{H} , then the self adjoint elements $A = A^*$ can be interpreted via the Spectral Theorem as real valued observations (on a noncommutative state space). Moreover, an arbitrary $A \in \mathfrak{A}$ can be uniquely decomposed as $A = A_1 + iA_2$ with A_1, A_2 self adjoint. Obviously, it holds that $A_1 = \frac{1}{2}(A + A^*)$ and $A_2 = \frac{1}{2i}(A - A^*)$. We think of A_1 and A_2 as the real and imaginary part of the observable A .

If A and B are self adjoint then $(AB)_1 = \frac{1}{2}(AB + BA)$ and $(AB)_2 = \frac{1}{2i}(AB - BA)$. For $A = A_1 + iA_2$ and $B = B_1 + iB_2$ we have

$$\begin{aligned} AB &= (A_1 + iA_2)(B_1 + iB_2) = (A_1B_1 - A_2B_2) + i(A_1B_2 + A_2B_1) = \\ &= \frac{1}{2}(A_1B_1 + B_1A_1 - A_2B_2 - B_2A_2 + A_1B_2 - B_2A_1 + A_2B_1 - B_1A_2) \\ &\quad + \frac{1}{2i}(A_1B_1 - B_1A_1 - A_2B_2 + B_2A_2) + \frac{i}{2}(A_1B_2 + B_2A_1 + A_2B_1 + B_1A_2) = \\ &= \frac{1}{2}(A_1B_1 + B_1A_1 - A_2B_2 - B_2A_2 + A_1B_2 - B_2A_1 + A_2B_1 - B_1A_2) \\ &\quad + \frac{i}{2}(-A_1B_1 + B_1A_1 + A_2B_2 - B_2A_2 + A_1B_2 + B_2A_1 + A_2B_1 + B_1A_2) = \\ &= ((A_1B_1)_1 - (A_2B_2)_1 + (A_1B_2)_2 - (B_1A_2)_2) \\ &\quad + i(-(A_1B_1)_2 + (A_2B_2)_2 + (A_1B_2)_1 + (B_1A_2)_1) \end{aligned}$$

⁶⁵which doesn't exist as such as only the algebra of observables exists as a mathematical object

112 States and Observables in Quantum Theory (à la Pták and Pulmannová)

Let $L = P(\mathbb{H})$ be the orthomodular lattice of projectors on Hilbert space \mathbb{H} , i.e. closed linear subspaces of \mathbb{H} ordered by \subseteq . For $A \in L$ its *orthocomplement* is given by $A' = \{x \in \mathbb{H} \mid \langle x \mid y \rangle = 0 \text{ for all } y \in A\}$. Obviously, we have $A'' = A$ and $B' \leq A'$ whenever $A \leq B$, i.e. the operation $(-)'$ is an involution on L which, moreover, satisfies the law of *orthomodularity* saying that $B = A \vee (A' \wedge B)$ whenever $A \leq B$. We say that $A, B \in L$ are compatible iff $A \leq B'$ (iff $B \leq A'$).

If X is a topological space let $\mathcal{B}(X)$ be the least σ -algebra containing all open sets of X . A (*quantum*) *observable* on X is a function $a : \mathcal{B}(X) \rightarrow L$ s.t.

$$(O1) \quad a(X) = 1$$

$$(O2) \quad a(X \setminus E) = a(E)' \text{ for all } E \in \mathcal{B}(X)$$

$$(O3) \quad a(\bigcup_{n \in \mathbb{N}} E_n) = \bigvee_{n \in \mathbb{N}} a(E_n) \text{ for all sequences } (E_n)_{n \in \mathbb{N}} \text{ in } \mathcal{B}(X).$$

Condition (O3) can be replaced by the requirements that a preserves suprema of ω -chains and binary suprema. Thus, the set of (quantum) observables on X are closed under suprema of ω -chains w.r.t. the pointwise order, i.e. form an ω -cpo.

Notice that the cpo $\mathcal{P}_q(X)$ of quantum observables (on X) should be interpreted as the space of X -valued observations on the quantum system represented by \mathbb{H} . One can check (I think!) that \mathcal{P}_q is a monad on ω -cpos. Morphisms in the Kleisli category from X to Y , i.e. maps $f : X \rightarrow \mathcal{P}_q(Y)$, may be considered as functions from X to Y parameterized over the quantum system \mathbb{H} .

A *state* (on L) is a function $s : L \rightarrow [0, 1]$ such that

$$(S1) \quad s(\top) = 1$$

$$(S2) \quad s(A') = 1 - s(A)$$

$$(S3) \quad s(\bigvee_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}} s(A_n) \text{ for all sequences } (A_n)_{n \in \mathbb{N}} \text{ which are pairwise compatible, i.e. } A_i \leq B'_j \text{ for all } i, j \in \mathbb{N} \text{ with } i \neq j.$$

Obviously, for an observable a on X and a state s their composite $s \circ a : \mathcal{B}(X) \rightarrow [0, 1]$ is a probability measure on X .

A state is *pure* iff there is an atom $a \in L$ with $s(a) = 1$. One may consider so called *partial states* which are functions $s : L \rightarrow [0, 1]$ satisfying (S3) and the condition $s(A') = s(\top) - s(A)$. For partial states $1 - s(\top)$ is interpreted as the probability of divergence. Obviously, partial states form a cpo with \perp w.r.t. the pointwise order.

Quantum Theory in Topological Domain Theory

Separable Hilbert space $H = \ell_2$ is a complete separable metric space and thus naturally lives within \mathbf{QCB}_0 , the Σ -extensional spaces within $\mathbf{Mod}(K_2)$. Closed

subsets of H are in 1-1-correspondence with elements of Σ^H since for every $p \in \Sigma^H$ the set $p^{-1}(\perp)$ is a closed subset of H and for every closed subset A of H there is a unique $p \in \Sigma^H$ with $A = p^{-1}(\perp)$. The set $L = \{p \in \Sigma^H \mid p^{-1}(\perp) \text{ is closed under addition and scalar multiplication}\}$ is a $\neg\neg$ -closed subset of Σ^H and thus in QCB_0 . Notice, however, that the operation $(-)^{\perp}$ sending $A \in L$ to its orthocomplement $A^{\perp} = \{x \in H \mid \forall y \in A. \langle x \mid y \rangle = 0\}$ is antitonic and thus does not correspond a QCB_0 -map on L . However, it holds in the internal logic of the function realizability topos $\text{RT}(K_2)$ that $\forall A \in L. \neg\neg\exists! B \in L. \forall x \in H. x \in B \leftrightarrow \forall y \in A. \langle x \mid y \rangle = 0$. Thus, for $A \in L$ its orthocomplement A^{\perp} is just an element of $\mathcal{P}(H) = \Omega^H$ and not necessarily of L . However, one may check(?) that for all $A \in L$ we have $A^{\perp\perp} = A$. One may show that for $A, B \in L$ we have $A \oplus B = (A \cup B)^{\perp\perp} \in L$.

For every object X of QCB_0 we may consider quantum observables on X as certain QCB_0 functions $a : \Sigma^X \rightarrow L$, namely those which are lattice homomorphisms.

The unit interval $I = [0, 1]$ ordered by \geq is a continuous lattice and thus an object of QCB_0 . Then states of L may be considered as certain QCB_0 functions $s : L \rightarrow I$, namely those with $s(0_L) = 0$, $s(1_L) = 1$ and $s(A \vee B) + s(A \wedge B) = s(A) + s(B)$.

113 Ideas on Higher Order Quantum Computing

Let \mathcal{Q} be the category of finite dimensional Hilbert spaces (over complex numbers) with partial isometries (i.e. self adjoint operators f with $fff = f$) as morphisms. (If one prefers total computations one may take unitary operators instead of partial isometries!) Obviously \oplus endows \mathcal{Q} with a symmetric monoidal structure (in both cases!).

For defining composition in the subsequently defined $\mathcal{G}(\mathcal{Q})$ we need a trace operation in \mathcal{Q} . Let $f : X \oplus Z \rightarrow Y \oplus Z$. Its trace $\text{Tr}(f) : X \rightarrow Y$ should satisfy the *feedback equation*

$$f(x \oplus z) = \text{Tr}(f)(x) \oplus z$$

for all $x \in X$. It is unlikely that for arbitrary morphism f the feedback equation has a unique solution z for all $x \in X$. Such a z has to satisfy

$$z = f_{XZ}(x) \oplus f_{ZZ}(z)$$

and one might attempt to construct it as

$$f_{XX} + \sum_{n \in \mathbb{N}} f_{ZX} f_{ZZ}^n f_{XZ}$$

applied to x . But, alas, there is no reason why this series should converge in general!

Notice that this is no problem if one works instead with the monoidal category $\mathcal{Q} = (\text{Rel}, \oplus, 0)$ where a trace is given by

$$\text{Tr}(f) = f_{XX} \cup \bigcup_{n \in \mathbb{N}} f_{XX} f_{ZZ}^n f_{XZ}$$

which works as the required union always exists.

One may apply the GoI (Geometry of Interaction) construction to \mathcal{Q} thus arriving at a category $\mathcal{G}(\mathcal{Q})$ whose objects are pairs $A = (A^+, A^-)$ of objects of \mathcal{Q} and whose morphisms from A to B are \mathcal{Q} -morphisms $f : A^+ \oplus B^- \rightarrow A^- \oplus B^+$. If $f : A \rightarrow B$ and $g : B \rightarrow C$ then their composite $gf : A \rightarrow C$ is given by the trace of

$$A^+ \oplus C^- \oplus B^- \oplus B^+ \cong A^+ \oplus B^- \oplus B^+ \oplus C^- \xrightarrow{f \oplus g} A^- \oplus B^+ \oplus B^- \oplus C^+ \cong A^- \oplus C^+ \oplus B^- \oplus B^+$$

in \mathcal{Q} .

Abramsky and Coecke suggest to work with $\mathcal{Q} = (\text{Rel}, \times, 1)$ or $\mathcal{Q} = (\text{FDVec}, \otimes, \mathbb{C})$ where a trace of $f : X \otimes Z \rightarrow Y \otimes Z$ is given by

$$x \text{Tr}(f) y \quad \text{iff} \quad \exists z \in Z (x, z) f (y, z)$$

and

$$\text{Tr}(f)(v_i) = \sum_k f_{ikjk} \cdot w_j$$

where $f(v_i \otimes e_k) = \sum_{j, \ell} f_{ikj\ell} \cdot w_j \otimes e_\ell$ with $\{v_i\}$, $\{w_j\}$ and $\{e_k\}$ orthonormal bases of X , Y and Z respectively. Here the sum is not a problem as it is finite!

114 Kleene's definition of Countable Functionals in his 1959 paper "Countable Functionals"

For $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ we define $\alpha|\beta = n$ iff $\exists m (\alpha(\bar{\beta}(n)) = m+1 \wedge \forall k < m \alpha(\bar{\beta}(k)) = 0)$. The family of pure types is defined inductively as follows: $Tp(0) = \mathbb{N}$ and $Tp(n+1) = \mathbb{N}^{Tp(n)}$, the set of *all* set-theoretic functions from $Tp(n)$ to \mathbb{N} .

In [Kl59] he defined for every $n \in \mathbb{N}$ a subset $Ct(n)$ of $Tp(n)$ of "countable functionals" of type n as follows

- (1) $Ct(0) = Tp(0)$
- (1) $Ct(1) = Tp(1)$ and for $\alpha \in Ct(1)$ the set of its associates is $\{\alpha\}$
- (2) for $n > 1$ an $F \in Tp(n)$ is in $Ct(n)$ iff F has an associate $\alpha \in \mathbb{N}^{\mathbb{N}}$ meaning that for every $f \in Ct(n-1)$ and every associate β of f it holds that $F(f) = \alpha|\beta$.

Obviously, the hierarchy $(Ct(n))_{n \in \mathbb{N}}$ coincides with the pure types in $RT(K_2) \downarrow \nabla$ where $\Gamma \dashv \nabla : \mathbf{Set} \leftrightarrow RT(K_2)$. The extensional collapse of this hierarchy in $RT(K_2) \downarrow \nabla$ coincides with the hierarchy of pure type in $RT(K_2)$ which nowadays is referred to as the countable functionals.

115 Exact and Regular Completions

Let \mathcal{C} be a category with finite limits. Its *exact completion* \mathcal{C}_{ex} has as objects pseudo-equivalence relations $r_0, r_1 : R \rightarrow X$ (not jointly monic in general) and as morphisms from $r_0, r_1 : R \rightarrow X$ to $s_0, s_1 : S \rightarrow Y$ equivalence classes $[f]$ of maps $f : X \rightarrow Y$ such that there exists a map $\bar{f} : R \rightarrow S$ with $f \circ r_i = s_i \circ \bar{f}$ for $i=0, 1$ where f and f' are equivalent iff there exists a map $g : X \rightarrow S$ with $f = s_0 \circ g$ and $f' = s_1 \circ g$. One embeds \mathcal{C} into \mathcal{C}_{ex} via a functor $y : \mathcal{C} \rightarrow \mathcal{C}_{\text{ex}}$ sending X to the span $\text{id}_X, \text{id}_X : X \rightarrow X$ and $f : X \rightarrow Y$ to the equivalence class $[f]$. One can show that for every exact category \mathcal{E} and finite limit preserving functor $F : \mathcal{C} \rightarrow \mathcal{E}$ there exists an (up to isomorphism) exact functor $\bar{F} : \mathcal{C}_{\text{ex}} \rightarrow \mathcal{E}$ with $f = \bar{F} \circ y$. One can show that an exact category \mathcal{E} is an exact completion of a finite limit category iff its full subcategory \mathcal{C} of (regular) projectives is closed under finite limits and every object in \mathcal{E} is covered by a (regular) projective in which case $\mathcal{E} \simeq \mathcal{C}_{\text{ex}}$.

The *regular completion* \mathcal{C}_{reg} of a finite limit category is constructed similarly but where objects are regular equivalence relations. Instead of regular equivalence relations on X one may work with maps $p : X \rightarrow U$ whose kernel pair induces a regular equivalence relation on X . So we may take as objects of \mathcal{C}_{reg} maps $h : X \rightarrow U$ and as morphisms from $p : X \rightarrow U$ to $q : Y \rightarrow V$ equivalence classes $[f]$ of maps $f : X \rightarrow Y$ with $qfr_0 = qfr_1$ (where r_0, r_1 is the kernel pair of p) where f and f' are equivalent iff $qf = qf'$. One can show that a regular category \mathcal{E} is a regular completion iff its full subcategory \mathcal{C} of (regular) projectives is closed under finite limits and every object of \mathcal{E} is covered by an object in \mathcal{C} and appears as subobject of some object in \mathcal{C} in which case $\mathcal{E} \simeq \mathcal{C}_{\text{reg}}$.

Exact and regular completions are a possible way of constructing realizability toposes and categories of assemblies. Let \mathcal{A} be a pca and $\mathbf{PAsm}(\mathcal{A})$ and $\mathbf{Asm}(\mathcal{A})$ be the categories of *partitioned assemblies* and *assemblies* over \mathcal{A} , respectively. Due to a seminal paper by E. Robinson and G. Rosolini one gets the realizability topos $\mathbf{RT}(\mathcal{A})$ as $\mathbf{PAsm}(\mathcal{A})_{\text{ex}}$ and $\mathbf{Asm}(\mathcal{A})$ as $\mathbf{PAsm}(\mathcal{A})_{\text{reg}}$. Later M. Menni has shown that $\mathbf{Asm}(\mathcal{A})_{\text{ex}}$ is also a topos, namely the topos given by the tripos $I(\bar{\mathcal{A}})^{(-)}$ over \mathbf{Set} where $\bar{\mathcal{A}}$ is the ordered pca of nonempty subsets of \mathcal{A} and $I(\bar{\mathcal{A}})$ consists of downward closed subsets of $\bar{\mathcal{A}}$. The topos $\mathbf{Asm}(\mathcal{A})_{\text{ex}}$ was introduced by J. van Oosten in his paper *Extensional Realizability* (APAL 1997) where he showed that the extensional realizability topos $\mathbf{Ext}(\mathcal{A})$ as introduced by R. Grayson appears as subtopos of $\mathbf{Asm}(\mathcal{A})_{\text{ex}}$. The topos $\mathbf{Ext}(\mathcal{A})$ is induced by the subtripos $J(\bar{\mathcal{A}})^{(-)}$ of $I(\bar{\mathcal{A}})^{(-)}$ where $J(\bar{\mathcal{A}})$ consist of those elements of $I(\bar{\mathcal{A}})$ which are closed under pushouts and directed unions, i.e. downward closures in $I(\bar{\mathcal{A}})$ of \mathcal{A}/R for some per R on \mathcal{A} .⁶⁶

Menni's proof that $\mathbf{Asm}(\mathcal{A})_{\text{ex}}$ is a topos employs his characterization of finite limit categories whose exact completion is a topos as those finite limit categories which have weak dependent products and a *generic proof*, i.e. a map $\theta : \Theta \rightarrow \Lambda$ such that for every map $f : Y \rightarrow X$ there is a map $\nu_f : X \rightarrow \Lambda$ such that there are maps $f \rightarrow \nu_f^* \theta$ and $\nu_f^* \theta \rightarrow f$ in the slice over X . Since $\mathbf{Asm}(\mathcal{A})$

⁶⁶Alas, in this paper and also in his book he forgot to mention closure under directed unions. But he confirmed to me privately that closure under directed unions has to be added.

is locally cartesian closed it has in particular weak dependent products. A generic proof for $\mathbf{Asm}(\mathcal{A})$ can be obtained as follows. Let $\Lambda = \nabla(\mathcal{P}(\mathcal{P}(\mathcal{A})))$ and Θ be the assembly whose underlying set is $\{(u, U) \mid \emptyset \neq u \in U\}$ and with $\|(u, U)\| = u$. The map θ sends (u, U) to U . For $f : Y \rightarrow X$ in $\mathbf{Asm}(\mathcal{A})$ take $\nu_f(x) = \{\|y\| \mid f(y) = x\}$.⁶⁷

Notice that the construction of the exact completion is reminiscent of the *setoid* construction in type theory. The existence of a generic proof in (a model of) type theory reminds one of Aczel's Axiom claiming that (in all contexts) for all types A there is an $a \in U$ such that there are maps $A \rightarrow \mathbf{El}(a)$ and $\mathbf{El}(a) \rightarrow A$. But, alas, the universe U as given by $\mathbf{Mod}(\mathcal{A})$ does presumably NOT validate Aczel's axiom.⁶⁸ Still, under the assumption of Aczel's axiom the setoid construction gives rise to a topos.

There is also an ex/reg-completion which to every regular category \mathcal{C} associates the exact category $\mathcal{C}_{\text{ex/reg}}$ whose objects are pairs $((X, R))$ where X is an object of \mathcal{C} and $R \rightrightarrows X \times X$ is an equivalence relation on X and morphism from (X, R) to (Y, S) are relations $F \rightrightarrows X \times Y$ such that $x'Rx \wedge xFy \wedge ySy' \vdash x'Fy'$, $xFy \wedge xFy' \vdash ySy'$ and $\exists y:Y. xFy$.⁶⁹ Composition in $\mathcal{C}_{\text{ex/reg}}$ is relational composition in \mathcal{C} and the identity on (X, R) is given by R . Carboni, Freyd and Scedrov have shown that $\mathbf{Asm}(\mathcal{A})_{\text{ex/reg}}$ is equivalent to $\mathbf{RT}(\mathcal{A})$.⁷⁰

The construction of \mathcal{C}_{ex} can be performed in two steps namely as regular completion followed by ex/reg completion. This is informative in case of $\mathcal{C} = \mathbf{Asm}(\mathcal{A})$ where $\mathbf{Asm}(\mathcal{A})_{\text{reg}}$ can be obtained as follows: objects are pairs (X, R) where $X \in \mathbf{Asm}(\mathcal{A})$ and R is a per on $|X|$ and morphisms from (X, R) to (Y, S) are morphisms $f : X \rightarrow Y$ with $f(x)Sf(y)$ whenever xRy where f and g will be identified iff $f(x)Sg(x)$ for $x \in |X|$.

116 Quotients and Choice imply Classical Logic and Maietti's Question

In their draft for a book on type theory Coquand, Dybjer, Palmgren and Setzer have come up with the following proof. Let P be a proposition. On N_2 define the equivalence relation $a \sim b \equiv a = b \vee P$. Let $q : N_2 \rightarrow Q$ be a quotient of N_2 modulo \sim . By Choice there exists a function $s : Q \rightarrow N_2$ with $q \circ s = \text{id}_Q$. Since equality on N_2 is decidable we have $s(0) =_{N_2} s(1) \vee \neg s(0) =_{N_2} s(1)$. In the first case since s is monic it follows that $0 =_Q 1$, i.e. $0 \sim 1$ and thus P . In the second case we have $\neg 0 =_Q 1$, i.e. $\neg 0 \sim 1$ and thus $\neg P$. Thus from choice for Q it follows that $P \vee \neg P$.

⁶⁷For $\mathbf{PAsm}(\mathcal{A})$ a generic proof is constructed similarly: the underlying set of Λ is $\mathcal{P}(\mathcal{A})$ together with some constant map to \mathcal{A} , the underlying set of Θ is $\{(a, u) \mid a \in u \in \mathcal{P}(\mathcal{A})\}$ with $\|(a, u)\| = a$ and $\theta(a, u) = a$. For $f : Y \rightarrow X$ in $\mathbf{PAsm}(\mathcal{A})$ one takes $\nu_f(x) = \{\|y\| \mid f(y) = x\}$.

⁶⁸In Menni's argument sketched above Λ is $\nabla(\mathcal{P}(\mathcal{P}(\mathcal{A})))$ and *not* $\nabla(\mathbf{PER}(\mathcal{A}))$!

⁶⁹Notice that in general such an F need not contain the graph $\langle \text{id}_X, f \rangle$ of some morphism $f : X \rightarrow Y$.

⁷⁰**Question** In $\mathbf{Asm}(\mathcal{A})$ for every strongly inaccessible cardinal κ there exists a generic κ -small family. Does that imply that also in $\mathbf{RT}(\mathcal{A}) \simeq \mathbf{Asm}(\mathcal{A})_{\text{ex/reg}}$ there exists a generic κ -small family?

This proof is a considerable simplification of Diaconescu’s proof that toposes with choice are boolean. In particular there is no reference to power types whatsoever.

M. Maietti asked the question whether power types together with choice imply classical logic if one does not postulate the principle of *propositional extensionality* claiming that equivalent propositions or predicates are equal. One might rephrase this as the question whether $\text{ECC} + \text{AC} \vdash \text{PEM}$. This is essentially unsolved but there is the following observation of Benno van den Berg. From section 115 we know that $\text{RT}(\mathcal{A})$ is a subtopos of $\mathbf{Asm}(\mathcal{A})_{\text{ex}}$. Pulling back the subobject fibration of $\mathbf{Asm}(\mathcal{A})_{\text{ex}}$ along the inclusion $\mathbf{Asm}(\mathcal{A}) \hookrightarrow \text{RT}(\mathcal{A}) \hookrightarrow \mathbf{Asm}(\mathcal{A})_{\text{ex}}$ we get a tripos⁷¹ over $\mathbf{Asm}(\mathcal{A})$ which validates choice since this tripos is equivalent to the poset reflection of the fundamental fibration of $\mathbf{Asm}(\mathcal{A})_{\text{ex}}$ which validates AC since existential quantification is interpreted via Σ . Thus $\text{HAH} + \text{AC}$ does not imply classical logic. Notice, however, that van den Berg’s tripos does not validate Lawvere comprehension and thus cannot be used for interpreting ECC (Extended Calculus of Constructions).

117 Axiom of Choice in Toposes

A topos \mathcal{E} validates AC iff every epi is split. Since in a topos epis are stable under pullbacks AC is equivalent to the requirement that for every epi $e : X \rightarrow Y$ and object A the function $\mathcal{E}(A, e) : \mathcal{E}(A, X) \rightarrow \mathcal{E}(A, Y)$ is onto. One can show that for toposes AC is stable under slicing. Suppose \mathcal{E} is a topos validating AC and $I \in \mathcal{E}$. For showing that \mathcal{E}/I satisfies AC suppose $e : a \rightarrow b$ is epic in \mathcal{E}/I . Since $\Sigma_I : \mathcal{E}/I \rightarrow \mathcal{E}$ is a left adjoint it preserves regular epis and thus all epis. Thus $e : A \rightarrow B$ is epic in \mathcal{E} and, therefore, there exists a section s of e in \mathcal{E} . But we also have $s : b \rightarrow a$ since $a \circ s = b \circ e \circ s = b$ and thus $s : b \rightarrow a$ is a section of $e : a \rightarrow b$ in \mathcal{E}/I .

A topos \mathcal{E} validates IAC (the internal AC) iff for every object A the functor $(-)^A$ preserves epis. Since every functor preserves split epis AC entails IAC. The reverse implication does not hold since for a nontrivial group \mathbb{G} the presheaf topos $\widehat{\mathbb{G}}$ validates IAC but not AC since the terminal projection of the representable object is epic but has no splitting.⁷² However, for a topos satisfying AC is equivalent to the requirement that it validates IAC and 1 is projective.

R. Diaconescu (1975) has shown that a topos validating IAC is boolean. Suppose $P \in \Omega$. We define $A_0, A_1 \in \mathcal{P}(2)$ as

$$A_0 = \{x \in 2 \mid (x = 0 \wedge P) \vee x = 1\} \quad A_1 = \{x \in 2 \mid x = 0 \vee (x = 1 \wedge P)\}$$

⁷¹there is a generic proposition since $\mathbf{Asm}(\mathcal{A})$ has a generic proof as shown in section 115

⁷²In some toposes there are even K -finite objects A for which $(-)^A$ does not preserve epis. Consider for example the Sierpiński topos \mathbf{Set}^2 for which we have $\Delta \dashv \Gamma \dashv \nabla : \mathbf{Set} \hookrightarrow \mathbf{Set}^2$. Let A be the object in \mathbf{Set}^2 with $A(0) = \{0, 1\}$ and $A(1) = \{*\}$. Let $B = \Delta(2)$ and e the obvious epi from B to A whose component at 0 is the identity on $2 = \{0, 1\}$. Notice that $\Gamma(B^A)$ contains 2 elements whereas $\Gamma(A^A)$ contains 4 elements. Thus $\Gamma(e^A)$ is not epic from which it follows that e^A is not epic either (since Γ has a right adjoint and thus preserves epis). Notice that A is K -finite since $A(0) \rightarrow A(1)$ is surjective.

However, if A is finite, i.e. is K -finite and has decidable equality, then $(-)^A$ preserves epis.

and put $A = \{A_0, A_1\}$. Obviously, for all $X \in A$ there exists $x \in 2$ with $x \in X$. Thus by IAC there is a function $f : A \rightarrow 2$ with $F(X) \in X$ for all $X \in A$. We have either $f(A_0) = 0$ or $f(A_0) = 1$. If $f(A_0) = 0$ then $0 \in A_0$ and thus P . So let us suppose w.l.o.g. that $f(A_0) = 1$. We have either $f(A_1) = 0$ or $f(A_1) = 1$. If $f(A_1) = 1$ then $1 \in A_1$ and thus P . So let us suppose w.l.o.g. that $f(A_1) = 0$. Suppose P holds. Then $A_0 = \{0, 1\} = A_1$ and thus $A_0 = A_1$ from which it follows that $0 = f(A_1) = f(A_0) = 1$ which clearly is impossible. So we conclude $\neg P$. Thus we have shown that in any case $P \vee \neg P$ holds. Thus the topos is boolean.

Using this one can show that a locally small cocomplete topos \mathcal{E} validates AC iff it is equivalent to one of the form $\mathbf{Sh}(B)$ for a complete boolean algebra B . It is well known that for a complete boolean algebra B the topos $\mathbf{Sh}(B)$ validates AC. For the reverse direction suppose that \mathcal{E} is a locally small cocomplete topos satisfying AC. Then by Diaconescu's result \mathcal{E} is boolean. We next show that in \mathcal{E} subobjects of 1 generate. Let $f, g : A \rightarrow B$ with $f \neq g$. Then the equalizer E of f and g is a proper subobject of A . Thus its complement E' is different from 0 and so is the support U of E' . By AC there is a map $s : U \rightarrow E'$. Composing with the inclusion of E' into A we obtain a map $a : U \rightarrow A$. Suppose $fa = ga$. Then a factors through E and E' and thus through 0 . But then U is isomorphic to 0 contradicting our assumption. Thus we have shown that $fa \neq ga$. Since in \mathcal{E} subobjects of 1 generate \mathcal{E} is equivalent to $\mathbf{Sh}(B)$ where $B = \mathbf{Sub}_{\mathcal{E}}(1)$.

118 Wellpointed Toposes

Let \mathcal{E} be a wellpointed topos. Suppose $m : U \rightarrow 1$ with U not initial. Then there exists two different maps $f, g : U \rightarrow \Omega$ whose equaliser is the initial object. Since \mathcal{E} is wellpointed there exists $u : 1 \rightarrow U$ with $fu \neq gu$ and thus $U \cong 1$. Thus, using PEM on the metalevel we have shown that \mathcal{E} is 2-valued, i.e. the only global elements of Ω are \top and \perp . Since $\neg\neg\top = \top$ and $\neg\neg\perp = \perp$ it follows by wellpointedness that $\neg\neg = \text{id}_{\Omega}$, i.e. that \mathcal{E} is boolean.

For a pca \mathcal{A} the category $\mathbf{Asm}(\mathcal{A})$ is well pointed. Since in general subobject lattices of $\mathbf{Asm}(\mathcal{A})$ are not boolean it follows that $\mathbf{Asm}(\mathcal{A})$ is not a topos (though it comes close to it since it is equivalent to the category of $\neg\neg$ -separated objects of $\mathcal{E}ff(\mathcal{A})$). If $\mathbf{Asm}(\mathcal{A})$ is boolean then it has a subobject classifier given by $\nabla(2)$ and thus is a boolean topos from which it follows that $\nabla(2) \cong 2 = 1+1$ which, however, entails that \mathcal{A} is trivial.

If a realizability topos $\mathbf{RT}(\mathcal{A})$ is boolean then it is equivalent to $\mathbf{Asm}(\mathcal{A})$ and thus \mathcal{A} is trivial by the observation above. If $\mathbf{RT}(\mathcal{A})$ is wellpointed then it is boolean and thus \mathcal{A} is trivial.

It is well known that Grothendieck toposes can be characterized as those locally small toposes which have copowers of 1 and a small generating family. Thus, a wellpointed locally small topos \mathcal{E} has copowers of 1 iff \mathcal{E} is a Grothendieck topos. Actually, in this case \mathcal{E} is a localic topos and since $\mathbf{Sub}_{\mathcal{E}}(1)$ has precisely two elements it holds that $\mathcal{E} \simeq \mathbf{Set}$.

Forcing models are toposes $\mathbf{Sh}(B)/U$ where B is a complete boolean algebra and U is an ultrafilter on B . Such toposes are wellpointed and locally small. Accordingly, a forcing model $\mathbf{Sh}(B)/U$ is a Grothendieck topos if and only if $\mathbf{Sh}(B)/U$ is equivalent to \mathbf{Set} . In case \mathbf{Set} validates the Continuum Hypothesis CH then forcing models $\mathbf{Sh}(B)/U$ not validating CH cannot be Grothendieck toposes as otherwise they were equivalent to \mathbf{Set} and would validate CH (since CH is a statement of higher order arithmetic whose truth is preserved by any equivalence).

119 Ω is Dedekind finite but need not be K -finite

Let $\top : 1 \rightarrow \Omega$ be the subobject classifier in a topos. Then every monic $f : \Omega \rightarrow \Omega$ is an isomorphism.⁷³ Consider the diagram

$$\begin{array}{ccccccc}
 V & \xlongequal{\quad} & V & \longrightarrow & U & \longrightarrow & 1 \\
 \downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow g \lrcorner & & \downarrow \top \\
 U & \xrightarrow{\quad} & 1 & \xrightarrow{\quad} & \Omega & \xrightarrow{\quad} & \Omega \\
 & \downarrow \text{!}_U & & \downarrow \top & & & \downarrow f
 \end{array}$$

from which it follows that $ffg = f\top!_U = g$ since both maps classify $V \rightarrow U$. But then the square

$$\begin{array}{ccc}
 U & \xlongequal{\quad} & U \\
 g \downarrow \lrcorner & & \downarrow g \\
 \Omega & \xrightarrow{\quad} & \Omega \\
 & \downarrow ff &
 \end{array}$$

is a pullback since ff is monic. Thus $f = fff$ since both maps classify g from which it follows that $ff = \text{id}_\Omega$, i.e. f is an isomorphism (with $f^{-1} = f$).

This shows that in a topos Ω is always *Dedekind finite*, i.e. all monic endomap on Ω are isomorphisms. On the other hand e.g. in the topos \mathbf{Set}^ω of “sets developing in discrete time”, the subobject classifier is not K -finite (since $\Omega(n)$ is infinite for all $n \in \omega$).⁷⁴ Thus, in a topos there may be Dedekind finite objects which are not K -finite.

120 For the free topos \mathcal{F} with nno the Heyting lattice $\text{Sub}_{\mathcal{F}}(1)$ is not complete (D. Scott)

Notice that $\text{Sub}_{\mathcal{F}}(1)$ is the Lindenbaum-Tarski algebra of Higher Order Heyting Arithmetic HAH. If $\text{Sub}_{\mathcal{F}}(1)$ were complete then $\text{Sub}_{\mathcal{F}}(1)_{\neg\neg} \cong \text{Sub}_{\mathcal{F},\neg\neg}(1)$ were

⁷³The map f need not be the identity map as e.g. $\neg : \Omega \rightarrow \Omega$ in a boolean topos.

⁷⁴Notice, however, that in the topos of trees $\mathbf{Set}^{\omega, \text{op}}$ the subobject classifier Ω is K -finite since all $\Omega(n)$ is finite and all transition maps are onto.

complete as well. Since $\mathcal{F}_{\neg, \neg}$ is the free boolean topos with nno the boolean lattice $\mathbf{Sub}_{\mathcal{F}_{\neg, \neg}}(1)$ is the Lindenbaum-Tarski algebra of Higher Order Peano Arithmetic (PAH) which is countable and non-atomic (by Gödel's Second Incompleteness Theorem). But countable non-atomic boolean algebras are all isomorphic to $\mathbf{Clp}(2^\omega)$, the clopen subsets of Cantor space, which is known to be not complete (since every open set is a union of clopen sets). Thus, we have derived a contradiction from the assumption that $\mathbf{Sub}_{\mathcal{F}}(1)$ is complete.

121 Remarks on Realizability Toposes

121.1 $\mathbf{Mod}(\mathcal{A})$ is not internally complete in $\mathbf{RT}(\mathcal{A})$

The object $2 = 1+1$ is modest and thus $!_2 : 2 \rightarrow 1$ is a family of modest sets. We will show that $d_2 = \Pi_{\top}!_2 : \tilde{2} \rightarrow \Omega$ (classifying $\eta_2 : 2 \rightarrow \tilde{2}$) is not (even) a family of (quasi-)modest sets. Thus $\mathbf{Mod}(\mathcal{A})$ (and also $\mathbf{QMod}(\mathcal{A})$) is not internally complete in $\mathbf{RT}(\mathcal{A})$.

Suppose d_2 were a family of (quasi-)modest sets. Then there is a diagram

$$\begin{array}{ccc} R & \longrightarrow & \Omega \times \Delta(\mathcal{A}) \\ \sigma \downarrow & & \downarrow \pi \\ \tilde{2} & \xrightarrow{d_2} & \Omega \end{array}$$

with $R \subseteq_{\neg\neg} \Omega \times \Delta(\mathcal{A})$. Thus, in $\mathbf{RT}(\mathcal{A})$ from $u = \sigma(p, a)$ it follows that $d_2(u) = p$. As σ is epic it holds that $\forall u \in \tilde{2}. \exists a \in \Delta(\mathcal{A}). u = \sigma(d_2(u), a)$. Thus, as $\mathbf{RT}(\mathcal{A})$ validates the uniformity principle for $\Delta(\mathcal{A})$ it holds that $\exists a \in \Delta(\mathcal{A}). \forall u \in \tilde{2}. u = \sigma(d_2(u), a)$. Thus, there is an $a \in \mathcal{A}$ such that $\forall u \in \tilde{2}. u = \sigma(d_2(u), a)$ from which it follows that $\eta_2(0) = \sigma(d_2(\eta_2(0)), a) = \sigma(\top, a) = \sigma(d_2(\eta_2(0)), a) = \eta_2(1)$. As η_2 is monic this implies $0 = 1$. Contradiction!

Notice, however, that $\mathbf{Mod}(\mathcal{A})$ (and also $\mathbf{QMod}(\mathcal{A})$) is internally complete w.r.t. $\mathbf{Asm}(\mathcal{A})$.

121.2 On the size of projective covers of $\Omega_{\mathbf{RT}(\mathcal{A})}$

Let $e : C \rightarrow \Omega$ with C projective in $\mathbf{RT}(\mathcal{A})$. We have also $q : \nabla(\mathcal{P}(\mathcal{A})) \rightarrow \Omega$ with $\llbracket q(b) = a \rrbracket = b \leftrightarrow a$ for $a, b \in \mathcal{P}(\mathcal{A})$. As C is projective there exists $f : C \rightarrow \nabla(\mathcal{P}(\mathcal{A}))$ with $q \circ f = e$. Thus, we have

$$\llbracket e(c) = a \rrbracket = f(c) \leftrightarrow a$$

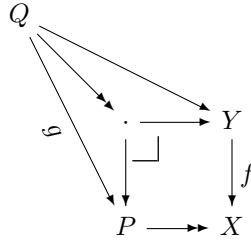
uniformly in $c \in |C|$ and $a \in \mathcal{P}(\mathcal{A})$. Thus, since e is epic it holds that $\forall a : \Omega. \exists c : C. f(c) \leftrightarrow a$. From a realizer for this proposition we can extract $e_1, e_2 \in \mathcal{A}$ such that for all $a \in \mathcal{P}(\mathcal{A})$ there exists $c_a \in |C|$ with $e_1 \Vdash a \rightarrow f(c_a)$ and $e_2 \Vdash f(c_a) \rightarrow a$. Instantiating a by \mathcal{A} we get that $e_1 e \downarrow$ for all $e \in \mathcal{A}$ and instantiating a by $\{e\}$ we get that $e_2(e_1 e) = e$. Thus, for $a \in \mathcal{P}(\mathcal{A})$ and $e \in \mathcal{A}$ we have $e_1 e \in f(c_a)$ iff $e \in a$. Accordingly, for different $a, a' \in \mathcal{P}(\mathcal{A})$ we have $\{e \in \mathcal{A} \mid e_1 e \in f(c_a)\} = a \neq a' = \{e \in \mathcal{A} \mid e_1 e \in f(c_{a'})\}$ from which it follows that $f(c_a) \neq f(c_{a'})$ and thus $c_a \neq c_{a'}$. Thus, the assignment $a \mapsto c_a$ gives rise to an injective function from $\mathcal{P}(\mathcal{A})$ into $|C|$, i.e. $2^{|\mathcal{A}|} \leq |C|$.

With almost the same argument one shows that in $\mathbf{RT}(\mathcal{A})$ every projective cover of the object $\Omega^+ = \{p \in \Omega \mid \neg\neg p\}$ is at least of size $2^{|\mathcal{A}|}$. Thus, for nontrivial pca's \mathcal{A} (which are necessarily infinite) the object Ω^+ in $\mathbf{RT}(\mathcal{A})$ is *not* \aleph_0 -small since every projective cover of Ω^+ contains at least $2^{|\mathcal{A}|} \geq \aleph_0$ elements. Thus in nontrivial realizability toposes there exist objects which are not \aleph_0 -small although they admit only one global element.

Question Does this generalize to arbitrary strongly inaccessible cardinals κ ?

122 Small Maps in Realizability Toposes

Let \mathcal{A} be a pca (e.g. K_1 corresponding to Kleene's number realizability) and $\mathcal{E} = \text{RT}(\mathcal{A})$ be the ensuing realizability topos. Joyal and Moerdijk in their booklet [JM95] on *Algebraic Set Theory* (AST) have defined for every strongly inaccessible⁷⁵ cardinal κ a class \mathcal{S}_κ of so-called κ -small maps consisting of all maps $f : Y \rightarrow X$ such that there is a diagram



where P and Q are projective and $|\Gamma(g)^{-1}(p)| < \kappa$ in \mathbf{Set} for all $p \in \Gamma(P)$. Joyal and Moerdijk have shown that \mathcal{S}_κ satisfies their axioms required for classes of small maps. In particular they have shown that \mathcal{S}_κ satisfies the following two properties (where we write \mathcal{S} as abbreviation for \mathcal{S}_κ)

Descent If e is a (regular) epi and $e^*f \in \mathcal{S}$ then $f \in \mathcal{S}$.

Representability There exist a map $\pi : E \rightarrow U$ in \mathcal{S} such that every map $f : Y \rightarrow X$ in \mathcal{S} fits into a diagram

$$\begin{array}{ccccc} Y & \longleftarrow & D & \longrightarrow & E \\ f \downarrow & & \downarrow & & \downarrow \pi \\ X & \longleftarrow & C & \longrightarrow & U \\ & & e & & \end{array}$$

where both square are pullbacks and e is a (regular) epi.

A map π validating this requirement will be called **weakly generic** for \mathcal{S} .

Obviously, from Representability together with Descent it follows that a map f is in \mathcal{S} iff there exists a regular epi e such that e^*f can be obtained as pullback of π iff (as follows by Kripke-Joyal)

$$(\forall x:X)(\exists a:U_\kappa) Y_x \cong E_a$$

holds in the internal language of \mathcal{E} .

In this note we discuss the following

Open Problem

Does there exist a **generic** map for \mathcal{S} , i.e. a map $\pi : E \rightarrow U$ in \mathcal{S} such that every $f : Y \rightarrow X$ in \mathcal{S} (and not just some pullback of f along some (regular) epi) can be obtained as pullback of π along some map from X to U .

⁷⁵i.e. κ is regular and $2^\lambda < \kappa$ whenever $\lambda < \kappa$

Obviously, if $\pi : E \rightarrow U$ is generic for \mathcal{S} then a map $f : Y \rightarrow X$ is in \mathcal{S} iff there exists a map $g : X \rightarrow U$ such that $(\forall x:X) Y_x \cong E_{g(x)}$ holds in the internal language of \mathcal{E} . Notice that the existence of a such a map g is stronger than the mere requirement that $(\forall x:X)(\exists a:U) Y_x \cong E_a$ since typically \mathcal{E} will not validate an appropriate axiom of choice.

Thus, since \mathcal{E} in general does not validate sufficiently strong choice principles there is no reason to assume that in general there do exist generic small maps. Since realizability toposes validate only a very restricted kind of choice there is no reason why they should admit generic κ -small maps. However, since the existence of generic small maps is *not equivalent* to the full axiom of choice the Open Problem is not settled yet!

Actually, the question appears as a quite natural one if – like the author of this note – looks at it from the point of view of categorical semantics for (dependent) type theories where type theoretic universes are axiomatized as classes \mathcal{S} of so-called “display” maps which are required to be stable under pullbacks, closed under various type theoretic constructions (like dependent sums Σ and dependent products Π) and for which there exists a generic map $\pi : E \rightarrow U$ for \mathcal{S} . **But** on the other hand in categorical (semantics of) type theory the descent axiom is usually not considered at all. Nevertheless descent does hold in the most interesting realizability models for type theory, namely categories $\mathbf{Asm}(\mathcal{A})$ of assemblies for a pca \mathcal{A} when taking for \mathcal{S} the class of those maps $f : Y \rightarrow X$ where each fibre of $\Gamma(f)$ has cardinality $< \kappa$ (where κ is some strongly inaccessible cardinal with $\mathcal{A} \in V_\kappa$). Moreover, for this \mathcal{S} there also exists a generic family $\pi : E \rightarrow U$ with $U = \nabla(\mathbf{Asm}_\kappa(\mathcal{A}))$ where $\mathbf{Asm}_\kappa(\mathcal{A})$ is the set of all assemblies X whose underlying set $|X|$ is an element of V_κ .

Thus, taking into account that $\mathbf{Asm}(\mathcal{A})$ appears as the full subcategory of $\mathbf{RT}(\mathcal{A})$ on $(\neg\neg)$ -separated objects it is quite natural by analogy to ask whether in $\mathbf{RT}(\mathcal{A})$ there exists a generic family for κ -small maps.

We first discuss what the Descent axiom is needed for and to which extent it is compatible with the existence of a generic small map. Then we discuss the question whether generic κ -small maps exist in realizability toposes.

122.1 The benefits of descent and its compatibility with the existence of a generic small map

In [JM95] it was shown that for every object A families of small subobjects of A are classified by a universal such family $\in_A \rightarrow A \times \mathcal{P}_\mathcal{S}(A)$. Joyal and Moerdijk construct $\mathcal{P}_\mathcal{S}(A)$ as $(\Sigma a:U)A^{E_a}$ modulo the equivalence relation

$$\langle a, f \rangle \sim \langle b, g \rangle \quad \equiv \quad (\forall i:E_a)(\exists j:E_b)f(i)=g(j) \ \& \ (\forall j:E_b)(\exists i:E_a)f(i)=g(j)$$

and define \in_A as

$$x \in_A [(a, f)]_\sim \quad \equiv \quad (\exists i:E_a)f(i)=a$$

For showing that $\in_A \rightarrow A \times \mathcal{P}_\mathcal{S}(A)$ is a family of small subobjects of A one needs the descent axiom for \mathcal{S} since one cannot choose representatives out of equivalence classes w.r.t. \sim .

In his LICS'99 paper [Si99] A. Simpson has shown that from the existence $\mathcal{P}_{\mathcal{S}}$ it follows that \mathcal{S} satisfies the descent axioms. Thus, the descent axiom is actually necessary for constructing $\mathcal{P}_{\mathcal{S}}$.

On the other hand in [Si99] it has also been shown that whenever $\mathcal{P}_{\mathcal{S}}(V) \rightarrow V$ then every $f : Y \rightarrow X$ in \mathcal{S} with $X \subseteq V$ can be obtained as pullback of $\in_V; \pi_2 : \in_V \rightarrow \mathcal{P}_{\mathcal{S}}(V)$ along some map $X \rightarrow \mathcal{P}_{\mathcal{S}}(V)$. Thus within the full subcategory \mathcal{E}_V of \mathcal{E} on subobjects of V there exists a generic small map, namely $\in_V; \pi_2 : \in_V \rightarrow \mathcal{P}_{\mathcal{S}}(V)$, where within \mathcal{E}_V the notion of smallness is given by $\mathcal{S}_V = \mathcal{E}_V \cap \mathcal{S}$.

Although not mentioned in [JM95] the descent axiom for \mathcal{S} has the (pleasant) consequence that the full subfibration $\mathcal{P}_{\mathcal{E}, \mathcal{S}}$ of the fundamental fibration $\mathcal{P}_{\mathcal{E}} = \partial_1 : \mathcal{E}^2 \rightarrow \mathcal{E}$ on those objects which are in \mathcal{S} is *definable* in the sense of Bénabou which can be seen as follows. For an arbitrary map $f : Y \rightarrow X$ in \mathcal{E} we may consider the following subobject

$$S = \{x \in X \mid (\exists a:U) Y_x \cong E_a\}$$

of X . The corresponding inclusion map $m : S \hookrightarrow X$ has to meet the following two requirements

- 1) m^*f is in \mathcal{S}
- 2) every map $g : W \rightarrow X$ with $g^*f \in \mathcal{S}$ factors through m (necessarily in a unique way).

Actually, the second requirement follows quite easily as follows. Suppose $g^*f \in \mathcal{S}$, i.e. $(\forall w:W)(\exists a:U) Y_{g(w)} \cong E_a$, from which it follows that $(\forall w:W) g(w) \in S$, i.e. that g factors through m . The first requirement that $m^*f \in \mathcal{S}$ can be seen as follows. By definition of S it holds that $(\forall x:S)(\exists a:U) Y_x \cong E_a$ which – in presence of the descent axiom – is equivalent to $m^*f \in \mathcal{S}$.

However, it is not clear why there should exist a map $h : S \rightarrow U$ with $g^*f \cong h^*\pi$ since this map would choose for every $x \in S$ an element $h(x) \in U$ with $Y_x \cong E_{h(x)}$ and it is not clear how this can be effected in general.

We discuss now how the Descent axiom may be used more directly for overcoming this drawback. We may consider the subobject Z of $(\Sigma x; X)(\Sigma a:U) Y_x^{E_a}$ consisting of all $(x, a, f) \in (\Sigma x: X)(\Sigma a: U) Y_x^{E_a}$ such that f is an isomorphism. Obviously, the image of $\text{pr}_1 : Z \rightarrow X$ is (isomorphic to) $m : S \hookrightarrow X$. Let $e : Z \rightarrow S$ be the unique map with $\text{pr}_1 = me$. Then, clearly, we have

$$\begin{array}{ccccc} E & \longleftarrow & \cdot & \longrightarrow & Y \\ \pi \downarrow & & \text{pr}_2^* f \downarrow & & m^* f \downarrow & & f \downarrow \\ U & \longleftarrow & Z & \xrightarrow{e} & S & \xrightarrow{m} & X \end{array}$$

where all squares are pullbacks from which it follows that m^*f is in \mathcal{S} . Although $\text{pr}_2^*\pi$ appears as pullback of π (along pr_2) it is not clear at all whether m^*f can be obtained as pullback of π along some map $h : S \rightarrow U$ since this would amount to a splitting of e .

Summarizing the discussion we observe that

1. descent is a desirable and often even necessary property of a class of small maps
2. in presence of a generic small map descent entails choice principles which are not always available in an intuitionistic context.
3. due to 1. and 2. it appears as necessary to weaken the assumption of a generic small map to the requirement of just the existence of a weakly generic map
4. in type theory (with a proof irrelevant representation of logic) descent has not been considered because it is concerned with type equality⁷⁶ rather than isomorphism of types as in the above applications of the descent property.

Nevertheless, in a wide range of situations small maps do admit a generic small map, namely κ -small maps in Grothendieck toposes over a site of cardinality $< \kappa$ as shown in [St05].

122.2 Generic κ -small maps in realizability toposes

As already observed in the previous section the technical difficulty is essentially the following. Suppose $\pi : E \rightarrow U$ is a weakly generic map for \mathcal{S}_κ , i.e. a map $Y \rightarrow X$ is in \mathcal{S}_κ iff $(\forall x:X)(\exists a:U) Y_x \cong \text{El}(a)$. Then it is not clear whether $(\forall x:X)(\exists a:U) Y_x \cong \text{El}(a)$ guarantees the existence of a map $g : X \rightarrow U$ with $g^*\text{El} \cong f$, i.e. $(\forall x:X) Y_x \cong \text{El}(g(x))$.

In [JM95] one can find a characterisation of κ -smallness which in a somewhat simplified but equivalent way can be formulated as follows. Let K be a set with $|K| = \kappa$, e.g. $K = V_\kappa$. Then map $f : Y \rightarrow X$ is κ -small iff the following statement folds in the internal logic of the realizability topos

$$\forall x \in X \exists P \in \mathcal{P}(\nabla(K)) [\exists e : P \twoheadrightarrow f^{-1}(x) \wedge \neg \exists e : \neg \neg P \twoheadrightarrow \nabla(K)]$$

i.e. iff every fibre $f^{-1}(x)$ can be covered by an assembly Q with $|\Gamma(Q)| < \kappa$. This observation gives rise to the following weakly generic map $\pi_\kappa : E_\kappa \rightarrow U_\kappa$ for \mathcal{S}_κ where U_κ is

$$\{(P, R) \in \mathcal{P}(\nabla(K)) \times \mathcal{P}(\nabla(K)^2) \mid \neg \exists e : \neg \neg P \twoheadrightarrow \nabla(K) \wedge R \text{ equiv. rel. on } P\}$$

and π_κ is the projection of $\coprod_{(P,R) \in U_\kappa} P/R$ on U_κ .

The underlying idea of this construction is to view⁷⁷ $\text{RT}_\kappa(\mathcal{A})$, the realizability topos constructed in V_κ , as a category internal to $\text{RT}(\mathcal{A})$ whose type of

⁷⁶In order to interpret type equality categorical models of type theory are required to be endowed with a functorial choice of pullbacks but has to drop the assumption that small display maps are closed under isomorphism. It is usually not required but holds in most models that for family of types $(X_i)_{i \in I}$ for a universe U the predicate $\{i \in I \mid X_i \in U\}$ exists as a predicate in the model.

⁷⁷here we tacitly assume that $\mathcal{A} \in V_\kappa$

objects is $U_\kappa \subseteq \mathcal{P}(\nabla(K)) \times \mathcal{P}(\nabla(K)^2)$. Actually, for this purpose it is appropriate to observe that the weakly generic family π_κ considered above is isomorphic to the following one.

The underlying set of U is the set of all $R \in \mathcal{P}(V_\kappa \times V_\kappa \times \mathcal{A}) \cap V_\kappa$ whose transpose $\tilde{R} : V_\kappa \times V_\kappa \rightarrow \mathcal{P}(\mathcal{A})$ is an equivalence relation, i.e. realizably symmetric and transitive. Writing $E(R)$ for the set of realizers for “ R symmetric and transitive” we define equality of U as follows

$$E_U(R_1, R_2) = E(R_1) \wedge E(R_2) \wedge \forall x, y \in V_\kappa [R_1(x, y) \leftrightarrow R_2(x, y)]$$

The underlying set of E consists of all pairs (R, x) such that $(x, x, a) \in R$ for some $a \in \mathcal{A}$ and

$$E_E((R_1, x_1), (R_2, x_2)) = E_U(R_1, R_2) \wedge R_1(x_1, x_2)$$

A further more type theoretic description of a weakly generic κ -small map starts from $\pi : \coprod_{A \in \mathbf{Asm}_\kappa(\mathcal{A})} A \rightarrow \nabla(\mathbf{Asm}_\kappa(\mathcal{A}))$, the canonical generic map for κ -small assemblies within $\mathbf{Asm}(\mathcal{A})$. Let $U_\kappa = \coprod_{A \in \nabla(\mathbf{Asm}_\kappa(\mathcal{A}))} \mathbf{ER}(A)$ where $\mathbf{ER}(A) = \{R \in \mathcal{P}(A^2) \mid R \text{ is an equiv. rel. on } A\}$. There is a canonical equivalence relation R_κ on

$$F_\kappa = \coprod_{A \in \mathbf{Asm}_\kappa(\mathcal{A})} A \times_{\nabla(\mathbf{Asm}_\kappa(\mathcal{A}))} \coprod_{A \in \nabla(\mathbf{Asm}_\kappa(\mathcal{A}))} \mathbf{ER}(A)$$

namely $R_\kappa((A_1, a_1, R_1), (A_2, a_2, R_2)) \equiv \bigcup \{R_1(a_1, a_2) \mid R_1 = R_2\}$ in the slice over U_κ giving rise to $E_\kappa = F_\kappa / R_\kappa$ as in

$$\begin{array}{ccc} E_\kappa & \longleftarrow F_\kappa & \longrightarrow \coprod_{A \in \mathbf{Asm}_\kappa(\mathcal{A})} A \\ & \searrow \cong & \downarrow \\ & & U_\kappa \longrightarrow \nabla(\mathbf{Asm}_\kappa(\mathcal{A})) \end{array}$$

Suppose $\pi : E \rightarrow U$ were a generic κ -small map, i.e. π is κ -small and all κ -small maps can be obtained as pullback of π . Then U cannot be $\neg\neg$ -separated for the following reason. The subobject classifier $\top : 1 \rightarrow \Omega$ is certainly κ -small. If one could obtain \top as pullback of π along some $f : \Omega \rightarrow U$ then $\forall p, q : \Omega (\neg\neg p = q \rightarrow p = q)$ which is certainly wrong since it would entail $\forall p : \Omega (\neg\neg p \rightarrow p)$ contradicting the fact that $\mathbf{RT}(\mathcal{A})$ is non-boolean for nontrivial pcas \mathcal{A} .

Thus, generic κ -small maps $\pi : E \rightarrow U$ cannot have separated codomain. For this reason one can always find a weakly generic κ -small map $\pi' : E' \rightarrow U'$ which is not generic for κ -small maps since if $\pi : E \rightarrow U$ is weakly generic and $e : U' \rightarrow U$ with U' projective then $\pi' = e^* \pi$ is weakly generic with separated codomain and thus cannot be strongly generic.

But, alas, this does not answer the question whether for all strongly inaccessible κ there exists *some* generic κ -small map $\pi_\kappa : E_\kappa \rightarrow U_\kappa$.

Now let us return to the particular case of $\text{RT}(\mathcal{A})$. A. Simpson has shown that $\text{RT}(\mathcal{A})$ is equivalent of $V(\mathcal{A})$, the McCarty realizability model of IZF. For a strongly inaccessible cardinal κ a weakly generic family $\pi_\kappa : E_\kappa \rightarrow U_\kappa$ is given by $U_\kappa = V_\kappa(\mathcal{A})$, the initial ZF-algebra for \mathcal{S}_κ , and $E_\kappa(a) = \{b \in V_\kappa \mid b \in a\}$. Now if $f : Y \rightarrow X$ is a κ -small map in $V(\mathcal{A})$ with $X, Y \subseteq V_\kappa(\mathcal{A})$ we can obtain f as pullback of π_κ along the map $\chi : X \rightarrow V_\kappa(\mathcal{A})$ defined $\chi(x) = f^{-1}(x) = \{y \in Y \mid f(y) = x\}$ for $x \in X$.

However, if κ' is a strongly inaccessible cardinal $> \kappa$ then we may consider $f = i^* \pi_{\kappa'}$ where $i : X \hookrightarrow V_{\kappa'}(\mathcal{A})$ is the inclusion of

$$X = \{a \in V_{\kappa'}(\mathcal{A}) \mid \exists b \in V_\kappa(\mathcal{A}) a \cong b\}$$

into $V_{\kappa'}$. Obviously, the map f κ -small but it is very unlikely that $f \cong g^* \pi_\kappa$ for some $g : X \rightarrow V_\kappa(\mathcal{A})$ since this g would choose for every $a \in X$ an element $g(a)$ out of the set $\{b \in V_\kappa(\mathcal{A}) \mid a \cong b\}$ which seems to require a kind of choice principle not validated by $V(\mathcal{A})$ (e.g. if $\mathcal{A} = K_1$). **It would be nice if one could actually disprove the existence of such a map g !**

Notice in case of $\mathbf{Asm}(\mathcal{A})$ the situation is quite different. For every strongly inaccessible cardinal κ we may consider $\pi_\kappa : E_\kappa \rightarrow U_\kappa$ where $U_\kappa = \nabla(\mathbf{Asm}_\kappa(\mathcal{A}))$, $|E_\kappa| = \coprod_{X \in \mathbf{Asm}_\kappa(\mathcal{A})} X$ with $\|(X, x)\|_{E_\kappa} = \|x\|_X$ and π_κ is the projection on the first component. A map $f : Y \rightarrow X$ arises as pullback of π_κ along some $\chi : X \rightarrow U_\kappa$ iff for all $x \in X$ the cardinality of the fibre $f^{-1}(x)$ is less than κ . Moreover, the class \mathcal{S}_κ of such maps is definable (in the sense of Bénabou) since for every map $f : Y \rightarrow X$ in $\mathbf{Asm}(\mathcal{A})$ we may consider the $\neg\neg$ -closed subobject $S = \{x \in X \mid |f^{-1}(x)| < \kappa\}$ of X for which one readily checks that $g : Z \rightarrow X$ factors through S iff $g^* f \in \mathcal{S}_\kappa$. Thus, in particular, the property of being isomorphic to an object in $\mathbf{Asm}_\kappa(\mathcal{A})$ is $\neg\neg$ -closed.

The reason why in $\mathbf{Asm}(\mathcal{A})$ things are so easy is that

- (i) an assembly X is isomorphic to an assembly in $\mathbf{Asm}_\kappa(\mathcal{A})$ if and only if $|X|$ has cardinality $< \kappa$ and
- (ii) if $|X|$ has cardinality $< \kappa$ then an isomorphic copy X' in $\mathbf{Asm}_\kappa(\mathcal{A})$ can be obtained quite easily by simply renaming the elements of $|X|$ into elements of the set $\text{card}(|X|)$ via some isomorphism between $|X|$ and $\text{card}(|X|)$ guaranteed by the classical metatheory in which AC available.

We **could** exhibit a generic family in \mathcal{S}_κ if we could prove the following

Conjecture 122.1 *If an object (X, E_X) in $\text{RT}(\mathcal{A})$ is isomorphic to some object in $\text{RT}_\kappa(\mathcal{A})$ then there exists an equivalence relation \sim on X satisfying the conditions*

- (1) $E_X(x_1, x_2) = E_X(x'_1, x'_2)$ whenever $x_1 \sim x'_1$ and $x_2 \sim x'_2$
- (2) $|X/\sim| < \kappa$

where $\text{RT}_\kappa(\mathcal{A})$ stands for the full subcategory of $\text{RT}(\mathcal{A})$ on those objects X in $\text{RT}(\mathcal{A})$ where the underlying set $|X|$ is an element of V_κ .

Using AC in the metatheory there exist maps $\iota : |X/\sim| \xrightarrow{\cong} X/\sim$ and $\gamma : |X/\sim| \rightarrow X$ with $a = [\gamma(a)]_R$ for all $a \in X/\sim$. Reindexing along $c_X = \gamma \circ \iota$ gives rise to an object $(|X/\sim|, E_X \circ (c_X \times c_X))$ in $\mathbf{RT}_\kappa(\mathcal{A})$ *trivially isomorphic* to (X, E_X) .

Actually, considering the unit $\eta_X : X \rightarrow \nabla(\Gamma(X))$ for which it holds that $\Gamma(\eta_X)$ is a bijection it **would** suffice to prove

Conjecture 122.2 *If (X, E_X) is an object in $\mathbf{RT}(\mathcal{A})$ with precisely one global element then there exists an equivalence relation \sim on X satisfying the conditions*

$$(1') \quad E_X(x_1, x_2) = E_X(x'_1, x'_2) \text{ whenever } x_1 \sim x'_1 \text{ and } x_2 \sim x'_2$$

$$(2') \quad |X/\sim| < \kappa.$$

because if (X, E_X) is isomorphic to some object in $\mathbf{RT}_\kappa(\mathcal{A})$ then $|X/\sim_X| < \kappa$ where $x \sim_X x'$ iff $E_X(x, x') \neq \emptyset$ and one can apply Conjecture 122.2 to the restrictions of (X, E_X) to the equivalence classes of \sim_X .

Notice that conditions (1) and (1') guarantee a sufficient amount of uniformity allowing one to **leave realizers unchanged**.

Alas — as pointed out by Benno van den Berg — Conjecture 122.2 is blatantly wrong as shown by the following counterexample. Let λ be a cardinal with $\kappa < \lambda$. Then we may define an $X \in \mathbf{RT}(\mathcal{A})$ with $|X| = \lambda$ and $E_X(\alpha, \beta) = \{0\} \cup \{1 \mid \alpha \leq \beta\}$. Then there cannot exist an equivalence relation \sim on $|X|$ with $E_X(x_1, x_2) = E_X(x'_1, x'_2)$ whenever $x_1 \sim x'_1$ and $x_2 \sim x'_2$ and $|X|/\sim < \kappa$ as otherwise there would exist $\alpha < \beta < \lambda$ with $\alpha \sim \beta$ and thus $\{0\} = E_X(\beta, \alpha) = E_X(\alpha, \beta) = \{0, 1\}$.

Thus, there is no easy way of **uniformly transforming** a κ -small object X in $\mathbf{RT}(\mathcal{A})$ into an isomorphic copy X' whose underlying set $|X'|$ has cardinality less than κ .

As noticed by Jaap van Oosten it is already not clear⁷⁸ whether every X in $\mathbf{RT}(\mathcal{A})$ with $|\Gamma(X)| = 1$ is isomorphic to some X' whose underlying set has cardinality less than κ . Of course, in van den Berg's counterexample above this is easily possible since it is obviously isomorphic to the terminal object. But there could be much more complicated objects X in $\mathbf{RT}(\mathcal{A})$ with $|\Gamma(X)| = 1$ for which such an argument is not possible. On the other hand it might be the case that every X with $|\Gamma(X)| = 1$ appears as subobject of $\Omega^+ = \{p \in \Omega \mid \neg\neg p\}$.

References

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⁷⁸but see Section 123 for a recent counterexample

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123 There are class many objects X in $\mathcal{E}ff$ with $|\Gamma(X)| = 1$ (Jonas Frey, Jan. 2008)

In January 2008 J. Frey has come up with a construction which for every cardinal κ exhibits an object X_κ in $\mathcal{E}ff$ with $|\Gamma(X_\kappa)| = 1$ and $|\mathcal{E}ff(\nabla(2), X_\kappa)| \geq \kappa$.

Put $|X_\kappa| = \kappa \times 2$ and $E_\kappa(\langle i, n \rangle, \langle j, m \rangle) = [\langle i, n \rangle = \langle j, m \rangle] \vee (\{n\} \times \{m\})$ (where $[\cdot = \cdot]$ stands for Lawvere equality, i.e. $[x = y] = \{0 \mid x = y\}$). Obviously, we have that

- (1) X_κ has precisely one global element and
- (2) X_κ arises quotient of $\nabla(\kappa \times 2)$ via the epi $e_\kappa : \nabla(\kappa \times 2) \twoheadrightarrow X_\kappa$ given by E_κ .

Thus every map $f : 2 \rightarrow \kappa \times 2$ induces the morphism $e_\kappa \circ \nabla(f) : \nabla(2) \rightarrow X_\kappa$ given by $F_f(n, \langle j, m \rangle) \equiv E_\kappa(f(n), \langle j, m \rangle)$.

For every $i \in \kappa$ let $f_i : 2 \rightarrow \kappa \times 2 : n \mapsto \langle i, n \rangle$ inducing the morphism $e_\kappa \circ \nabla(f_i) : \nabla(2) \rightarrow X_\kappa$ given by $F_{f_i}(n, \langle k, m \rangle) \equiv E_\kappa(f_i(n), \langle k, m \rangle)$.

Suppose $e_\kappa \circ \nabla(f_i) = e_\kappa \circ \nabla(f_j)$. Then $E_\kappa(f_i(n), \langle k, m \rangle) \vdash E_\kappa(f_j(n), \langle k, m \rangle)$ is uniformly realizable and thus $E_\kappa(f_i(n), f_i(n)) \vdash E_\kappa(f_j(n), f_i(n))$ is also uniformly realizable. Thus $E_\kappa(f_j(n), f_i(n))$ is uniformly realizable (since $[f_i(n) = f_i(n)] \vdash E_\kappa(f_i(n), f_i(n))$ is uniformly realizable). Thus there is a number realizing both $E_\kappa(f_j(0), f_i(0))$ and $E_\kappa(f_j(1), f_i(1))$ from which (due to the particular way how E_κ is defined) it follows that $i = j$.

Thus we have shown that $\mathcal{E}ff(\nabla(2), X_\kappa)$ contains at least κ many elements.

From this it follows that for every cardinal κ there exists a cardinal κ' such that $X_{\kappa'}$ is not isomorphic to X_λ for $\lambda \leq \kappa$ (choose κ' such that it is strictly greater than the cardinality of $\bigcup_{\lambda \leq \kappa} \mathcal{E}ff(\nabla(2), X_\lambda)$).

From this it follows that there are as many non-isomorphic objects X in $\mathcal{E}ff$ with $|\Gamma(X)| = 1$ as there are ordinal numbers.

124 Elementary characterization κ -small objects and maps in $\mathcal{E}ff$

Suppose $X = (|X|, E_X)$ is an object in $\mathcal{E}ff$. Then X is covered by the $(\dashv\dashv)$ -separated object \tilde{X} with underlying set $|X|$ and $\|x\|_{\tilde{X}} = E_X(x) = E_X(x, x)$ by the map $q = [Q] : \tilde{X} \twoheadrightarrow X$ with $Q(x', x) \equiv E_X(x', x)$. Now if $e : C \twoheadrightarrow X$ is a regular epi and C is a projective object with $|\Gamma(C)| < \kappa$ then there exists a map $h : C \rightarrow \tilde{X}$ with $e = q \circ h$. Thus, we can cover X already by the restriction of q to the regular subobject \hat{X} of \tilde{X} where $x \in |\hat{X}|$ iff $x = h(y)$ for some $y \in |C|$.

Thus X is κ -small iff X can be covered by some κ -small assembly and the latter can be chosen as a regular subobject of \tilde{X} . Let us write \hat{q} for the restriction of q to \tilde{X} . Although one can always choose a trivial realizer (namely identity) for the surjectivity of q realizers for the surjectivity of \hat{q} may be very difficult to find and certainly do depend on the choice of $e : C \rightarrow X$. Thus, it seems to be very unlikely that realizers for \hat{q} can be chosen uniformly in κ -small X .

Next we give a more concrete characterization of κ -smallness of maps. By definition a map $f : Y \rightarrow X$ is κ -small iff f fits into a diagram

$$\begin{array}{ccccc}
 D & \xrightarrow{e_2} & e_1^* Y & \longrightarrow & Y \\
 & \searrow \varrho & \downarrow e_1^* f & \lrcorner & \downarrow f \\
 & & C & \xrightarrow{e_1} & X
 \end{array}$$

where C and D are projective objects and $\Gamma(g)$ has κ -small fibres. W.l.o.g. one may choose C as the canonical projective cover of X whose underlying set consists of pairs $\langle x, n \rangle$ with $x \in |X|$ and $n \in E_X(x)$ and $\|\langle x, n \rangle\|_C = \{n\}$. Now let $q : \tilde{Y} \rightarrow Y$ be the canonical cover of Y by an assembly as described in the previous paragraph. Then we have

$$\begin{array}{ccccc}
 \hat{Y} & \xrightarrow{i} & e_1^* \tilde{Y} & \longrightarrow & \tilde{Y} \\
 & \nearrow n & \downarrow e_1^* q & \lrcorner & \downarrow q \\
 D & \xrightarrow{e_2} & e_1^* Y & \longrightarrow & Y \\
 & \searrow \varrho & \downarrow e_1^* f & \lrcorner & \downarrow f \\
 & & C & \xrightarrow{e_1} & X
 \end{array}$$

where i is a regular mono and, therefore, the map $\hat{q} = e_1^* q \circ i : \hat{Y} \rightarrow e_1^* Y$ is a regular epi. Since $\Gamma(g)$ has κ -small fibres and \hat{q} is epic for every $x \in |X|$ and $n \in E_X(x)$ we can choose a subset $\hat{Y}_{x,n} \subseteq |Y_x|$ of cardinality $< \kappa$ such that

$$\forall \langle x, n \rangle \in |C| \forall y \in |Y_x| \forall m \in E_Y(y) \exists y' \in \hat{Y}_{x,n} \ e \cdot \langle n, m \rangle \in E_Y(y', y)$$

for some $e \in \mathbb{N}$.⁷⁹ Since by assumption \mathcal{A} has cardinality $< \kappa$ (and κ is regular) for every $x \in |X|$ the set $\hat{Y}_x = \bigcup_{n \in \mathbb{N}} \hat{Y}_{x,n}$ has also cardinality $< \kappa$.

⁷⁹Recall that for $x \in |X|$ the object Y_x is obtained by restricting E_Y to the set $|Y_x| = \{y \in |Y| \mid F(y, x) \neq \emptyset\}$ where $f = [F]$.

Thus we have

$$\forall x \in |X| \forall n \in E_X(x) \forall y \in |Y_x| \forall m \in E_Y(y) \exists y' \in \widehat{Y}_x \ e \cdot \langle n, m \rangle \in E_Y(y', y)$$

for some $e \in \mathbb{N}$ and some $|X|$ -indexed family \widehat{Y} such that \widehat{Y}_x is a subset of $|Y_x|$ of cardinality $< \kappa$ for all $x \in |X|$. Obviously, this condition is also sufficient for f being κ -small and thus characterizes κ -smallness of f .

Summarizing the above discussion we have the following

Theorem (explicit characterization of κ -smallness)

An object X of $\mathcal{E}ff$ is κ -small iff there exists a subset \widehat{X} of $|X|$ of cardinality $< \kappa$ such that

$$\forall x \in |X| \forall n \in E_X(x) \exists x' \in \widehat{X} \ e \cdot n \in E_X(x', x)$$

for some $e \in \mathbb{N}$.

A map $f : Y \rightarrow X$ in $\mathcal{E}ff$ is κ -small iff there exists an $|X|$ -indexed family \widehat{Y} such that \widehat{Y}_x is a subset of $|Y_x|$ of cardinality $< \kappa$ and

$$\forall x \in |X| \forall n \in E_X(x) \forall y \in |Y_x| \forall m \in E_Y(y) \exists y' \in \widehat{Y}_x \ e \cdot \langle n, m \rangle \in E_Y(y', y)$$

for some $e \in \mathbb{N}$.

125 Full Inclusions of Groupoids

Let $f : X \rightarrow Y$ be a functor between groupoids and

$$\begin{array}{ccc}
 X \times_Y X & \xrightarrow{p_2} & X \\
 \downarrow p_1 & \lrcorner & \downarrow f \\
 X & \xrightarrow{f} & Y
 \end{array}
 \begin{array}{c}
 \\
 \xRightarrow{\iota} \\
 \\
 \end{array}$$

a bicomma square, i.e. $X \times_Y X$ is the category whose objects are triples (x_1, x_2, i) with $i : fx_1 \rightarrow fx_2$ in X and whose morphisms from (x_1, x_2, i) to (y_1, y_2, j) are pairs (α_1, α_2) making the diagram

$$\begin{array}{ccc}
 fx_1 & \xrightarrow{i} & fx_2 \\
 f\alpha_1 \downarrow & & \downarrow f\alpha_2 \\
 fy_1 & \xrightarrow{j} & fy_2
 \end{array}$$

commute. The functor p_k sends (x_1, x_2, i) to x_k and (α_1, α_2) to α_k . The natural transformation ι is given by $\iota_{(x_1, x_2, i)} = i$. Let $d : X \rightarrow X \times_Y X$ be the unique functor with $p_k \circ d = \text{id}_X$ and $\iota f = \text{id} : \text{Id}_X \Rightarrow \text{Id}_X$.

Lemma

The functor $f : X \rightarrow Y$ is a full inclusion iff d is an equivalence.

Proof. Suppose f is a full inclusion. Then $p_1 d = \text{Id}_X$ and $d p_1$ is isomorphic to $\text{Id}_{X \times_Y X}$ via φ whose component at (x_1, x_2, i) is given by

$$\begin{array}{ccc}
 fx_1 & \xlongequal{\quad} & fx_1 \\
 \parallel & & \downarrow f\alpha \\
 fx_1 & \xrightarrow{i} & fx_2
 \end{array}$$

where $\alpha : x_1 \rightarrow x_2$ is the unique map with $f\alpha = i$.

For the reverse direction suppose d is an equivalence. Then there exists $\tilde{d} \dashv d$ whose unit η and counit ε are natural isomorphisms anyway since X and Y are groupoids by assumption. For fullness suppose $i : fx_1 \rightarrow fx_2$. Let $x = \tilde{d}(x_1, x_2, i)$ and $\eta_{(x_1, x_2, i)} = (\eta_1, \eta_2) : (x_1, x_2, i) \rightarrow d\tilde{d}(x_1, x_2, i)$, i.e.

$$\begin{array}{ccc}
 fx_1 & \xrightarrow{i} & fx_2 \\
 f\eta_1 \downarrow & & \downarrow f\eta_2 \\
 fx & \xlongequal{\quad} & fx
 \end{array}$$

then $i = f(\eta_2^{-1}\eta_1)$. For faithfulness suppose $\alpha_1, \alpha_2 : x_1 \rightarrow x_2$ with $f\alpha_1 = i = f\alpha_2$. For $k = 1, 2$ we have $(\alpha_k, \text{id}_{x_2}) : (x_1, x_2, i) \rightarrow dx_2$. Since $\tilde{d} \dashv d$ there exists a unique $\gamma_k : x \rightarrow x_2$ with $d\gamma_k\eta_{(x_1, x_2, i)} = (\alpha_k, \text{id}_{x_2})$, i.e. $\gamma_k\eta_1 = \alpha_k$ and $\gamma_k\eta_2 = \text{id}_{x_2}$. But then we have $\gamma_1 = \eta_1^{-1} = \gamma_2$ and thus also $\alpha_1 = \gamma_1\eta_1 = \gamma_2\eta_1 = \alpha_2$.

126 Groupoids in the Effective Topos

(Rosolini Spring 2009, talk at PSSL 88 on 4th April)

Let \mathcal{E} be the effective topos $\mathcal{E}ff$. It is not known although expected that there is no generic family of separated discrete objects in \mathcal{E} . In order to remedy this situation Pino has suggested to consider the category \mathcal{F} of groupoids in \mathcal{E} whose object of objects is projective and whose hom-sets are modest. Morphisms in \mathcal{F} are internal functors which are considered as equal iff there externally exists an internal isomorphism between them. The Lemma of the previous Section 125 tells us that monos in \mathcal{F} are given by full inclusions of groupoids from which it follows that the inclusion $\mathcal{E}ff = \mathcal{E} \hookrightarrow \mathcal{F}$ is logical.

Now in the topos⁸⁰ \mathcal{F} one may consider the universe $el : El \rightarrow Un$ which is defined as follows. Un is the groupoid in \mathcal{E} whose object of objects is $\nabla(\text{PER})$ and whose hom-sets are defined as $Un(R, S) = \{i : \mathbb{N}/R \rightarrow \mathbb{N}/S \mid i \text{ iso}\}$ which is a modest set. El is the groupoid in \mathcal{E} whose object of objects is given by the partitioned assembly whose elements are pair (R, n) with $R \in \text{PER}$ and nRn and where $\|(R, n)\| = \{n\}$. The hom-set $El((R, n), (S, m))$ is $\{i : Un(R, S) \mid i([n]_R) = [m]_S\}$. The morphism $el : El \rightarrow Un$ is given by the functor sending (R, n) to R and whose morphism part is given by inclusions $El((R, n), (S, m)) \hookrightarrow Un(R, S)$. A *parametric family of modest sets* in \mathcal{F} is a map which can be obtained as pullback of el in \mathcal{F} . One can show that for $X \in \mathcal{E}$ a map $f : Y \rightarrow X$ is a parametric family of modest sets, i.e. $f \cong c^*el$ for some $c : X \rightarrow Un$ in \mathcal{F} , if and only if $f : Y \rightarrow X$ is a family of separated discrete objects in $\mathcal{E}ff$. Thus $el : El \rightarrow Un$ is a generic family for such families which, however, does not exist within \mathcal{E} but only in its extension \mathcal{F} . Thus $f : Y \rightarrow X$ is a family of modest sets in the sense of [HRR] iff $f \cong c^*el$ for some $c : X \rightarrow Un$ factoring through $\nabla(\text{PER})$. Such c map every iso in X to an identity and thus may be considered as “locally constant”. The subobject classifier $\top : 1 \rightarrow \Omega$ is a typical example of a parametric family of modest sets which is not locally constant since it is classified by the inclusion $\Omega \hookrightarrow Un$ which does not factor through $\nabla(\text{PER})$ (as otherwise all monos were $\neg\neg$ -closed).

Apparently, the discrete groupoid embeds into Un via an obvious inclusion $i : \nabla(\text{PER}) \hookrightarrow Un$. Pulling back el along this i gives rise to a map in $\mathcal{E}_{\neg\neg}$

⁸⁰As observed by Stekelenburg in June 2012 the category \mathcal{F} presumably is not even regular. Pino suggested that the monos are the full inclusions of groupoids and the regular epis are the groupoid morphisms which are surjective on objects. This, however, is wrong for the following simple reason. Let 1 be the trivial group and $f, g : A \rightarrow B$ different homomorphisms of abelian groups. The unique group homomorphism $e : 1 \rightarrow A$ is surjective on objects and $f \circ e = g \circ e$ but there is no isomorphism between f and g since otherwise f and g were equal because B is abelian.

isomorphic to the usual generic family of modest sets in $\mathcal{E}_{\rightarrow}$.

Outlook It may be expected that replacing modest sets by κ -small objects of $\mathcal{E}ff$ one may construct an extension \mathcal{F}_κ of $\mathcal{E}ff$ containing a generic family of κ -small maps. The idea is that \mathcal{F}_κ consists of groupoids internal to $\mathcal{E}ff$ whose object of objects is projective and whose hom-sets are κ -small.

127 Why small maps should be a definable class

Let \mathcal{E} be a (pre)topos. Let \mathcal{S} be a class of morphisms in \mathcal{E} satisfying the conditions

- (A1) \mathcal{S} contains all iso and is closed under composition
- (A2) \mathcal{S} is stable under pullbacks along arbitrary morphism in \mathcal{E}
- (A3) if $ge = f \in \mathcal{S}$ and e is epic then $g \in \mathcal{S}$
- (A4) if $f, g \in \mathcal{S}$ then $f + g \in \mathcal{S}$
- (A5) $0 \rightarrow 1$ and $1 + 1 \rightarrow 1$ are in \mathcal{S} .

One might ask whether \mathcal{S} contains all Kuratowski finite maps. In his (unpublished) paper *Definability, Finiteness, Projectivity and Choice* (from early 1990ies) Jean Bénabou has come up with the following counterexample for $\mathcal{E} = \mathbf{Set}$. Let \mathcal{S} be the collection of all maps $f : J \rightarrow I$ such that there exists $n \in \mathbb{N}$ with $\text{card}(f^{-1}(i)) \leq n$ for all $i \in I$. Obviously, this class satisfies (A), i.e. the axioms (A1) - (A5), but does *not* contain the generic Kuratowski finite map $k : K \rightarrow N$ where $K = \{(i, n) \in \mathbb{N}^2 \mid i < n\}$ and $k(i, n) = n$.

However, if \mathcal{E} is a topos and the class \mathcal{S} is definable (as a full subfibration of the fundamental fibration $P_{\mathcal{E}} = \partial_1 : \mathcal{E}^2 \rightarrow \mathcal{E}$) then \mathcal{S} contains all Kuratowski finite maps. The reason is that for all $X \in \mathcal{E}$ there is a generic family of small subsets of X which can be constructed as follows. Let $\in_X : E(X) \rightarrow P(X) \times X$ classify $\text{Sub}_{\mathcal{E}}(- \times X)$ and $E(X) \rightarrow P(X)$ be $e_X = \pi \circ \in_X : E(X) \rightarrow P(X)$. Then, since \mathcal{S} is assumed to be definable there exists a greatest subobjects $m_X : S(X) \rightarrow P(X)$ of the powerset $P(X)$ such that $m_X^* e_X \in \mathcal{S}$ and all $r : Y \rightarrow P(X)$ with $r^* e_X \in \mathcal{S}$ factor through m_X . Let us write $e_S(X) : E_S \rightarrow S(X)$ for the pullback of e_X along m_X . From the axioms (A) it follows using Kripke-Joyal semantics that $S(X)$ satisfies the closure conditions

$$\emptyset \in S(X) \wedge (\forall x : X. \{x\} \in S(X)) \wedge (\forall a, b \in S(X). a \cup b \in S(X))$$

Since $K(X)$, the object of Kuratowski finite subsets of X , is the least subobject of $P(X)$ satisfying these closure conditions it follows that $K(X) \subseteq S(X)$. Now if $f : X \rightarrow I$ is a family of Kuratowski finite sets then the map $f^{-1} : I \rightarrow P(X)$ factors through $K(X) \subseteq S(X)$ from which it follows that $f \cong (f^{-1})^* e_X$ is in \mathcal{S} as desired.

128 K -finite and finite sets in (presheaf) toposes

For an object X in a topos \mathcal{E} let $K(X)$ be the least subobject of $\mathcal{P}(X)$ that is closed under \cup and contains all singleton sets and the empty set. An object X is called K -finite (Kuratowski-finite) iff $K(X) = \mathcal{P}(X)$. An object is called finite if X is K -finite and, moreover, has decidable equality.

Let us further assume that \mathcal{E} has a natural numbers object. Then there exists a weakly generic family $k : K \rightarrow N$ of finite objects where $K = \{\langle m, n \rangle \in N^2 \mid m < n\}$ and $k(\langle m, n \rangle) = n$. It can be shown that a map $f : Y \rightarrow X$ is a family of finite (K -finite) sets iff it fits into a diagram

$$\begin{array}{ccc} Y & \longleftarrow \cdot \longrightarrow & K \\ f \downarrow & & \downarrow e^* f \\ X & \longleftarrow \cdot \longrightarrow & N \\ & e & g \end{array}$$

where e is epic, the left square is a pullback and the right square is a pullback (a quasi-pullback⁸¹). Thus f is a family of finite objects iff it is “locally” a pullback of k and f is a family of K -finite sets if it is “locally” the quotient of a pullback of k . Using Kripke-Joyal semantics these properties can be characterised in logical terms as follows: $Y \rightarrow X$ is a family of finite objects iff

$$\forall x \in X \exists n \in N. Y_x \cong K_n$$

holds in the internal language of \mathcal{E} and $Y \rightarrow X$ is a family of K -finite objects iff

$$\forall x \in X \exists n \in N, e \in Y_x^{K_n}. e \text{ epic}$$

holds in the internal language of \mathcal{E} . Of course, in general there is no reason whatsoever why for a family $f : Y \rightarrow X$ of finite (K -finite) objects there should exist a map $g : X \rightarrow N$ such that $f \cong g^*k$ (f is a quotient of g^*k in \mathcal{E}/X).

If \mathbb{C} is a small category then an object A in the presheaf topos $\widehat{\mathbb{C}} = \mathbf{Set}^{\mathbb{C}^{\text{op}}}$ is K -finite iff all $A(I)$ are finite and all $A(u)$ are surjective. Obviously, the object A is finite iff all $A(I)$ are finite and all $A(u)$ are bijections.

In the topos $\mathcal{T} = \widehat{\omega}$ of trees the truth value object $\Omega_{\mathcal{T}}$ is K -finite but not finite. But, what may be surprising, it is not even the case that Ω appears as quotient of a finite object since the fibres of $\Omega_{\mathcal{T}}$ can get arbitrarily big (but still finite). Thus the terminal projection of $\Omega_{\mathcal{T}}$ cannot be obtained as quotient of some pullback of k along some $1 \rightarrow N$.

If \mathbb{C} is a groupoid then, of course, in $\widehat{\mathbb{C}}$ the notions of “finite” and “ K -finite” coincide since the logic of $\widehat{\mathbb{C}}$ is boolean and thus equality of all objects is decidable. If \mathbb{G} is a nontrivial finite group then the representable object $G = y(*)$ is finite but it cannot be obtained as pullback of k along some $1 \rightarrow N$ since the latter is in the image of $\Delta : \mathbf{Set} \rightarrow \widehat{\mathbb{C}}$ whereas G is not. Notice, moreover, that

⁸¹a square is a “quasi-pullback” iff the mediating arrow to the pullback is epic

G does not have a global element although it is finite and well-supported, i.e. $G \rightarrow 1$ is epic.

As A. Blass recently observed (July 2007) the argument above shows that for a non-trivial finite group \mathbb{G} there is no generic family of (K -)finite objects in $\widehat{\mathbb{G}} = \mathbf{Set}^{\mathbb{G}^{\text{op}}}$ which, moreover, is definable in the language of higher order arithmetic because $\Delta : \mathbf{Set} \rightarrow \mathbf{Set}^{\mathbb{G}^{\text{op}}}$ is a logical functor and thus any such family would be of the form $\Delta(u)$ and the above argument already shows that no such family can be generic for families of finite objects.

Notice that in every topos with nno N for a family $A \rightarrow I$ of finite objects it holds in the internal logic that $\forall i:I. \exists^1 n:N. A_i \cong K_n$. By AUC it follows that $\exists^1 f:N^I. \forall i:I. A_i \cong K_{f(i)}$. Thus, there exists also a unique (external) morphism $f : I \rightarrow N$ with $\forall i:I. A_i \cong K_{f(i)}$. However, there is an implicit existential quantifier in the statement $A_i \cong K_{f(i)}$ which need not be resolvable by a(n external) Skolem function g which for every $i \in I$ chooses an isomorphism $g(i) \in \text{Iso}(A_i, K_{f(i)})$. This is illustrated by the terminal projection $G = y(*) \rightarrow 1$ for which there does exist a global element $n : 1 \rightarrow N$ (namely $n = |\mathbb{G}|$) such that $G \cong K_n$ holds in the internal logic although the existential quantifier implicit in $G \cong K_n$ is not witnessed by a global element of $\text{Iso}(G, K_n)$.

129 Linear Ordinals may form a set in IZF

In the 1980ies G. Rosolini (and possibly McCarty) showed that in Friedman's realizability model for IZF the linear ordinals form a set. Obviously, linear ordinals have decidable equality. First one observes that in $\mathcal{E}ff$ every decidable object is modest and thus subcountable.⁸² Thus, since isomorphic ordinals are equal in the Friedman realizability model for IZF one can index the class of linear ordinals by a subobject of $\mathcal{P}(\mathbb{N}^2)$ from which it follows by Replacement that they form a set.

B. van den Berg has used this as a counterexample to the claim that IZF proves that every inflationary map on a directed complete partial order (dcpo) has a fixpoint. Let X the set of linear ordinals (in Friedman's realizability model for IZF) ordered by \subseteq . If D is a directed subset of X then its union $\bigcup D$ is again linearly ordered (for $x, y \in X$ pick an $\alpha \in X$ with $x, y \in \alpha$ and conclude using linearity of α). Thus (X, \subseteq) is a dcpo but the successor map $f : \bigcup X \rightarrow \bigcup X : x \mapsto x \cup \{x\}$ is inflationary and doesn't have a fixpoint. This doesn't contradict Pataia's Fixpoint Theorem⁸³ because f is not monotonic.

Obviously, the union of all linear ordinals is an ordinal but not linear itself since it is greater than all linear ordinals.

The Friedman realizability model shows that it is consistent with IZF that all linear ordinals are subcountable. In ZF, however, one easily shows that there exists an uncountable ordinal (since countable ordinals form a set and their union thus is an uncountable ordinal) which, of course, is linear (since ZF proves all ordinals to be linear). Thus, the existence of uncountable linear ordinals is independent from IZF. This fact has been exploited by Friedman and Scedrov for showing that IZF lacks definable witnesses for certain existential statements (unlike IzF obtained from IZF by just claiming Replacement instead of Collection).

⁸²If X has decidable equality then it is, of course, also $\neg\neg$ -separated. One easily sees that assemblies with decidable equality are modest because different objects cannot have a common realizer.

⁸³saying that every monotone endomap on a dcpo has a (least) fixpoint

130 Generic Sets vs. Generic Ultrafilters

Let M be a (countable) transitive model of $\text{ZF}(C)$. For every $x \in M$ we write $\mathcal{P}_M(x)$ as an abbreviation for $\mathcal{P}(x) \cap M$.

Let P be a poset in M . For every $X \in \mathcal{P}(M)$ we define⁸⁴

$$\neg\neg X := \{p \in P \mid \forall q \leq p \exists r \leq q \ r \in X\}$$

and observe that $\neg\neg X \in M$ whenever $X \in M$. A subset D of P is called *dense* iff $\neg\neg D = P$ and a subset C of P is called $(\neg\neg)$ -closed iff $\neg\neg C = C$. We write $\mathcal{B}(P)$ for the set of all $\neg\neg$ -closed subsets of P and $\mathcal{B}_M(P)$ for $\mathcal{B}(P) \cap M$. One easily checks that $\mathcal{B}(P)$ is a complete boolean algebra w.r.t. \subseteq where $\bigvee \mathcal{X} = \neg\neg \bigcup \mathcal{X}$ for all $\mathcal{X} \subseteq \mathcal{B}(P)$. Obviously, $\mathcal{B}_M(P)$ is closed under complementation and finite suprema and infima taken in $\mathcal{B}(P)$. Moreover, for every $\mathcal{X} \in \mathcal{P}_M(\mathcal{B}_M(P))$ both $\bigvee \mathcal{X}$ and $\bigwedge \mathcal{X}$ are elements of $\mathcal{B}_M(P)$. In other words $\mathcal{B}_M(P)$ is an M -complete boolean algebra.

From now on we assume that the poset P is *refined*, i.e. $\downarrow p \in \mathcal{B}(P)$ for all $p \in P$.⁸⁵ Under this assumption $\{\downarrow p \mid p \in P\}$ forms a basis for $\mathcal{B}(P)$ and $\mathcal{B}_M(P)$.

Definition 130.1 *An M -generic set is a filter G in P such that $G \cap D \neq \emptyset$ for all dense $D \in \mathcal{P}_M(P)$. An M -generic ultrafilter is a filter \mathcal{U} in $\mathcal{B}_M(P)$ such that $\mathcal{X} \cap \mathcal{U} \neq \emptyset$ for all $\mathcal{X} \in \mathcal{P}_M(\mathcal{B}_M(P))$ with $\bigvee \mathcal{X} \in \mathcal{U}$. \diamond*

Next we establish a 1-1-correspondence between these two notions.

Lemma 130.1 (J. L. Bell)

If G is an M -generic set then $\mathcal{U}_G = \{C \in \mathcal{B}_M(P) \mid G \cap C \neq \emptyset\}$ is an M -generic ultrafilter and if \mathcal{U} is an M -generic ultrafilter then $G_{\mathcal{U}} = \{p \in P \mid \downarrow p \in \mathcal{U}\}$ is an M -generic set.

Moreover, these two constructions are mutually inverse, i.e. $G_{\mathcal{U}_G} = G$ for all M -generic sets G and $\mathcal{U}_{G_{\mathcal{U}}} = \mathcal{U}$ for all M -generic ultrafilters \mathcal{U} , thus establishing a 1-1-correspondence between M -generic sets and M -generic ultrafilters.

Proof: Suppose G is M -generic.

Obviously \mathcal{U}_G is a subset of $\mathcal{B}_M(P)$ upward closed w.r.t. \subseteq and $\emptyset \notin \mathcal{U}_G$. Suppose $C_1, C_2 \in \mathcal{U}_G$. Then there exist $p_1 \in G \cap C_1$ and $p_2 \in G \cap C_2$. As G is a filter there exists $p \in G$ with $p \leq p_1, p_2$ and thus $p \in C_1 \cap C_2$ since the C_i are downward closed. Thus $C_1 \cap C_2 \in \mathcal{U}_G$. Thus we have shown that \mathcal{U}_G is a filter.

⁸⁴ $\neg\neg X = \{p \in P \mid \forall q \leq p \ q \notin X\}$

⁸⁵Thus, the poset P is refined iff

$$\forall p, q ((\forall r \leq q \exists r' \leq r \ r' \leq p) \rightarrow q \leq p)$$

iff

$$\forall p, q ((\forall r \leq q \ r \downarrow p) \rightarrow q \leq p)$$

iff

$$\forall p, q (q \not\leq p \rightarrow \exists r \leq q \ r \not\downarrow p) .$$

Let $C \in \mathcal{B}_M(P)$. As $C \cup \neg C$ is dense its intersection with G is nonempty and thus $C \in \mathcal{U}_G$ or $\neg C \in \mathcal{U}_G$. But it cannot be that both C and $\neg C$ are in \mathcal{U}_G as otherwise there existed $p \in G \cap C$ and $q \in G \cap \neg C$ from which there follows the existence of $r \in G$ with $r \leq p, q$ and thus $r \in C \cap \neg C$ which is impossible. Thus, for every $C \in \mathcal{B}_M(P)$ either $C \in \mathcal{U}_G$ or $\neg C \in \mathcal{U}_G$.

Suppose $\mathcal{X} \in \mathcal{P}_M(\mathcal{B}_M(P))$ with $\bigvee \mathcal{X} \in \mathcal{U}_G$. Thus, by the above observation we have $\neg \bigvee \mathcal{X} \notin \mathcal{U}_G$, i.e. $G \cap \neg \bigvee \mathcal{X} = \emptyset$. Thus, as $\bigcup \mathcal{X} \cup \neg \bigcup \mathcal{X}$ is dense we have $G \cap \bigcup \mathcal{X} \neq \emptyset$ from which it follows that for some $C \in \mathcal{X}$ we have $G \cap C \neq \emptyset$, i.e. $C \in \mathcal{U}_G$.

Suppose \mathcal{U} is an M -generic ultrafilter.

Suppose $p \in G_{\mathcal{U}}$ and $p \leq q$. Then $\mathcal{U} \ni \downarrow p \subseteq \downarrow q$ from which it follows that $\downarrow q \in \mathcal{U}$, i.e. $q \in G_{\mathcal{U}}$. Thus $G_{\mathcal{U}}$ is upward closed w.r.t. \leq .

Suppose $p, q \in G_{\mathcal{U}}$. Then both $\downarrow p$ and $\downarrow q$ are elements of \mathcal{U} from which it follows that $\downarrow p \cap \downarrow q \in \mathcal{U}$. Let $\mathcal{C} = \{r \in P \mid r \in \downarrow p \cap \downarrow q\}$. As $\mathcal{C} \in M$ and $\bigvee \mathcal{C} \in \mathcal{U}$ we have $\mathcal{C} \cap \mathcal{U} \neq \emptyset$. Thus, there is an $r \leq p, q$ with $\downarrow r \in \mathcal{U}$, i.e. $r \in G_{\mathcal{U}}$. Thus $G_{\mathcal{U}}$ is a filter.

Suppose $D \in \mathcal{P}_M(P)$ is dense. Then $\mathcal{D} = \{\uparrow p \mid p \in D\} \in M$ and $\bigvee \mathcal{D} = 1_{\mathcal{B}_M(P)} \in \mathcal{U}$. Thus $\mathcal{D} \cap \mathcal{U} \neq \emptyset$ from which it follows that there is some $p \in D$ with $\downarrow p \in \mathcal{U}$, i.e. $p \in G_{\mathcal{U}}$. Thus $G_{\mathcal{U}}$ is M -generic.

Finally we check that the two constructions are mutually inverse.

Suppose G is M -generic and $p \in P$. Then $p \in G_{\mathcal{U}_G}$ iff $\downarrow p \in \mathcal{U}_G$ iff $G \cap \downarrow p \neq \emptyset$ iff $p \in G$ (where the last equivalence holds since G is upward closed).

Suppose \mathcal{U} is an M -generic ultrafilter and $C \in \mathcal{B}_M(P)$. Then $C \in \mathcal{U}_{G_{\mathcal{U}}}$ iff $C \cap G_{\mathcal{U}} \neq \emptyset$ iff $\exists p(p \in C \wedge \downarrow p \in \mathcal{U})$ iff $C \in \mathcal{U}$ (where the last equivalence holds since $\mathcal{C} = \{\downarrow p \mid p \in C\} \in M$, $\bigvee \mathcal{C} = C$ and \mathcal{U} is an M -generic ultrafilter). \square

Now given a refined poset P and an M -generic ultrafilter \mathcal{U} we may construct a $\mathcal{B}_M(P)$ -valued model of $\text{ZF}(\mathcal{C})$ which when factored by \mathcal{U} is isomorphic to the so-called ‘‘generic extension’’ $M[G_{\mathcal{U}}]$ as considered by set theorists following P. J. Cohen’s original notion of forcing.

131 Reducing mutual recursive type definitions to iterated single recursive type definitions

Let \mathcal{C} be some category of domains and strict maps between them. Let us suppose that \mathcal{C} is algebraically compact in the sense of Freyd. Now given a functor $F : \mathcal{C}^{\text{op}} \times \mathcal{C} \times \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{C}$ it can be augmented to a functor $F^{\S} = \langle F^-, F^+ \rangle : \mathcal{C}^{\text{op}} \times \mathcal{C} \times \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{C}^{\text{op}} \times \mathcal{C}$ where $F^+ = F$ and $F^-(X^-, X^+, P^-, P^+) = F(X^+, X^-, P^+, P^-)$. Freyd has shown that $\mathcal{C}^{\S} = \mathcal{C}^{\text{op}} \times \mathcal{C}$ is algebraically compact and thus F^{\S} is a **covariant** functor form $\mathcal{C}^{\S} \times \mathcal{C}^{\S}$ to \mathcal{C}^{\S} . Thus, by a result of Freyd there exists a (covariant) functor $H : \mathcal{C}^{\S} \rightarrow \mathcal{C}^{\S}$ such that $i_P : F^{\S}(H(P), P) \rightarrow H(P)$ is a free algebra for the functor $F^{\S}(-, P) : \mathcal{C}^{\S} \rightarrow \mathcal{C}^{\S}$. Thus we have the object part of H . Its morphism part is defined as follows: given $f : P \rightarrow Q$ in \mathcal{C}^{\S} we define $H(f) : H(P) \rightarrow H(Q)$ as the least map $g : H(P) \rightarrow H(Q)$ making the diagram

$$\begin{array}{ccc} F^{\S}(H(Q), Q) & \longrightarrow & H(Q) \\ \uparrow F^{\S}(g, f) & & \uparrow g \\ F^{\S}(H(P), P) & \longrightarrow & H(P) \end{array}$$

commute.

Now given another $G : \mathcal{C}^{\text{op}} \times \mathcal{C} \times \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{C}$ we may form G^{\S} in a way analogous to the construction of F^{\S} . Then by Bekić's Lemma the free solution of

$$A = F(A, A, B, B) \quad B = G(A, A, B, B)$$

is given by the (isomorphic) components of $A = H(B)$ and $B = \mu Y.G^{\S}(H(Y), Y)$, respectively.

Warning Notice that for $H(P) = \langle H^-(P), H^+(P) \rangle$ we have

$$i_P^+ : F(H^-(P), H^+(P), P) \xrightarrow{\cong} H^+(P) \quad i_P^- : H^-(P) \xrightarrow{\cong} F(H^+(P), H^-(P), P^+, P^-)$$

and not

$$i_P^+ : F(H^+(P), H^+(P), P) \xrightarrow{\cong} H^+(P) \quad i_P^- : H^-(P) \xrightarrow{\cong} F(H^-(P), H^-(P), P^+, P^-)$$

for which reason it will **in general not** hold that $H^+(P) = \mu X.F(X, X, P)$ and $H^-(P) = \mu X.F(X, X, P^+, P^-)$ as one might hope.

Thus, I don't see any possibility to extend $\mu X.F(X, X, P^-, P^+)$ to a functor covariant in P and contravariant in P^- .

132 A Constructive Theory not having the Existence Property

Usually it is considered as a characteristic property of constructive theories that they satisfy the following *existence property* : if $\vdash \exists x A(x)$ then there is a closed term t such that $\vdash A(t)$. But that this need not be necessarily the case is shown by the following counterexample due to G. Kreisel.

Let Prf be a binary primitive recursive predicate where $\text{Prf}(n, \ulcorner A \urcorner)$ iff n is a code of a derivation of A in HA. We define $A(x) \equiv \text{Prf}(x, \ulcorner \perp \urcorner) \vee \forall y \neg \text{Prf}(y, \ulcorner \perp \urcorner)$. Since HA is consistent it holds constructively that $\exists x A(x)$ and thus the theory $\text{HA} + \exists x A(x)$ is constructively valid. However, this constructively valid theory does not have the existence property.

Obviously, we have $\text{HA} + \exists x A(x) \vdash \exists x A(x)$. Suppose that $\text{HA} + \exists x A(x) \vdash A(t)$ for some closed term t , i.e. $\text{HA} \vdash \exists x A(x) \rightarrow A(t)$. Since HA is consistent we have $\text{HA} \vdash \neg \text{Prf}(t, \ulcorner \perp \urcorner)$. Thus $\text{HA} \vdash \exists x A(x) \rightarrow \forall y \neg \text{Prf}(y, \ulcorner \perp \urcorner)$. From this it follows that $\text{HA} \vdash \text{Prf}(x, \ulcorner \perp \urcorner) \vee \forall y \neg \text{Prf}(y, \ulcorner \perp \urcorner) \rightarrow \forall y \neg \text{Prf}(y, \ulcorner \perp \urcorner)$ and, therefore, also $\text{HA} \vdash \text{Prf}(x, \ulcorner \perp \urcorner) \rightarrow \forall y \neg \text{Prf}(y, \ulcorner \perp \urcorner)$. But then $\text{HA} \vdash \text{Prf}(x, \ulcorner \perp \urcorner) \rightarrow \perp$ and thus $\text{HA} \vdash \forall x \neg \text{Prf}(x, \ulcorner \perp \urcorner)$ which is impossible by Gödel's Incompleteness Theorem.

133 The n.c.i. of being Cauchy vs. Partial Cauchy

Let (a_n) be a sequence of rational numbers which is Cauchy, i.e.

$$\forall n. \exists m. \forall k \geq m. \forall i, j \in [m, k]. |a_i - a_j| < 2^{-n}$$

whose no counterexample interpretation is

$$\forall n. \forall f \geq \text{id}_{\mathbb{N}}. \exists m. \forall i, j \in [m, f(m)]. |a_i - a_j| < 2^{-n}$$

i.e.

$$\forall n. \forall f \geq \text{id}_{\mathbb{N}}. \exists m. \underbrace{\text{diam}(\{a_m, \dots, a_{f(m)}\})}_{d_m^{(f)}} < 2^{-n}$$

which says that for every $f \geq \text{id}_{\mathbb{N}}$ the sequence $(d_n^{(f)})_{n \in \mathbb{N}}$ gets arbitrarily closed to 0. This is weaker than Richman's notion of (a_n) being *partial Cauchy* meaning that for every $f \geq \text{id}_{\mathbb{N}}$ the sequence $(d_n^{(f)})_{n \in \mathbb{N}}$ converges to 0.

Actually, it suffices to require that for every $f \geq \text{id}_{\mathbb{N}}$ the sequence $(d_n^{(f)})_{n \in \mathbb{N}}$ is Cauchy. Consider $g : \mathbb{N} \rightarrow \mathbb{N}$ with $g(2n) = 2n$ and $g(2n+1) = f(2n+1)$. Then $(d_n^{(g)})_{n \in \mathbb{N}}$ is also Cauchy and thus converges. But we have

$$\lim d_n^{(f)} = \lim d_{2n+1}^{(f)} = \lim d_{2n+1}^{(g)} = \lim d_{2n}^{(g)} = 0$$

showing that actually $(d_n^{(f)})_{n \in \mathbb{N}}$ converges to 0.

134 Kreisel's No Counterexample Interpretation vs. Gödel's Functional Interpretation

First observe that for every formula A in prenex normal form its Dialectica interpretation A^D gives rise to the Skolem normal form A^S of A where existential quantifiers are replaced by Skolem functions. Kreisel (in 1951) defined the no counterexample interpretation (n.c.i) of A as $A^H \equiv \neg(\neg A)^S$ (where negation of formulas in prenex normal form is achieved by à la de Morgan). Notice that for Π_2^0 formulas $A \equiv \forall x \exists y R(x, y)$ we have $A^H \equiv A$ whereas for Σ_2^0 formulas $A \equiv \exists x \forall y R(x, y)$ its n.c.i. is $A^H \equiv \forall f \exists x R(x, f(x))$. If A is classically provable then one can exhibit a type 2 functional F such that $A(F(f), f(F(f)))$ is provable constructively. Such an F is said to witness A^H .

Alternatively one may consider for every formula A the translation A^{ND} , i.e. the Dialectica interpretation of A^N , the negative translation of A . Recall that if $A^D \equiv \exists x \forall y A_D(x, y)$ then we have

$$(\neg A)^D \equiv \exists f \forall x \neg A_D(x, f(x)) \quad (\neg \neg A)^D \equiv \exists F \forall f A_D(F(f), f(F(f)))$$

Thus $(\neg \neg A)^D$ claims the existence of a witness for the n.c.i. of A^D . One easily sees that if A is a Π_2^0 or a Σ_2^0 formula then A^{ND} claims the existence of a witness for A^H .

However, already for Σ_3^0 formula these two translations give different results. Consider $A \equiv \exists x \forall y \exists z R(x, y, z)$. Then $A^H \equiv \forall f \exists x, z R(x, f(x), z)$ whereas A^{ND} claims the existence of a witness for $\forall F \exists x, f R(x, F(x, f), f(F(x, f)))$ where F ranges over type 2 functionals.

As emphasized by U. Kohlenbach Kreisel's n.c.i. is *not modular* (there are severe problems with interpreting *modus ponens*) whereas $(-)^{ND}$ is modular but one has to pay the prize of taking higher type functionals into account. Since $(\neg \neg A)^D \equiv \exists F \forall f A_D(F(f), f(F(f)))$ claims the existence of a witness F for the n.c.i of $\exists x \forall y A_D(x, y)$ one may say that $(-)^{ND}$ provides the correct modular version of Kreisel's no counterexample interpretation.

Kreisel introduced his n.c.i. in 1951 whereas Gödel found his functional interpretation already around 1940 though it was published first in 1958 (i.e. almost 20 years later). It is surprising that apparently Kreisel did not know about Gödel's functional interpretation when he invented his n.c.i. because Kreisel had intensive contact with Gödel.

A typical example of a Π_3^0 statement is

$$A \equiv \forall k \exists p \forall m \left(m > p \rightarrow |q_m - q_p| < \frac{1}{k} \right)$$

stating that (a_n) is a Cauchy sequence (of rational numbers). The n.c.i. of A is given by $A^H \equiv \neg(\neg A)^S$. We have

$$\neg A \equiv \exists k \forall p \exists m \left(m > p \wedge |q_m - q_p| \geq \frac{1}{k} \right)$$

and thus its Skolemization is

$$(\neg A)^S \equiv \exists k f \forall p \left(f(p) > p \wedge |q_{f(p)} - q_p| \geq \frac{1}{k} \right)$$

and thus

$$A^H \equiv \neg(\neg A)^S \equiv \forall k f \exists p \left(f(p) > p \rightarrow |q_{f(p)} - q_p| < \frac{1}{k} \right)$$

Finally notice that in resolution style automated theorem proving a formula A (in prenex form) is proved by proving the Σ_1 formula $A^H \equiv \neg(\neg A)^S \equiv \exists \vec{x} A_0(\vec{f}, \vec{x})$ (A_0 quantifier free) for which by Herbrand's Theorem there are finitely many instances $\vec{t}_1, \dots, \vec{t}_m$ such that

$$A_0(\vec{f}, \vec{t}_1) \vee \dots \vee A_0(\vec{f}, \vec{t}_m)$$

is a propositional tautology.

Thus Kreisel's no counterexample interpretation lies at the heart of modern resolution style theorem proving.

135 Functional Interpretation vs. Modified Realizability combined with Friedman's Trick

When unwinding proofs of Π_2 statement in \mathbf{PA}_ω one can use either

- (1) Gödel-Gentzen negative translation (G -translation) followed by Gödel's functional ("Dialectica") interpretation or
- (2) Friedman's R -translation followed by Kreisel's modified realizability interpretation.

in order to obtain a constructive proof in \mathbf{HA}_ω . U. Kohlenbach has observed that for this purpose method (1) is in various respects better than method (2). It is the aim of this note to illustrate this claim by considering some examples.

If the classical proof uses just Markov's principle then one can apply Gödel's functional interpretation immediately since it validates

$$\neg\neg\exists x:\sigma.P(x) \rightarrow \exists x:\sigma.P(x)$$

for all primitive recursive predicates P and all types σ . From this one sees that the functional interpretation of Markov's principle cannot be simply unbounded search as it is the case for Kleene's number or function realizability interpretation of Markov's principle for base type ι . This is impossible with Kreisel's modified realizability since it (intentionally so!) does not interpret Markov's principle (even) for base type.

Next we consider the case where the classical derivation uses some lemmas of form Σ_2 (or Π_3 which is the form of convergence statements in analysis) which can be proved only classically

When applying the G -translation then all premisses are translated to the negative fragment where realizers (in the sense of modified realizability) do not carry any computational information.⁸⁶ This is dramatically different when one applies Friedman's R -translation and applies Friedman's trick, i.e. instantiates the generic propositional constant by the desired Σ_1 formula.

We will see in the following examples that it requires quite some effort i.e. computational complexity, to realize these R -translations of the lemmas in Σ_2 form. The examples also show that this effort is much less for the functional interpretation of the G -translation of these lemmas.

This difference does not show up when considering classical proofs of Σ_1 statements from axioms of the form Π_1 (or Π_2) where modified realizability is somewhat simpler. Thus, one might conclude that the certainly more complicated functional interpretation pays off only when applied to proofs using lemmas of the form Σ_2 or Π_3 .

⁸⁶As opposed to functional interpretation where interpretations of negative formulas in general does carry algorithmic information, e.g. in case of $\neg\forall n \neg P(x)$,

135.1 Drinker's Theorem

Let $P(x)$ be a prime formula. Then

$$\begin{aligned} & \lambda f : \forall x : N. \neg_R \forall y : N. (\neg_R \neg_R P(x) \rightarrow \neg_R \neg_R P(y)) \\ & \quad f \ 0 \ (\lambda y : N. \lambda h : \neg_R \neg_R P(0). \lambda : \neg_R P(y). \\ & \quad \quad f \ y \ (\lambda z : N. \lambda k : \neg_R \neg_R P(y). \lambda k' : \neg_R P(z). \ k g)) \\ & \in \neg_R \forall x : N. \neg_R \forall y : N. (\neg_R \neg_R P(x) \rightarrow \neg_R \neg_R P(y)) \end{aligned}$$

i.e. we have constructed a proof term for the R -translation of Drinker's Theorem $\exists x \forall y (P(x) \rightarrow P(y))$ which is the typical example of a Σ_2 statement which can be proved classically but in general not constructively.⁸⁷ Notice that this realizer will not get simpler when instantiating R by \perp .

Now let us look at the functional interpretation of the G -translation

$$\neg \forall x \neg (P(x) \rightarrow P(y))$$

of Drinker's Theorem. For this purpose one has to come up with a functional Φ such that

$$\forall f (P(\Phi(f)) \rightarrow P(f(\Phi(f))))$$

which can be achieved easily by defining

$$\Phi(f) = \mathbf{if} \ P(a) \rightarrow P(f(a)) \ \mathbf{then} \ a \ \mathbf{else} \ f(a)$$

where a is any constant of base type (e.g. 0). This Φ does the job since (even constructively) it holds that $(P(a) \rightarrow P(f(a))) \vee (P(f(a)) \rightarrow P(f(f(a))))$ as can be seen easily by case analysis on $P(a)$ and $\neg P(a)$ (which is admissible since P is decidable).

Obviously, this Φ is much simpler than the above somewhat lengthy λ -term.

135.2 Proving $\exists x \forall y (f(x) \leq f(y))$ using the Least Number Principle

Applying the Least Number Principle

$$\text{(LNP)} \quad \exists x A(x) \rightarrow \exists x (A(x) \wedge \forall y (A(y) \rightarrow x \leq y))$$

to $A(x) \equiv \exists z x = f(z)$ gives rise to

$$\exists x (\exists z x = f(z)) \rightarrow \exists x ((\exists z x = f(z)) \wedge \forall y ((\exists z y = f(z)) \rightarrow x \leq y))$$

from which it follows that $\exists x \forall y (f(x) \leq f(y))$.

⁸⁷Consider a prim. rec. predicate P such that $\forall n P(n)$ is true but not provable, e.g. $P(n) \equiv \neg \text{Prf}(n, \ulcorner 0=1 \urcorner)$. Now if $\exists x \forall y (P(x) \rightarrow P(y))$ were constructively provable then $\forall y (P(y) \rightarrow P(y))$ is constructively provable as well for some $n \in \mathbb{N}$. Thus, since $P(n)$ is constructively provable we obtain a constructive proof of $\forall y P(y)$ which is impossible by assumption.

135.3 Realizing the G -translation of the Least Number Principle

Let $A(x)$ be an arbitrary predicate. We write A^G for the Gödel-Gentzen negative translation of A .

By induction we can prove intuitionistically that

$$\forall x((\forall y (y < x \rightarrow \neg A^G(y))) \rightarrow \neg A^G(x)) \rightarrow \forall x \neg A^G(x)$$

From this it follows that

$$\forall x ((\forall y (A^G(y) \rightarrow x \leq y)) \rightarrow \neg A^G(x)) \rightarrow \forall x \neg A^G(x)$$

since $y < x \rightarrow \neg A^G(y)$ is intuitionistically equivalent to $A^G(y) \rightarrow x \leq y$ (because \leq is decidable and $x < y$ is intuitionistically equivalent to $\neg x \leq y$). Thus, by contraposition we get

$$\neg \forall x \neg A^G(x) \rightarrow \neg \forall x ((\forall y (A^G(y) \rightarrow x \leq y)) \rightarrow \neg A^G(x))$$

equivalent to

$$\neg \forall x \neg A^G(x) \rightarrow \neg \forall x \neg (A^G(x) \wedge (\forall y (A^G(y) \rightarrow x \leq y)))$$

which is the G -translation of LNP_A , i.e. the formula $\exists x A(x) \rightarrow \exists x (A(x) \wedge \forall y (A(y) \rightarrow x \leq y))$.

If A is a Σ_1 formula then the formula

$$\forall x ((\forall y (y < x \rightarrow \neg A^G(y))) \rightarrow \neg A^G(x)) \rightarrow \forall x \neg A^G(x)$$

is negative and its uniquely determined modified realizer carries no algorithmic information.

135.4 Realizing the R -translation of the Least Number Principle for Σ_1 predicates

Let $A(x)$ be an arbitrary predicate. We write A^R for the Gödel-Gentzen negative translation of A which is defined like the G -translation but replacing \neg by $\neg_R \equiv (-) \rightarrow R$ and defining $P^R \equiv \neg_R \neg_R P$ for prime formulas P .

By induction we can prove intuitionistically that

$$\forall x((\forall y (y < x \rightarrow \neg_R A^R(y))) \rightarrow \neg_R A^R(x)) \rightarrow \forall x \neg_R A^R(x)$$

From this it follows that

$$\forall x ((\forall y (A^R(y) \rightarrow \neg_R \neg_R x \leq y)) \rightarrow \neg_R A^R(x)) \rightarrow \forall x \neg_R A^R(x)$$

since $y < x \rightarrow \neg_R A^R(y)$ is intuitionistically equivalent to $A^R(y) \rightarrow \neg_R \neg_R x \leq y$ (as one easily sees by analysis of the cases $y < x$ and $x \leq y$). Thus, by contraposition we get

$$\neg_R \forall x \neg_R A^R(x) \rightarrow \neg_R \forall x ((\forall y (A^R(y) \rightarrow \neg_R \neg_R x \leq y)) \rightarrow \neg_R A^R(x))$$

equivalent to

$$\neg_R \forall x \neg_R A^R(x) \rightarrow \neg_R \forall x \neg_R (A^R(x) \wedge (\forall y (A^R(y) \rightarrow \neg_R \neg_R x \leq y)))$$

which is the R -translation of LNP_A , i.e. the formula $\exists x A(x) \rightarrow \exists x (A(x) \wedge \forall y (A(y) \rightarrow x \leq y))$.

If A and R are Σ_1 formulas then for showing that the formula

$$\forall x ((\forall y (y < x \rightarrow \neg_R A^R(y))) \rightarrow \neg_R A^R(x)) \rightarrow \forall x \neg_R A^R(x)$$

is modified realizable one has to apply the primitive recursor of Gödel's T at type level 3 which is the type of potential realizers for $\neg_R A_R$ (after reducing $\neg_R \neg_R \neg_R$ to \neg_R) if both A and R are Σ_1 formulas.

135.5 Functional interpretation of $(\text{LNP}_A)^G$ for Σ_1 predicates A

If one derives using classical logic a Σ_1 formula $\exists n P(n)$ from an instance of LNP where $A(x)$ is a Σ_1 predicate $\exists y R(x, y)$ then when unwinding such a proof via functional interpretation one can totally avoid the R -translation since functional interpretation validates Markov's principle at all types.

Applying the G -translation to such a classical proof we obtain an intuitionistic proof of $(\text{LNP}_A)^G \rightarrow \neg \neg \exists x P(x)$. Since under functional interpretation $\neg \neg \exists x P(x)$ is actually identical to $\exists x P(x)$ it suffices to give a functional interpretation of LNP_A .

As we have seen in section 3 it suffices to give a functional interpretation of

$$\forall x ((\forall y (y < x \rightarrow \neg A^G(y))) \rightarrow \neg A^G(x)) \rightarrow \forall x \neg A^G(x)$$

Since $A^G(x) \equiv \neg \exists y \neg R(x, y) \iff \neg \forall y \neg R(x, y) \iff \neg \forall y R(x, y)$ for functional interpretation of the induction principle for $A^G(x)$ we need the primitive recursor just at type 1.

Thus we conclude that

unwinding classical proofs of Σ_1 sentences from LNP_A with A of form Σ_1 functional interpretation is much more economical than modified realizability.

135.6 Functional interpretation of $(\exists x \forall y (f(x) \leq f(y)))^G$

Now we consider the case where a Σ_1 sentence has been derived from $A_f \equiv \exists x \forall y (f(x) \leq f(y))$ using classical logic. When using Friedman's trick followed by modified realizability we need primitive recursion at type level 2 as we have seen in sections 2 and 4. We show now that when using functional interpretation it suffices to employ the primitive recursor just at type level 0.

Suppose we are given a classical proof of

$$\exists x \forall y (f(x) \leq f(y)) \rightarrow \exists x P(x)$$

where $P(x)$ is primitive recursive. Then by G -translation we get an constructive proof of

$$(\exists x \forall y (f(x) \leq f(y)))^G \rightarrow \neg \neg \exists x P(x)$$

For unwinding this proof via functional interpretation we just have to come up with a functional interpretation of the premiss, i.e. of $\neg \forall x \neg \forall y (f(x) \leq f(y))$, which amounts to exhibiting a type 2 functional Φ such that

$$\forall g (f(\Phi(g)) \leq f(g(\Phi(g))))$$

Such a Φ can be defined as $\Phi(g) = \Psi(g, 0)$ where

$$\Psi(g, n) = \mathbf{if} \ f(g(n)) \leq f(g(g(n))) \ \mathbf{then} \ g(n) \ \mathbf{else} \ \Psi(g, g(n))$$

Although not given in prim. rec. form it can be easily turned into such since $\Psi(g, n)$ terminates after $\leq n+1$ unfoldings of the recursive definition of Ψ . Thus Ψ and consequently also Φ can be define in the fragment of Gödel's T which employs the primitive recursor at type level 0.

135.7 Modified Realization of $(\exists x \forall y (f(x) \leq f(y)))^G$

Suppose one has classically proven a Σ_1 statement $\exists x P(x)$ from $\exists x \forall y (f(x) \leq f(y))$.

Applying the R -translation of section 4 one arrives at a constructive proof of $(\exists x P(x))^R$ from $(\exists x \forall y (f(x) \leq f(y)))^R \equiv \neg_R \forall x \neg_R \forall y \neg_R (f(x) \leq f(y))$. The latter formula cannot be further simplified since $\neg_R \neg_R (f(x) \leq f(y))$ is not equivalent to $f(x) \leq f(y)$.

Thus, it is better to first apply the G -translation giving rise to a constructive derivation of $(\exists x P(x))^G$ from $(\exists x \forall y (f(x) \leq f(y)))^G$ which is equivalent to $\neg \forall x \neg \forall y (f(x) \leq f(y))$ since \leq is a decidable predicate. Now to this latter derivation one can apply Friedman's original R -translation, i.e. to replace any occurrence of a prime formula Q by $Q \vee R$ and \perp by R . Thus, it suffices to come up with a constructive proof of the original Friedmann translation of $\exists x \forall y (f(x) \leq f(y))$, i.e. $\neg_R \forall x \neg_R \forall y (f(x) \leq f(y) \vee R)$ which is certainly simpler than the previous $\neg_R \forall x \neg_R \forall y \neg_R \neg_R (f(x) \leq f(y))$.

For proving $\neg_R \forall x \neg_R \forall y (f(x) \leq f(y) \vee R)$ assume that

$$(1) \ \forall x \neg_R \forall y (f(x) \leq f(y) \vee R).$$

We will now prove by induction that

$$(2) \ \forall x \forall y (x \leq f(y) \vee R)$$

Obviously, we have $\forall y (0 \leq f(y) \vee R)$. Suppose as induction hypothesis that $\forall y (x \leq f(y) \vee R)$. We have to show that $\forall y (x+1 \leq f(y) \vee R)$. For arbitrary $y \in N$ if not already $x+1 \leq f(y)$ then we have to consider the following two remaining cases

Case $x = f(y)$: from (1) we know that $\neg_R \forall z (f(y) \leq f(z) \vee R)$ and thus $\neg_R \forall z (x \leq f(z) \vee R)$ from which it it follows by induction hypothesis that R

Case $f(y) < x$: since by induction hypothesis we have $x \leq f(y) \vee R$ it follows that R and thus $x+1 \leq f(y) \vee R$.

Now having established (2) we instantiate x by $f(0)+1$ and y by 0 and thus get $f(0)+1 \leq f(0) \vee R$. Since the first clause of the disjunction is wrong it follows that R .

Thus, since our proof for the original Friedman translation of $(\exists x \forall y (f(x) \leq f(y)))^G$ involves induction just for a Π_2 predicate we can conclude that there is a modified realizer for it using the primitiv recursor only at type levels ≤ 1 (which is 2 levels better than what one might expect from section 4).

For sake of concreteness we exhibit a modified realizer for $\neg_R \forall x \neg_R \forall y (f(x) \leq f(y) \vee R)$. Suppose $\phi \text{ mr } \forall x \neg_R \forall y (f(x) \leq f(y) \vee R)$. We now construct a $\psi \text{ mr } \forall x \forall y (x \leq f(y) \vee R)$ by putting

$$\psi 0 = \lambda x:N.0$$

and

$$\psi(x+1)(y) = \begin{cases} 0 & \text{if } x < f(y) \\ \phi(y)(\psi(x)) + 1 & \text{if } x = f(y) \\ \psi(x)(y) & \text{if } f(y) < x \end{cases}$$

where somewhat nonstandardly for prime P and Q we define $x \text{ mr } P \vee \exists x Q(x)$ as $(x=0 \wedge P) \vee (x>0 \wedge Q(x-1))$. Finally, putting

$$\Phi(\phi) = \psi(f(0)+1)(0) - 1$$

we arrive at a modified realizer for $\neg_R \forall x \neg_R \forall y (f(x) \leq f(y) \vee R)$ as desired.

We now construct a slightly simpler modified realizer for

$$\neg_R \forall x \neg_R \forall y (f(x) \leq f(y) \vee R)$$

where $R \equiv \exists u P(u)$ for some primitive predicate P . In order to simplify things let us say that $x \text{ mr } Q \vee R$ iff $Q \vee P(x)$.

Suppose

$$(1) \ \varphi \text{ mr } \forall x \neg_R \forall y (f(x) \leq f(y) \vee R).$$

We now show that for $\psi : N \rightarrow N \rightarrow N$ with

$$\psi(0) = \lambda y:N.0 \quad \psi(x+1) = \lambda y:N. \varphi y(\psi(x))$$

we have $\psi \text{ mr } \forall x \forall y (x \leq f(y) \vee R)$. We proceed by induction on x . For $x = 0$ we have $0 \leq f(y)$ and thus $\psi(0)(y) = 0$ modified realizes $f(0) \leq f(y) \vee R$. Suppose as induction hypothesis that

$$(2) \ \psi(x) \text{ mr } \forall y (x \leq f(y) \vee R)$$

We have to show that $\psi(x+1) \text{ mr } \forall y (x+1 \leq f(y) \vee R)$. If $x+1 \leq f(y)$ then any number and thus in particular $\psi(x+1)(y)$ modified realizes $x+1 \leq f(y) \vee R$. Assume $x+1 > f(y)$, i.e. $f(y) \leq x$. We show that $\psi(x) \text{ mr } \forall z (f(y) \leq f(z) \vee R)$

from which it follows by (1) that $\psi(x+1) = \varphi(y)(\psi(x))$ modified realizes R and thus also $x+1 \leq f(y) \vee R$. If $f(y) \leq f(z)$ then any number and thus in particular $\psi(x)(z) \text{ mr } f(y) \leq f(z) \vee R$. Otherwise we have $f(z) < f(y)$ and thus also $f(z) < x$ from which it follows by induction hypothesis (2) that $\psi(x)(z) \text{ mr } R$ and thus also $\psi(x)(z) \text{ mr } f(y) \leq f(z) \vee R$.

Since $\psi \text{ mr } \forall x \forall y (x \leq f(y) \vee R)$ we get that $\psi(f(0)+1)(0) \text{ mr } f(0)+1 \leq f(0) \vee R$ and thus $\psi(f(0)+1)(0) \text{ mr } R$ since $f(0) < f(0)+1$.

136 A Modified Realizer for PHP₂

Let f be a variable of type $N \rightarrow 2$ and R a propositional variable. Then the *binary pigeon hole principle* is

$$\text{PHP}_2 \quad (A_0 \rightarrow R) \rightarrow (A_1 \rightarrow R) \rightarrow R$$

where $A_i \equiv \forall n(\forall k(k \geq n \rightarrow f(k) = i \rightarrow R)) \rightarrow R$.

Suppose $p_0 \text{ mr } A_0 \rightarrow R$ and $p_1 \text{ mr } A_1 \rightarrow R$. We will construct a term t with $t \text{ mr } R$. For this purpose it suffices to construct a term t_1 with $t_1 \text{ mr } A_1$ because then

$$(1) \quad t \equiv p_1(t_1) \text{ mr } R.$$

Suppose $m \in N$ and $g_1 \text{ mr } \forall k(k \geq n \rightarrow f(k) = 1 \rightarrow R)$. We now construct a term t_2 with $t_2 \text{ mr } A_0$ because then

$$(2) \quad t_1 \equiv \lambda m. \lambda g_1. p_0(t_2) \text{ mr } A_1.$$

Suppose $n \in N$ and $g_0 \text{ mr } \forall k(k \geq n \rightarrow f(k) = 0 \rightarrow R)$. We now construct a term t_3 with $t_3 \text{ mr } R$ because then

$$(3) \quad t_2 \equiv \lambda n. \lambda g_0. t_3 \text{ mr } A_0.$$

Now if $f(\max(n, m)) = 0$ then $g_0(n) \text{ mr } R$ (since $\max(n, m) \geq n$) and if $f(\max(n, m)) = 1$ then $g_1(m) \text{ mr } R$ (since $\max(n, m) \geq m$). Thus we have

$$(4) \quad t_3 \equiv \mathbf{if} \ f(\max(n, m)) = 0 \ \mathbf{then} \ g_0(n) \ \mathbf{else} \ g_1(m) \ \text{mr } R.$$

Sticking these partial solutions together we get that

$$(5) \quad t \equiv p_1(\lambda m. \lambda g_1. p_0(\lambda n. \lambda g_0. \mathbf{if} \ f(\max(n, m)) = 0 \ \mathbf{then} \ g_0(n) \ \mathbf{else} \ g_1(m))) \ \text{mr } R.$$

Thus, a modified realizer for PHP₂ is given by

$$\lambda p_0. \lambda p_1. p_1(\lambda m. \lambda g_1. p_0(\lambda n. \lambda g_0. \mathbf{if} \ f(\max(n, m)) = 0 \ \mathbf{then} \ g_0(n) \ \mathbf{else} \ g_1(m)))$$

where $g_0, g_1 : N \rightarrow R$ and $p_0, p_1 : (N \rightarrow (N \rightarrow R)) \rightarrow R$.

137 Storage Operators for Natural Numbers

Let $\text{Int}(x) \equiv \forall X [\forall y (X(y) \rightarrow X(sy)), X(0) \rightarrow X(x)]$. Then its negative translation $\text{Int}^*(x) \equiv \forall X [\forall y (\neg X(y) \rightarrow \neg X(sy)), \neg X(0) \rightarrow \neg X(x)]$. We want to show that $T = \lambda f \lambda n n(\lambda g g \circ s) f 0$ (with $0 \equiv \lambda f \lambda x x$ and $s \equiv \lambda k \lambda f \lambda z k f(fz)$) has type $\forall x (\neg \text{Int}(x) \rightarrow \neg \text{Int}^*(x))$.

For that purpose suppose $f : \neg \text{Int}(n)$ and $n : \text{Int}^*(n)$. Consider the predicate $F(x) \equiv \forall X [\forall y (\neg X(y) \rightarrow \neg X(sy)) \rightarrow \neg X(x) \rightarrow \neg X(n)]$. From $n : \text{Int}^*(n)$ it follows that $n : \forall y (\neg F(y) \rightarrow \neg F(sy)) \rightarrow \neg F(0) \rightarrow \neg F(n)$. Obviously, we have $0 : F(n)$ and $f : \neg F(0)$ since $F(0) \equiv \text{Int}(n)$. Thus $n(\lambda g g \circ s) f 0$ is of the desired type \perp if we can show that $\lambda g g \circ s$ is of type $\neg F(x) \rightarrow \neg F(sx)$. But for that purpose it suffices to show that $s : F(sx) \rightarrow F(x)$. For that purpose suppose $k : F(sx) \equiv \forall X [\forall y (\neg X(y) \rightarrow \neg X(sy)) \rightarrow \neg X(sx) \rightarrow \neg X(n)]$, $f : \forall y (\neg X(y) \rightarrow \neg X(sy))$ and $z : \neg F(x)$. But then $fz : \neg X(sx)$ and thus $k f(fz) : \neg X(n)$. Thus, we have $s \equiv \lambda k \lambda f \lambda z k f(fz) : F(sx) \rightarrow F(x)$ as desired.

There is another storage operator for Int which has the following simpler derivation. Suppose $f : \neg \text{Int}(n)$ and $n : \text{Int}^*(n)$. Instantiating X by $\neg \text{Int}$ we obtain $n : \forall y (\neg \neg \text{Int}(y) \rightarrow \neg \neg \text{Int}(sy)), \neg \neg \text{Int}(0) \rightarrow \neg \neg \text{Int}(n)$. Since $s : \text{Int}(y) \rightarrow \text{Int}(sy)$ we have $\lambda g \lambda h g(h \circ s) : \forall y (\neg \neg \text{Int}(y) \rightarrow \neg \neg \text{Int}(sy))$. Moreover, we have $\lambda h h 0 : \neg \neg \text{Int}(0)$. Thus $n(\lambda g \lambda h g(h \circ s))(\lambda h h 0) f : \perp$ and, therefore, we have $T' \equiv \lambda f \lambda n n(\lambda g \lambda h g(h \circ s))(\lambda h h 0) f : \forall n (\neg \text{Int}(n) \rightarrow \neg \text{Int}^*(n))$ from which it follows that T' is a storage operator for natural numbers. Though the derivation of T' is simpler it is slightly more complicated than T .

138 Krivine's Classical Realizability

Terms are defined inductively as

$$t ::= x \mid tt \mid \lambda xt \mid cc \mid k_\pi$$

where π ranges over the set Π of stacks, i.e. finite lists of closed terms. We write Λ_0 for the set of closed terms. A *position* is a pair $(t, \pi) \in \Lambda \times \Pi$ usually denoted as $t \star \pi$. Let \succ be the least reflexive and transitive relation on $\Lambda \times \Pi$ such that

$$\begin{aligned} ts \star \pi &\succ t \star s.\pi \\ \lambda xt \star s.\pi &\succ t[s/x] \star \pi \\ cc \star t.\pi &\succ t \star k_\pi.\pi \\ k_\pi \star t.\pi' &\succ t \star \pi \end{aligned}$$

A set $\perp \subseteq \Lambda \times \Pi$ is called *saturated* iff $p \succ q \in \perp$ implies $p \in \perp$. A set $\perp \subseteq \Lambda \times \Pi$ is called *strongly saturated* iff \perp is saturated and $\perp \ni p \succ q$ implies $q \in \perp$.

A model \mathcal{M} is given by an underlying set nonempty set $M = |\mathcal{M}|$ and a mapping $f^{\mathcal{M}} : M^n \rightarrow M$ for every n -ary function symbol f . For every closed term t we denote its interpretation in \mathcal{M} by $t^{\mathcal{M}}$.

With every formula A we inductively associate a subset $\|A\|$ of Π as follows

$$\begin{aligned}
\|\top\| &= \emptyset \\
\|R(t_1, \dots, t_n)\| &= R(t_1^M, \dots, t_n^M) \\
\|A \rightarrow B\| &= \{(t, \pi) \mid p \in |A| \text{ and } \pi \in \|B\|\} \\
\|\forall x A(x)\| &= \bigcup_{a \in M} \|A(a)\| \\
\|\forall X A(X)\| &= \bigcup_{R \in \mathcal{P}(\Pi)^{M^n}} \|A(R)\|
\end{aligned}$$

where $R \in \mathcal{P}(\Pi)^{M^n}$ and $|A| = \{t \mid \forall \pi \in \|A\| \ t \star \pi \in \perp\}$. Notice that $|\top| = \Lambda$. For $\perp = \forall X X$ we have $\|\perp\| = \Pi$ and thus $|\perp| = \{t \mid \forall \pi \in \Pi \ t \star \pi \in \perp\}$.

We always have $|A \rightarrow B| \subseteq |A| \rightarrow |B| = \{t \in \Lambda \mid \forall s \in |A| \ ts \in |B|\}$ since by saturatedness of \perp from $t \star s.\pi \in \perp$ it follows that $ts \star \pi \in \perp$. However, for strongly saturated \perp we also have $|A| \rightarrow |B| \subseteq |A \rightarrow B|$ since in presence of this stronger assumption from $\forall s \in |A| \forall \pi \in \|B\| \ ts \star \pi \in \perp$ it follows that $\forall s \in |A| \forall \pi \in \|B\| \ t \star s.\pi \in \perp$.

One might ask whether $\|A \rightarrow B\|$ contains all stacks π such that $t \star \pi \in \perp$ for all $t \in |A \rightarrow B|$. This, however, will not be the case in general when \perp is the empty set of configurations. If both $|A|$ and $\|B\|$ are nonempty then $\|A \rightarrow B\|$ is nonempty but does not contain the empty stack ε although $t \star \varepsilon \in \perp$ for all $t \in |A \rightarrow B| = \emptyset$.

138.1 Realizing Krivine's Axiom

Countable choice can be proved⁸⁸ from Krivine's Axiom

$$(KA) \quad \exists S \forall x [\forall n^{\text{Int}} A(x, S_{x,n}) \rightarrow \forall X A(x, X)]$$

For realizing (KA) we have to introduce the following extension of our programming language. For every $t \in \Lambda_0$ let n_t be a Gödel number for t , i.e. $\tau_{n_t} = t$ where (τ_n) is some Gödelization of Λ_0 . We introduce a new language construct χ^* with evaluation rule

$$\chi^* \star t.\pi \succ t \star \underline{n}_t.\pi$$

and will show that χ^* realizes KA.

Let $P_n(\Pi) = \{\pi \in \Pi \mid \tau_n \star \underline{n}.\pi \notin \perp\}$ for $n \in \mathbb{N}$. If $P_n(\perp) \cap \|\forall X A(x, X)\|$ is nonempty then there exists a predicate S with $P_n(\perp) \cap \|A(x, S)\|$ nonempty. Thus, by countable choice on the metalevel there exists a predicate S such that from $P_n(\perp) \cap \|\forall X A(x, X)\| \neq \emptyset$ it follows that $P_n(\perp) \cap \|A(x, S_{x,n})\| \neq \emptyset$.

We now show that χ^* realizes $\forall x [\forall n^{\text{Int}} A(x, S_{x,n}) \rightarrow \forall X A(x, X)]$. For that purpose suppose $x \in \mathbb{N}$ and $t \Vdash \forall n(\text{Int}(n) \rightarrow A(x, S_{x,n}))$ and $\pi \in \|\forall X A(x, X)\|$. We have to show that $\chi^* \star t.\pi \in \perp$. Let $n := n_t$. Thus $\tau_n = t$. Since \perp is saturated and $\chi^* \star t.\pi \succ \tau_n \star \underline{n}_t.\pi$ it suffices to show that $\tau_n \star \underline{n}.\pi \in \perp$. We argue by contradiction. Suppose $\tau_n \star \underline{n}.\pi \notin \perp$. Then $\pi \in P_n(\perp) \cap \|\forall X A(x, X)\|$. Thus there exists $\pi' \in P_n(\perp) \cap \|A(x, S_{x,n})\|$. But by assumption on t we have $t \star \underline{n}.\pi' \in \perp$ in contradiction with $\pi' \in P_n(\perp)$.

⁸⁸since it allows one to reduce $AC_{0,1}$ to $AC_{0,0}$ and the latter holds by the least number principle provable in classical second order arithmetic

139 An Axiomatic Account of Krivine’s Classical Realizability

Let Λ and Π be sets of “terms” and “stacks”, respectively. We assume that Λ contains the elements K , S and cc . Moreover, there is a binary application operation on Λ denoted by juxtaposition ts , a push operation $\Lambda \times \Pi \rightarrow \Pi$ denoted as $t.\pi$ and a unary operation $\Pi \rightarrow \Lambda$ denoted as k_π . Such a structure is called a *realizability structure*.

A subset \perp of $\Lambda \times \Pi$ is called *saturated* iff its complement $\perp^c = \Lambda \times \Pi \setminus \perp$ satisfies the following closure conditions

- (S1) $ts \star \pi$ in \perp^c implies $t \star s.\pi$ in \perp^c
- (S2) $K \star t.s.\pi$ in \perp^c implies $t \star \pi$ in \perp^c
- (S3) $S \star t.s.u.\pi$ in \perp^c implies $t \star u.su.\pi$ in \perp^c
- (S4) $cc \star t.\pi$ in \perp^c implies $t \star k_\pi.\pi$ in \perp^c
- (S5) $k_\pi \star t.\pi'$ in \perp^c implies $t \star \pi$ in \perp^c .

A *proposition* A is given by a subset $\|A\| \subseteq \Pi$. The set of *realizers* for A is given by $|A| = \{t \in \Lambda \mid \forall \pi \in \|A\| \ t \star \pi \in \perp\}$. Logic is interpreted as follows $\|A \rightarrow B\| = |A|. \|B\| = \{t.\pi \mid t \in |A|, \pi \in \|B\|\}$ and $\|\forall x A(x)\| = \bigcup_a \|A(a)\|$.

J.-L. Krivine thinks of this as “generalized Cohen forcing”. Let P be a poset (of so-called “forcing conditions”) which we assume to have finite infima and a least element 0.⁸⁹

Such a poset P gives rise to a realizability structure in the following way. Put $\Lambda = \Pi = P$, interpret both application and the push operation as binary meet \wedge , interpret k as the identity on P and the remaining constants K , S and cc as 1, the greatest element of P . One easily checks that for every downward closed subset \mathcal{D} of P the set $\perp_{\mathcal{D}} = \{(p, q) \in P \times P \mid p \wedge q \in \mathcal{D}\}$ is saturated.

Since \mathcal{D} is downward closed for arbitrary $\|A\| \subseteq P$ the set

$$|A| = \{p \in P \mid \forall q \in \|A\| \ (p, q) \in \perp_{\mathcal{D}}\} = \{p \in P \mid \forall q \in \|A\| \ p \wedge q \in \mathcal{D}\}$$

is always downward closed and contains \mathcal{D} as a subset. Moreover, for $\|\perp\| = P$ we have $|\perp| = \{p \in P \mid \forall q \in P \ p \wedge q \in \mathcal{D}\} = \mathcal{D}$ where the latter equality holds since P has a greatest element (and thus $p = p \wedge 1 \in \mathcal{D}$ iff $\forall q \in P \ (p \wedge q \in \mathcal{D})$ since \mathcal{D} is downward closed).

For propositions A and B we have

$$\begin{aligned} p \in |A \rightarrow B| & \quad \text{iff} \quad \forall q \in |A| \ \forall r \in \|B\| \ p \wedge q \wedge r \in \mathcal{D} \\ & \quad \text{iff} \quad \forall q \in |A| \ p \wedge q \in |B| \\ & \quad \text{iff} \quad \forall q \leq p \ (q \in |A| \rightarrow q \in |B|) \end{aligned}$$

⁸⁹Usually, one doesn’t assume such a least element 0 but one may always add it putting $p \wedge q$ if p and q don’t have a common lower bound in the original poset.

showing that implication works like in forcing over posets, i.e. like in the Heyting algebra of downward closed subsets of P . Thus, we get, in particular, that $|\neg A| = \{p \in P \mid \forall q \in |A| p \wedge q \in \mathcal{D}\} = \{p \in P \mid \forall q \leq p (q \in |A| \rightarrow q \in \mathcal{D})\}$ since $\neg A = A \rightarrow \perp$.

Thus, this interpretation of logic is a particular instance of Girard's phase semantics where the monoid is $(P, \wedge, 1)$ and $\perp = \mathcal{D}$. As usual one defines for $X \subseteq P$ its orthogonal as $X^\perp = \{p \in P \mid \forall q \in X p \perp q\}$ where $p \perp q$ iff $p \wedge q \in \mathcal{D}$. It is easy to show that $X \subseteq X^{\perp\perp}$ and $Y^\perp \subseteq X^\perp$ whenever $X \subseteq Y$ from which it follows that $X^\perp = X^{\perp\perp\perp}$. Since $|A| = \||A|\|^\perp$ and $|\neg A| = |A|^\perp$ we have $|\neg\neg A| = |A|^{\perp\perp} = \||A|\|^{\perp\perp\perp} = \||A|\|^\perp = |A|$.

If P is a meet semilattice and $\mathcal{D} = \{0_P\}$ then sets of the form $|A|$ are in 1-1-correspondence with subsets a of $P^\dagger = P \setminus \{0\}$ such that $p \in a$ whenever $\forall q \leq p \exists r \leq q r \in a$. The latter arise as interpretations of propositions in forcing over the poset P^\dagger . It is in this sense that J.-L. Krivine considers his classical realizability as a generalization of Cohen forcing over (nonempty) posets (with conditional finite meets).

We will now characterize those realizability structures induced by a \wedge -semilattice. First of all Λ has to be a \wedge -semilattice w.r.t. the binary operation of application, i.e. applications has to be associative, commutative and idempotent and there is a neutral element 1 for application. Secondly the map $k : \Pi \rightarrow \Lambda$ has to be bijective. Thus w.l.o.g. we will consider k as an identity. Accordingly, we write P for $\Lambda = \Pi$. Thirdly, the application and the push operation coincide, i.e. $ts = t.s$ for all $t, s \in P$.

Under these assumptions for a saturated set \perp we may define $\mathcal{D} = \{t \in \Lambda \mid (t, 1) \in \perp\}$.

Lemma 139.1 *If $t \in \mathcal{D}$ then $(t, s) \in \perp$ for all s .*

Proof: Suppose $t \in \mathcal{D}$, i.e. $(t, 1) \in \perp$, and $s \in P$. Then by (S5) it follows that $(1, t.s) \in \perp$ and thus $(t, s) = (1t, s) \in \perp$ by (S1). \square

We get as a consequence that \mathcal{D} is downward closed (where $x \leq y$ iff $xy = x$).

Lemma 139.2 *\mathcal{D} is downward closed, i.e. $y \leq x \in \mathcal{D}$ implies $y \in \mathcal{D}$.*

Proof: Suppose $x \in \mathcal{D}$, i.e. $(x, 1) \in \perp$. From Lemma 139.1 it follows that $(x, y.1) \in \perp$. From (S1) it follows that $(xy, 1) \in \perp$, i.e. $y = xy \in \mathcal{D}$. \square

Lemma 139.3 *For all $t, s \in P$ we have $(t, s) \in \perp$ iff $ts \in \mathcal{D}$.*

Proof: Suppose $(t, s) \in \perp$. Since $s.1 = s$ by our third assumption it follows that $(t, s.1) \in \perp$. Thus by (S1) we get $(ts, 1) \in \perp$, i.e. $ts \in \mathcal{D}$.

Suppose $ts \in \mathcal{D}$, i.e. $(ts, 1) \in \perp$. Then by (S5) it follows that $(1, ts.1) \in \perp$. Since by our third assumption we have $ts.1 = ts = t.s$ it follows that $(1, t.s) \in \perp$. Thus $(t, s) = (1t, s) \in \perp$. \square

Thus, under our three assumptions we have established a 1-1-correspondence between saturated sets \perp and downclosed subsets \mathcal{D} given by $(t, s) \in \perp$ iff $ts \in \mathcal{D}$.

Thus we have shown that the above three assumptions characterize those realizability structures corresponding to Cohen forcing.

140 Krivine’s Thread Model is “Nonstandard”

For the models considered in section 138 if the pole \perp is nonempty, i.e. contains a process $t \star \pi$, then the term $k_\pi t \in |\perp|$ because for every $\pi' \in \Pi$ we have $k_\pi t \star \pi' \succ t \star \pi \in \perp$. For this reason it is crucial to restrict attention to *proof-like term*, i.e. terms not containing any occurrence of k . We write PL for the set of closed proof like terms. We say that A holds in the realizability model iff $|A|$ contains a proof like term, i.e. $|A| \cap \text{PL} \neq \emptyset$.

But there is still the problem of finding a pole \perp such that $|\perp|$ is not inhabited by a proof like term, i.e. $\forall t \in \text{PL}. \exists \pi \in \Pi. t \star \pi \notin \perp$. This is not so easy to achieve. For this purpose Krivine has introduced a countable set $\Pi_0 = \{\pi_n \mid n \in \mathbb{N}\}$ of so-called *stack constants* from which stacks are constructed by pushing terms on them. Let $\{\xi_n \mid n \in \mathbb{N}\}$ be an enumeration of PL . The n -th *thread* is defined as $\vartheta_n = \{t \star \pi \mid \xi_n \star \pi_n \succ t \star \pi\}$. Now, when defining \perp as the complement of $\bigcup_{n \in \mathbb{N}} \vartheta_n$ we have $\xi_n \star \pi_n \notin \perp$, i.e. $\xi_n \notin |\perp|$.

Like for any pole we may consider for this particular one the associated classical realizability model for ZF, called the *thread model*. For every set I of the *ground model* we may consider $\mathfrak{J}(I)$ corresponding to \check{I} in traditional set theoretic notation and $\nabla(I)$ in (categorical) realizability. In forcing over a cBa B , \mathfrak{J} is (the object part of) Δ left adjoint to $\Gamma : \mathbf{Sh}(B) \rightarrow \mathbf{Set}$. Since Δ preserves colimits it follows that Δ preserves ordinals.

This, however, is not the case for the thread model. Krivine has shown that for the predicate $P(x) \equiv x \neq 0 \wedge x \neq 1$ the thread model validates $\exists x P(x)$, $\neg P(0)$ and $\neg P(1)$.⁹⁰ Moreover, for a (non-syntactical) predicate P on $\mathfrak{J}(\mathbb{N})$ validates $\exists! x P(x)$ and $\neg P(n)$ for all $n \in \mathbb{N}$. Thus, in the thread model $\mathfrak{J}(\mathbb{N})$ has global elements which do not correspond to elements of the underlying set \mathbb{N} .⁹¹ The set $\tilde{\mathbb{N}}$ of natural numbers in the thread model is isomorphic to a subobject of $\mathfrak{J}(\mathbb{N})$. Krivine has constructed a function $\delta : \mathbb{N} \rightarrow 2$ such that

$$(\dagger) \quad \forall x \in \mathfrak{J}(2) (x \neq 0 \rightarrow \exists n \in \tilde{\mathbb{N}} 0 \neq \delta(n) \leq x)$$

holds in the thread model when adding a further language construct. From this it follows that in any Tarski model for the theory validated by the thread model there must exist a nonstandard natural number. Let $u \in \mathfrak{J}(2)$ different from 0 and 1 for which (\dagger) guarantees the existence of $n \in \tilde{\mathbb{N}}$ with $\delta(n) = u$ which must be nonstandard since otherwise u were equal to 0 or 1 contradicting the assumption on u .

⁹⁰In boolean valued $\mathfrak{J}(2)$ also has nonstandard global elements but nevertheless it holds that $\llbracket \exists x : \mathfrak{J}(2) P(x) \rrbracket = \llbracket P(\check{0}) \vee P(\check{1}) \rrbracket$.

⁹¹This, however, also happens in boolean valued models of ZF where global elements of the set of natural numbers correspond to countable partitions of the cBa and there are always nontrivial ones if the cBa contains more than two elements. However, in boolean valued models one has $\llbracket \exists x \in \mathfrak{J}\mathbb{N} A(x) \rrbracket = \bigvee_{n \in \mathbb{N}} \llbracket A[\check{n}/x] \rrbracket$.

141 Classical Realizability Model for ZF

Let \mathcal{M} be the ground model for set theory (ZFC) and (Λ, Π, \dots) a classical realizability structure and \perp a pole for it both living in \mathcal{M} . We consider the induced classical realizability model $V_{\perp}^{(\Pi)}$ which has the same underlying set as \mathcal{M} . For sets a and b we define

$$\|b \not\subseteq a\| = \{\pi \in \Pi \mid \langle b, \pi \rangle \in a\}$$

and by transfinite recursion we define the relations \subseteq and $\not\subseteq$ as

$$\|a \subseteq b\| = \|\forall x (x \not\subseteq b \rightarrow x \not\subseteq a)\| = \bigcup_{c \in \text{dom}(a)} |c \not\subseteq b|. \|c \not\subseteq a\|$$

$$\|a \not\subseteq b\| = \|\forall x (a \subseteq x \rightarrow x \subseteq a \rightarrow x \not\subseteq b)\| = \bigcup_{c \in \text{dom}(b)} |a \subseteq c|. |c \subseteq a|. \|c \not\subseteq b\|$$

where as usual we write $|A|$ for $\|A\|^{\perp}$. Thus, the model of ZF depends on the choice of \perp for which reason we write $\mathcal{M}_{\perp}^{(\Pi)}$ for it.

With every natural number $n \in \omega$ we associate a name \tilde{n} defined inductively as follows

$$\tilde{0} = \emptyset \quad \text{and} \quad \widetilde{n+1} = \tilde{n} \cup (\{\tilde{n}\} \times (\{\tilde{n}\}. \Pi))$$

where \tilde{n} is the Church numeral $\lambda z \lambda s s^n z$ for n . The set of natural numbers in $\mathcal{M}_{\perp}^{(\Pi)}$ is then given by $\tilde{\omega} = \bigcup_{n \in \omega} \tilde{n}$. Writing 0 for $\tilde{0}$ and s for the endo map on

$\tilde{\omega}$ with $s(\tilde{n}) = \widetilde{n+1}$ one observes that the following sentences are realized (by proof-like objects)

- (1) $\forall x (x \not\subseteq 0)$
- (2) $0 \varepsilon \tilde{\omega}$
- (3) $(\forall x \in \tilde{\omega}) \forall y (y \varepsilon s(x) \leftrightarrow y \varepsilon x \vee y = x)$
- (4) $(\forall x \varepsilon \tilde{\omega}) s(x) \in \tilde{\omega}$
- (5) $\phi(0) \rightarrow (\forall x \varepsilon \tilde{\omega}) (\phi(x) \rightarrow \phi(s(x))) \rightarrow (\forall x \varepsilon \tilde{\omega}) \phi(x)$

where ϕ may contain the non extensional predicate ε .⁹² Given a sequence $(a_n)_{n \in \omega}$ of elements of \mathcal{M} we write \hat{a} for the endomap on \mathcal{M} mapping \tilde{n} to a_n and $b \in \mathcal{M} \setminus \text{dom}(\hat{a})$ to $\tilde{0}$.

Now for every predicate ϕ with parameters in $\mathcal{M}_{\perp}^{(\Pi)}$ we will construct a sequence a in \mathcal{M} such that the sentence

$$\forall n (\phi(\hat{a}(n)) \rightarrow n \not\subseteq \tilde{\omega}) \rightarrow \forall x \neg \phi(x)$$

⁹²As opposed to A. Miquel I think that (1)-(4) hold also when replacing ε by \in .

is realized by the proof like term $\lambda xy. \text{quote } y(xy)$ where **quote** is a new constant satisfying the reduction rule

$$\text{quote} \star t.u.\pi \succ u \star \overline{n_t}.\pi$$

where n_t is the Goedel number of term t (we write t_n for the closed term with Goedel number n). Thus, there is a proof like term realizing the sentence

$$\exists x \phi(x) \rightarrow \exists n M_\phi(n)$$

where

$$M_\phi(n) \equiv \phi(\widehat{a}(n)) \wedge (\forall m \varepsilon \tilde{\omega})(\phi(\widehat{a}(m)) \rightarrow m \notin n)$$

states that n is the least number m with $\phi(\widehat{a}(m))$. Now let $A = \bigcup_{n \in \omega} \text{dom}(a_n)$

and

$$a = \bigcup_{b \in A} (\{b\} \times \|\!(\forall n \varepsilon \tilde{\omega}) M_\phi(n) \rightarrow b \notin \widehat{a}(n)\|\!|)$$

Thus $\text{dom}(a) = A = \bigcup_{n \in \omega} \text{dom}(a_n)$ and

$$\|\!|b \notin a\|\!| = \|\!(\forall n \varepsilon \tilde{\omega}) M_\phi(n) \rightarrow b \notin \widehat{a}(n)\|\!|$$

Thus, the sentence

$$\forall x (x \varepsilon a \leftrightarrow (\exists n \varepsilon \tilde{\omega}) M_\phi(n) \wedge x \varepsilon \widehat{a}(n))$$

holds in $\mathcal{M}_\perp^{(\Pi)}$ and thus also

$$(\forall n \varepsilon \tilde{\omega}) M_\phi(n) \rightarrow \widehat{a}(n) \approx a$$

where $x \approx y$ stands for $x \subseteq y \wedge y \subseteq x$, i.e. x and y being extensionally equal. Accordingly, in $\mathcal{M}^{(\Pi)\perp}$ it holds that

$$\exists x \phi(x) \rightarrow \exists x (x \approx a \wedge \phi(x))$$

and thus also

$$\exists x \phi(x) \rightarrow \phi(a)$$

provided ϕ is *extensional*, i.e. built up from \exists and \approx .

Maps tracking functional relations in $V^{(\Pi)}$

Every element a of $V^{(\Pi)}$ may be considered as an object of the classical realizability topos when endowed with the restriction of the equality predicate on $V^{(\Pi)}$ to $\text{dom}(a)$. Under this identification $V_{\alpha+1}^{(\Pi)}$ is isomorphic to $\mathcal{P}(V_\alpha^{(\Pi)})$ which is a weakly complete object of the classical realizability topos.

Suppose $a \in V^{(\Pi)}$ and $V_\perp^{(\Pi)}$ validates $\forall x \in a \exists^1 y F(x, y)$. Then for some ordinal α we have $\forall x \in a \exists^1 y \in V_{\alpha+1}^{(\Pi)} F(x, y)$. Since $\mathcal{P}(V_\alpha^{(\Pi)})$ is weakly complete there exists a map $f : \text{dom}(a) \rightarrow V_{\alpha+1}^{(\Pi)}$ tracking the functional relation F .

142 Cohen Forcing over a Poset of Conditions

Let \mathbb{P} be a poset or, slightly more general, a preorder which typically has a greatest element 1. In most cases \mathbb{P} is a conditional meet-semilattice, i.e. whenever $r \leq p, q$ then $p \wedge q$ exists in \mathbb{P} . For such a \mathbb{P} one may consider the complete Heyting algebra $\Omega = \mathbf{dcl}(\mathbb{P})$ of downward closed subsets of \mathbb{P} ordered by subset inclusion. Then $B = \Omega_{\neg}$ is a complete Boolean algebra. Cohen forcing amounts to considering the B -valued model $V^{(B)}$ of ZFC. In topos-theoretic terms it corresponds to the interpretation of ZFC in the localic topos $\mathbf{Sh}(B) \simeq \mathbf{Sh}(\mathbf{dcl}(\mathbb{P}))_{\neg} \simeq \widehat{\mathbb{P}}_{\neg} = \mathbf{Set}^{\mathbb{P}^{\text{op}}}$. For $A \in \mathbf{dcl}(\mathbb{P})$ we have $p \in \neg A$ iff for all $q \leq p$ exists $r \leq q$ with $r \in A$. We have $\neg \downarrow p = \downarrow p$ iff for every q with $q \not\leq p$ there exists an $r \leq q$ such that $p \perp r$, i.e. p and r do not have a common lower bound. Such a poset \mathbb{P} is called *regular*.

One may embed V into $V^{(B)}$ by sending $x \in V$ to

$$\widehat{x} = \{\langle \widehat{y}, 1 \rangle \mid y \in x\}$$

in $V^{(B)}$ (i.e. we define the operation $\widehat{\cdot}$ by transfinite recursion over \in). Apparently we have $\widehat{\emptyset} = \emptyset$ and $\widehat{1} = \{\langle \emptyset, 1 \rangle\}$.

The set of truth values in $V^{(B)}$ is given by $\Omega = \{\langle \{\langle \emptyset, b \rangle\}, 1 \rangle \mid b \in B\}$. If \mathbb{P} is regular then $\Gamma = \{\langle \widehat{p}, \downarrow p \rangle \mid p \in \mathbb{P}\}$ is called the *generic set*.⁹³ In the topos $B\text{-Set}$ of B -valued sets Ω is given by the object whose underlying set is B and where $\llbracket a =_{\Omega} b \rrbracket = a \leftrightarrow b$ and Γ is given by the object with underlying set \mathbb{P} and where $\llbracket p =_{\Gamma} q \rrbracket = \{r \in \mathbb{P} \mid r \leq p = q\}$. In $\mathbf{Sh}(\mathbb{P})_{\neg}$ the object Ω is given by $\Omega(p) = \{a \in B \mid a \subseteq \downarrow p\}$ and $\Omega(q \leq p)(a) = a \cap \downarrow q$ and the object Γ is given by $\coprod_{p \in \mathbb{P}} y(p)$, i.e. the sum of representables. Thus Γ is obtained by $(\neg\neg)$ -sheafifying the presheaf G where $G(p) = \uparrow p$ and $G(q \leq p)$ is the inclusion of $\uparrow p$ into $\uparrow q$, i.e. G is isomorphic to $\coprod_{p \in \mathbb{P}} y(p)$ in presheaves over \mathbb{P} .

143 Ω vs. Generic Set in $\mathbf{Sh}(A)$

Let A be a complete Heyting algebra. The truth value object in $A\text{-Set}$ is given by $(A, \llbracket \cdot \sim_{\Omega} \cdot \rrbracket)$ where $\llbracket a \sim_{\Omega} b \rrbracket = a \leftrightarrow b$. The corresponding $\Omega \in \mathbf{Sh}(A)$ is given by $\Omega(a) = \{b \in A \mid b \leq a\}$.

The generic set Γ in $A\text{-Set}$ is given by $(A, \llbracket \cdot \sim_{\Gamma} \cdot \rrbracket)$ where $\llbracket a \sim_{\Gamma} b \rrbracket = a$ if $a = b$ and $\llbracket a \sim_{\Gamma} b \rrbracket = \perp$ otherwise. A partial element of Γ is (given by) a function $p : A \rightarrow A$ such that for all $a, b \in A$ it holds that

- i) $p(a) \leq a$ and
- ii) $p(a) \wedge p(b) = \perp$ whenever $a \neq b$.

⁹³since for a generic ultrafilter G on \mathbb{P} we have $\Gamma_G = G$ where for $x \in V^{(B)}$ the set $x_G = \{\widehat{y} \mid \exists b \in B \langle y, b \rangle \in x \text{ and } b \cap G \neq \emptyset\}$

The extent of p is $E(p) = \bigvee_{a \in A} p(a)$. If $p(a) = p(b)$ and $a \neq b$ then $p(a) \wedge p(b) = \perp$ and thus $p(a) = p(b) = \perp$. Accordingly, if $p(a) = p(b) \neq \perp$ then $a = b$. Thus, partial elements of Γ can equivalently be given by partial one-to-one functions $f : A \rightarrow A$ such that

- i') the domain of f is a partition of an element of A and
- ii') $a \leq f(a)$ for all $a \in \text{dom}(f)$

Thus, the corresponding $\Gamma \in \text{Sh}(A)$ can be described as follows. For $a \in A$ the set $\Gamma(a)$ consists of those partial elements f with $\bigvee \text{dom}(f) = a$ and for $b \leq a$ the restriction⁹⁴ $f \upharpoonright b$ is defined on all $c \wedge b$ with $c \in \text{dom}(p)$ and $c \wedge b \neq \perp$ and sends those elements to $f(c)$. Apparently Γ is isomorphic to $\coprod_{a \in A} \gamma(a)$ in $\text{Sh}(A)$.

144 The home of the generic set in $\text{Sh}(A)$

Suppose A is a cHa. Then in $\text{Sh}(A)$ we may consider the sheaf X_A where $X_A(u)$ is the set of all maps $f : A \rightarrow A$ such that f preserves all joins and all binary meets and $f(1_A) = u$, i.e. all frame morphisms from A to $\downarrow_A(u)$. For $v \leq u$ and $f \in X_A(u)$ we define $f \upharpoonright u(v) = f(v) \wedge u$. The *generic element* of X_A picks for every $u \in A$ the function $g_u : v \mapsto u \wedge v$ which corresponds to the “generic set” of the A -valued model of IZF.

In case of $A = \mathcal{O}(T)$ for some topological space T the sheaf $X_{\mathcal{O}(T)}$ is isomorphic to the sheaf X_T where $X_T(U)$ is the set of continuous functions from U to T and restriction to $V \subseteq U$ is given by the restriction of a continuous function on U to the open subset V . The generic element of X_T picks the inclusion of U into T for every $U \in \mathcal{O}(X)$.

I think this makes precise what R. Lubarsky has in mind with his “generic sets” in topological or Heyting valued models of IZF.

145 Boolean Completions of Regular Posets Exist and are Unique up to Isomorphism

Let P be poset. Then $\text{dcl}(P)$ is the poset of downward closed subsets of P partially order by \subseteq . One readily checks that $\text{dcl}(P)$ is a complete Heyting algebra where infima and suprema are given by intersections and unions, respectively, and implication is given by

$$A \rightarrow B = \{p \in P \mid \forall q \leq p (q \in A \Rightarrow q \in B)\}$$

for $A, B \in \text{dcl}(P)$. Since the least element 0 is the empty set we have

$$\neg A = A \rightarrow 0 = \{p \in P \mid \forall q \leq p (q \notin A)\}$$

⁹⁴If partial elements of Γ are taken as maps $p : A \rightarrow A$ then the restriction of p can be taken as $(p \upharpoonright b)(c) = p(c) \wedge b$.

and

$$\neg\neg A = \{p \in P \mid \forall q \leq p \exists r \leq q \ r \in A\}$$

for $A \in \text{dcl}(P)$. Then $\text{RO}(P)$, the set of regular opens of P , consists of those $A \in \text{dcl}(P)$ with $\neg\neg A = A$, i.e. those downward closed subsets of A for which $p \in A$ follows from $\forall q \leq p \exists r \leq q \ r \in A$. Obviously, for every poset P the poset $\text{RO}(P)$ is a complete Boolean algebra (cBa).

If P is a poset then a *boolean completion* of P is a pair (B, e) where B is a cBa and $e : P \rightarrow B \setminus \{0_B\}$ satisfying the conditions

(bc1) e preserves and reflects the order

(bc2) for all $x \in B \setminus \{0\}$ there exists a $p \in P$ with $e(p) \leq x$.

One easily checks that $\downarrow p = \{q \in P \mid q \leq p\}$ is a regular opens iff whenever $q \not\leq p$ then there exists an $r \leq q$ with $r \perp p$, i.e. $\neg \exists s (s \leq r \wedge s \leq p)$. A poset P called *regular* if this condition holds for all $p \in P$. Obviously, for a regular poset P a boolean completion of P is given by P together with the map $\downarrow : P \rightarrow \text{RO}(P) \setminus \{\emptyset\}$.

Suppose that P is a regular poset and (B, e) is a boolean completion of P then it is isomorphic to $(\text{RO}(P), \downarrow)$.

Lemma 1 For every $x \in B$ the set $P_x := \{p \in P \mid e(p) \leq x\} \in (\text{RO}(P), \downarrow)$ and $x = \bigvee e[P_x]$.

Proof We first show that P_x is a regular open. For this purpose suppose $\forall q \leq p \exists r \leq q \ e(r) \leq x$. We have to show that $p \in P_x$. For sake of contradiction assume $e(p) \not\leq x$. Then $e(p) \wedge \neg x \neq 0_B$. Thus, by (bc2) there exists $q \in P$ with $e(q) \leq e(p) \wedge \neg x$. Since $e(q) \leq e(p)$ it follows by (bc1) that $q \leq p$. Thus, there exists $r \leq q$ with $e(r) \leq x$. Since also $e(r) \leq e(q) \leq \neg x$ we have $e(r) \leq x \wedge \neg x = 0_B$ contradicting the assumption 0_B is not in the image of e .

Next we show that $x = \bigvee e[P_x]$. Obviously, we have $y := \bigvee e[P_x] \leq x$. It remains to show that $y = x$. For sake of contradiction suppose $y < x$. Then $x \wedge \neg y \neq 0_B$. Thus, by (bc2) there exists $p \in P$ with $e(p) \leq x \wedge \neg y$. Since $e(p) \leq x$ we have $p \in P_x$ and thus $e(p) \leq y$. Since also $e(p) \leq \neg y$ we have $e(p) \leq y \wedge \neg y = 0_B$ contradicting the assumption that 0_B is not in the image of e . \square

The maps

$$f : \text{RO}(P) \rightarrow B : U \mapsto \bigvee e[U]$$

and

$$g : B \rightarrow \text{RO}(P) : x \mapsto P_x$$

are both monotonic. By Lemma 1 we have $f \circ g = \text{id}_B$. If we can show that f is also injective then from $f \circ g \circ f = f$ it follows that also $g \circ f = \text{id}_{\text{RO}(P)}$ and thus f is an iso of posets with inverse g .

Notice, moreover, that $f(\downarrow p) = e(p)$ and thus also $g(e(p)) = \downarrow p$ if $g = f^{-1}$.

Lemma 2 The map $f : \text{RO}(P) \rightarrow B : U \mapsto \bigvee e[U]$ is injective.

Proof. We first show that for disjoint $U, V \in \text{RO}(P)$ we have $f(U \cap V) = f(U) \wedge f(V)$. Indeed for $U, V \in \text{RO}(P)$ with $U \cap V = \emptyset$ we have

$$f(U) \wedge f(V) = \bigvee_{p \in U} e(p) \wedge \bigvee_{q \in V} e(q) = \bigvee \{e(p) \wedge e(q) \mid p \in U, q \in V\} = 0_B$$

where the last equality follows from the following consideration: if $p \in U$ and $q \in V$ with $e(p) \wedge e(q) \neq 0_B$ then there exists $r \in P$ with $e(r) \leq e(p) \wedge e(q)$ and thus $r \leq p, q$ contradicting the assumption $U \cap V = \emptyset$.

For f being injective it suffices to show that $f(U) = x = f(P_x)$ implies $U = P_x$. Let $V := P_x \cap \neg U$. Since U and V are disjoint we have by the above consideration that $f(U) \wedge f(V) = f(U \cap V) = f(\emptyset) = 0_B$. Since $U \subseteq P_x$ we have $f(V) \leq f(P_x) = x = f(U)$. Thus $f(V) = f(U) \wedge f(V) = 0_B$. Now if $p \in V$ then $f(V) \geq f(\downarrow p) = e(p) \neq 0_B$. Thus $\emptyset = V = P_x \cap \neg U$ from which it follows that $P_x \subseteq U$ as desired. \square

Summarizing we have show that every Boolean completion (B, e) of P is isomorphic to the canonical Boolean completion $(\text{RO}(P), \downarrow)$. Thus, all Boolean completions of P are isomorphic.

Every cBa B is of the form $\text{RO}(P)$ for some regular poset P

Suppose B is a cBa. The $P = B \setminus \{0_B\}$ is obviously a regular poset w.r.t the order inherited from B . Thus, the map $i : B \rightarrow \text{RO}(P) : b \mapsto P \cap \downarrow b$ is one-to-one and preserves and reflects the order. If we can show that it is onto it follows that i is actually an isomorphism of cBa's.

Suppose U is in $\text{RO}(P)$. If U is empty then $U = i(0_B)$. If U is nonempty let $p = \bigvee U$. We now show that

$$(*) \quad \forall p' \leq p \forall q \leq p' \exists r \leq q \ r \in U$$

Suppose $q \leq p' \leq p$. Then $q = q \wedge p = q \wedge \bigvee_{b \in U} b = \bigvee_{b \in U} q \wedge b$. Since $q \neq 0_B$ there exists a $b \in U$ with $r := q \wedge b \neq 0_B$. Obviously, we have $r \leq q$. Since $r \leq b \in U$ we have also $r \in U$. Thus, we have proven (*). Now since $U \in \text{RO}(P)$ it follows from (*) that $i(p) \subseteq U$ and $U \subseteq i(p)$ since $U \leq p$. Thus $U = i(p)$.

Thus we have shown that $i : B \rightarrow \text{RO}(P)$ is onto as claimed and, therefore, an isomorphism.

146 Forcing an Ultrafilter on \mathbb{N}

If the metatheory is ZF (instead of ZFC) with DC (dependent choice) one may force the existence of a generic ultrafilter on \mathbb{N} by considering the poset \mathbb{P} of *infinite* subsets of \mathbb{N} ordered by \subseteq . Apparently, maximal filters in \mathbb{P} correspond to non-principal ultrafilters on \mathbb{N} . We describe the construction of the generic ultrafilter on \mathbb{N} in B -valued sets. First notice that up to isomorphism $\mathcal{P}(\mathbb{N})$ is given by the B -set whose underlying set is $B^{\mathbb{N}}$ and where $\llbracket X = Y \rrbracket = \bigwedge_{n \in \mathbb{N}} X(n) \leftrightarrow Y(n)$. A condition p may be considered as a predicate $i(p)$ on \mathbb{N} by putting $i(p)(n) = \{q \in \mathbb{P} \mid n \in p\}$. The generic ultrafilter \mathcal{U} is the given by

$$\llbracket X \in \mathcal{U} \rrbracket = \exists p \in \Gamma \forall n \in \mathbb{N} (n \in X \leftrightarrow n \in i(p))$$

for $X \in B^{\mathbb{N}}$. One can characterise $\llbracket X \in \mathcal{U} \rrbracket$ as the $\neg\neg$ -closure of the set of $q \in \mathbb{P}$ such that there exists a $p \in \mathbb{P}$ with $p \supseteq q$ satisfying the condition

$$\forall n \in \mathbb{N} \forall r \leq q (r \in X(n) \Leftrightarrow n \in p)$$

which in turn is equivalent to the requirement that the following two conditions hold

- (1) $\forall n \in \mathbb{N} (n \in q \Rightarrow q \in X(n))$ and
- (2) $\forall n \in \mathbb{N} (q \in X(n) \vee q \in \neg(n \in X))$

since one may choose $p = \{n \in \mathbb{N} \mid q \in X(n)\}$. We will show now that

Lemma For every $X \in B^{\mathbb{N}}$ and $p \in \mathbb{P}$ there exists a $q \subseteq p$ in \mathbb{P} satisfying condition (2).

For this purpose, however, we need a certain combinatorial lemma whose formulation requires the following auxiliary relation on \mathbb{P}

$$X \leq Y \quad \text{iff} \quad \exists n \forall k \geq n (k \in X \Rightarrow k \in Y) \quad (\text{iff } X \setminus Y \text{ is finite})$$

Forcing over the poset (\mathbb{P}, \subseteq) amounts to the same as forcing over the preorder (\mathbb{P}, \leq) . The reason is that for $A \in \text{dcl}(\mathbb{P})$ its negation $\neg A = \{p \in \mathbb{P} \mid \forall q \in A \ p \cap q \text{ finite}\}$ which is obviously downward closed w.r.t. \leq .

Combinatorial Lemma

Every sequence (A_n) in \mathbb{P} decreasing w.r.t. \leq has a lower bound in \mathbb{P} w.r.t. \leq .

Proof. Suppose (A_n) is decreasing w.r.t. \leq . Obviously, for every n the set $B_n := \bigcap_{k \leq n} A_k$ is infinite. Thus, for every n there is a least $m > n$ with $m \in B_n$ which we denote by $f(n)$. Let B be the image of f . The set B is infinite since for $f(n) \in B$ we have $f(n) > f(f(n)) \in B$. Moreover, we have $f(k) \in A_n$ for all $k \geq n$ and thus $B \setminus A_n$ is finite, i.e. $B \leq A_n$. Thus B is a lower bound for the sequence A_n w.r.t. \leq . \square

Now suppose $X \in B^{\mathbb{N}}$. Since B is boolean we have

$$(3) \quad \forall n \in \mathbb{N} \forall p \exists q \leq p (q \in X(n) \vee q \in \neg X(n))$$

Moreover, the set $\{q \in \mathbb{P} \mid q \in X(n) \vee q \in \neg X(n)\}$ is downward closed w.r.t. \leq . Suppose $p \in \mathbb{P}$. Then by dependent choice we can define a sequence (p_n) in \mathbb{P} decreasing w.r.t. \leq such that $p \geq p_0$ and $p_n \in X(n) \vee p_n \in \neg X(n)$ for all $n \in \mathbb{N}$. By the combinatorial Lemma there exists an $r \in \mathbb{P}$ with $r \leq p_n$ for all n . Since $r \leq p$ the set $r \setminus p$ is finite and thus w.l.o.g. we may assume that $r \subseteq p$. Notice that $\forall n (r \in X(n) \vee r \in \neg X(n))$ since $r \leq p_n$ for all $n \in \mathbb{N}$. Thus we have shown the Lemma.

The sets $r_0 = \{n \in r \mid r \in X(n)\}$ and $r_1 = \{n \in r \mid r \in \neg X(n)\}$ are disjoint and their union is \mathbb{N} . Thus, at least one of them is infinite. Put $q := r_0$ if r_0 is infinite and $q := r_1$ otherwise. Thus $q \in \mathbb{P}$ with $q \subseteq p$ and satisfies the conditions (1) and (2).

Thus we have shown that $\llbracket X \in \mathcal{U} \vee \neg X \in \mathcal{U} \rrbracket = 1_B$ for all $X \in \mathbb{N}$. Moreover, one easily shows that $\llbracket X, Y \in \mathcal{U} \Rightarrow X \cap Y \in \mathcal{U} \rrbracket$. Thus, the set \mathcal{U} validates the axioms for an ultrafilter in $B\text{-Set}$.

$B = \text{dcl}(\mathbb{P})_{\neg\neg}$ has no points

One easily shows that points of a complete Boolean algebra B correspond to atoms of B since if $p : B \rightarrow 2$ is a point then $a := \neg \bigvee p^{-1}(0)$ is an atom of B . But our $B = \text{dcl}(\mathbb{P})_{\neg\neg}$ has no atom since any $\downarrow p$ can be decomposed nontrivially as $\downarrow p = \downarrow p_1 \vee \downarrow p_2$ where p_1 and p_2 are infinite disjoint subsets of p with $p = p_1 \cup p_2$.

N and N^N in $\text{Sh}(B)$

The category of B -sets is known to be equivalent to $\text{Sh}(B)$ where $B = \text{dcl}(\mathbb{P})_{\neg\neg}$. Thus $\text{Sh}(B)$ arises as subtopos of the presheaf topos $\mathbf{Set}^{\tilde{\mathbb{P}}^{\text{op}}}$ where $\tilde{\mathbb{P}}$ is the poset obtained by factoring \mathbb{P} modulo the preorder \leq . The constant presheaf $\Delta(\mathbb{N})$ with value \mathbb{N} is the nno in $\mathbf{Set}^{\tilde{\mathbb{P}}^{\text{op}}}$ whereas the nno N in the sheaf topos looks as follows: $N(p)$ consists of disjoint covers of $\downarrow p$ indexed by subsets of \mathbb{N} and for $q \leq p$ restricting an element $(U_i)_{i \in I}$ of $N(p)$ to q gives $(U_i \cap \downarrow q)_{i \in I'}$ where $I' = \{i \in I \mid U_i \cap \downarrow q \neq \emptyset\}$. Let us write $\eta_N : \Delta(N) \rightarrow N$ for the reflection map. One can show that $\eta_N^{\Delta \mathbb{N}} : \Delta(\mathbb{N}^{\mathbb{N}}) \cong \Delta(\mathbb{N})^{\Delta(\mathbb{N})} \rightarrow N^{\Delta(\mathbb{N})} \cong N^N$ is dense w.r.t. the $\neg\neg$ -topology on $\tilde{\mathbb{P}}$.

Using the Combinatorial Lemma this can be seen as follows. Let $f \in N^{\Delta(\mathbb{N})}(p)$. Let $p_0 \leq p$ with $\downarrow p_0 \subseteq f(0)_{g(0)}$. Suppose we have already constructed $p_n \leq p_{n-1} \leq \dots \leq p_1 \leq p_0 \leq p$ with $\downarrow p_{i+1} \subseteq f(i+1)_{g(i+1)} \cap \downarrow p_i$. By the Combinatorial Lemma there exists a q with $q \leq p_n$ for all n . If we now restrict f to q then for all n the restriction of $f(n)$ to q is the non-mixed $\{g(n) \mapsto \downarrow q\}$ in $N(q)$. Thus the restriction of f to q is given by g in $\Delta(\mathbb{N}^{\mathbb{N}})(q)$.

147 Realizability for Analysis with an Ultrafilter on \mathbb{N}

In his 2008 paper *Structures de réalisabilité, RAM et ultrafiltre sur \mathbb{N}* J.-L. Krivine has constructed a realizability model for Analysis with Dependent Choice and the Axiom that there exists a selective⁹⁵ ultrafilter on $\mathcal{P}(\mathbb{N})$. His construction is based on the construction of a forcing model for this axiom employing the poset \mathbb{P} of infinite subsets of \mathbb{N} ordered by \subseteq . Apparently maximal filters in \mathbb{P} are precisely the non-principal ultrafilters on $\mathcal{P}(\mathbb{N})$. Krivine proceeds by constructing such a forcing model inside his realisability model for classical analysis with dependent choice.

For this purpose it is better to consider instead of \mathbb{P} the (meet semi)lattice $P = \mathcal{P}(\mathbb{N})$ with \cap as meet operation. For $p, q \in P$ we write simply pq instead of $p \cap q$. On P we consider the predicate

$$C(p) \equiv \forall n \exists m p(n + m)$$

expressing that p contains infinitely many elements. We often write I for the complement of C , i.e. for the property of being a finite subset of \mathbb{N} . Notice that $C(pq)$ may be understood as “ p and q are compatible” and $I(pq)$ as “ p and q are incompatible”. The above poset \mathbb{P} can then be recovered as the set of all p with $C(p)$. If $A \subseteq P$ then its intuitionistic negation is defined as $\sim A = \{p \in P \mid \forall q (q \in A \supset I(pq))\}$ which is obviously downclosed. Subsets of P of the form $\sim A$ are called classical propositions.

Let k -ary X range over predicates on \mathbb{N}^k and X^+ range over predicates on $P \times \mathbb{N}^k$. We now define forcing as follows⁹⁶

$$\begin{aligned} p \Vdash^f X(\vec{t}) &\equiv C(p) \rightarrow X(\vec{t}) \\ p \Vdash^f X^+(\vec{t}) &\equiv \forall q (C(pq) \rightarrow X^+(q, \vec{t})) \equiv \forall q (\neg X^+(q, \vec{t}) \rightarrow \neg C(pq)) \\ p \Vdash^f F \rightarrow G &\equiv \forall q (p \Vdash^f F \rightarrow pq \Vdash^f G) \\ p \Vdash^f \forall \xi F &\equiv \forall \xi p \Vdash^f F \quad \text{where } \xi \text{ is } x, x^{\text{int}}, X \text{ or } X^+ \\ p \Vdash^f J(q) &\equiv \neg C(pq) \equiv \forall r (C(pr) \rightarrow q \neq r) \end{aligned}$$

Notice that $J(q) = \{p \mid I(pq)\} = \{p \mid \forall r \in \{q\} I(pr)\} = \sim\{q\}$ and thus $\sim J(q) = \sim\sim\{q\}$. Thus $\sim J$ is what is usually called the “generic set” in forcing. Krivine calls J the “generic ideal”.

Now Krivine considers the above construction in some of his classical realizability models. Let $\mathcal{X} \in \mathcal{P}(\Pi)^{P \times \mathbb{N}^k}$. Then we have

⁹⁵An ultrafilter \mathcal{U} on $\mathcal{P}(\mathbb{N})$ is called *selective* iff for every equivalence relation on \mathbb{N} such that \mathcal{U} does not contain any equivalence class there is an element of \mathcal{U} choosing one element from each equivalence class.

⁹⁶Since $p \Vdash^f X^+(\vec{t})$ iff $p \in \sim\{q \in P \mid \neg X^+(q, \vec{t})\}$ it makes sense to write $q \notin X^+(\vec{t})$ for $\neg X^+(q, \vec{t})$ and thus $q \in X^+(\vec{t})$ for $X^+(\vec{t})$. But this notation is still a bit unfortunate since $q \in X^+(\vec{t})$ does not entail that $p \Vdash^f X^+(\vec{t})$.

$$\begin{aligned}
\xi \Vdash (p \Vdash^f \mathcal{X}(\vec{t})) & \text{ iff } \xi \Vdash \forall q (C(pq) \rightarrow \mathcal{X}(q, \vec{t})) \\
& \text{ iff } \forall q \forall \tau \in C(pq) \forall \pi \in \mathcal{X}(q, \vec{t}) \xi * \tau. \pi \in \perp \\
& \text{ iff } \forall q \forall \pi \in \mathcal{X}(q, \vec{t}) \forall \tau \in C(pq) \xi * \tau. \pi \in \perp
\end{aligned}$$

This observation suggest to consider another realizability structure where $\mathbf{\Lambda} = \mathbf{\Lambda} \times P$, $\mathbf{\Pi} = \mathbf{\Pi} \times P$ and $\mathbf{\Lambda} * \mathbf{\Pi} = (\mathbf{\Lambda} \times \mathbf{\Pi}) \times P$ and

$$(\xi, p) * (\pi, q) = (\xi * \pi, pq) \quad \text{and} \quad (\xi, p) \cdot (\pi, q) = (\xi \cdot \pi, pq)$$

and $\perp \subseteq \mathbf{\Lambda} * \mathbf{\Pi}$ defined as

$$(\xi, p) * (\pi, q) \in \perp \quad \text{iff} \quad \forall \tau \in C(pq) \xi * \pi^\tau \in \perp$$

where $\pi^\tau = \pi \cdot \tau$. If $A \subseteq \mathbf{\Pi}$ we write $(\xi, p) \Vdash A$ for $\forall (\pi, q) \in A (\xi, p) * (\pi, q) \in \perp$.
Now one may introduce new constants χ and χ' with reduction rules

$$\chi * \xi \cdot \pi^\tau \succ \xi * \tau \cdot \pi \quad \text{and} \quad \chi' * \xi \cdot \tau \cdot \pi \succ \xi * \pi^\tau$$

Then for $\mathcal{X} \in \mathcal{P}(\mathbf{\Pi})^{P \times \mathbb{N}^k}$ we have

$$\begin{aligned}
\xi \Vdash (p \Vdash^f \mathcal{X}(\vec{t})) & \text{ iff } \xi \Vdash \forall q (C(pq) \rightarrow \mathcal{X}(q, \vec{t})) \\
& \text{ iff } \forall q \forall \tau \in C(pq) \forall \pi \in \mathcal{X}(q, \vec{t}) \xi * \tau. \pi \in \perp \\
& \text{ iff } \forall q \forall \pi \in \mathcal{X}(q, \vec{t}) \forall \tau \in C(pq) \xi * \tau. \pi \in \perp \\
& \text{ implies } \forall q \forall \pi \in \mathcal{X}(q, \vec{t}) \forall \tau \in C(pq) \chi \xi * \pi^\tau \in \perp \\
& \text{ iff } \forall q \forall \pi \in \mathcal{X}(q, \vec{t}) (\chi \xi, p) * (\pi, q) \in \perp \\
& \text{ iff } (\chi \xi, p) \Vdash \mathcal{X}(\vec{t})
\end{aligned}$$

and

$$\begin{aligned}
(\xi, p) \Vdash \mathcal{X}(\vec{t}) & \text{ iff } \forall q \forall \pi \in \mathcal{X}(q, \vec{t}) (\xi, p) * (\pi, q) \in \perp \\
& \text{ iff } \forall q \forall \pi \in \mathcal{X}(q, \vec{t}) \forall \tau \in C(pq) \xi * \pi^\tau \in \perp \\
& \text{ implies } \forall q \forall \pi \in \mathcal{X}(q, \vec{t}) \forall \tau \in C(pq) \chi' \xi * \tau. \pi \in \perp \\
& \text{ implies } \forall q \forall \pi \in \mathcal{X}(q, \vec{t}) \forall \tau \in C(pq) \chi' \xi * \tau. \pi \in \perp \\
& \text{ iff } \forall q \forall \tau \in C(pq) \forall \pi \in \mathcal{X}(q, \vec{t}) \chi' \xi * \tau. \pi \in \perp \\
& \text{ iff } \chi' \xi \Vdash \forall q (C(pq) \rightarrow \mathcal{X}(q, \vec{t})) \\
& \text{ iff } \chi' \xi \Vdash (p \Vdash^f \mathcal{X}(\vec{t}))
\end{aligned}$$

148 Comments on van den Berg’s PhD Thesis *Predicative Topos Theory and Models of Constructive Set Theory* (Utrecht, May 2006)

In May 2006 Benno van den Berg (successfully) defended his PhD Thesis on *Predicative Topos Theory and Models of Constructive Set Theory*.

As a notion of “predicative topos” he suggested the notion of ΠW -pretopos, i.e. a locally cartesian closed pretopos with W -types. He also considers ML-categories (Martin-Löf categories) which are defined as locally cartesian closed categories with W -types and finite disjoint and stable sums and proves that for every ML-category \mathcal{D} its exact completion \mathcal{D}_{ex} is always a ΠW -pretopos.

There are plenty of examples of ΠW -pretoposes which are not toposes. The first kind are exact completions \mathcal{E}_{ex} of toposes \mathcal{E} without a *generic proof*. (M. Menni has shown that for a topos \mathcal{E} its exact completion \mathcal{E}_{ex} is a topos if and only if \mathcal{E} has a generic proof.) By a result of M. Menni a presheaf topos $\widehat{\mathbb{C}}$ has a generic proof iff \mathbb{C} is a groupoid. Thus $\widehat{\mathbb{C}}_{\text{ex}}$ is ΠW -pretopos but not a topos iff \mathbb{C} is not a groupoid. A second kind of examples are (by a theorem of Lietz and Streicher) realizability models $\mathbf{RC}(\mathcal{T})$ where \mathcal{T} is a typed pca with weak W -types but no universal type (as e.g. $\mathbf{PER}(K_1)$, $\mathbf{PER}(K_2)$, $\mathbf{PER}(\mathcal{P}\omega)$, A. Simpson’s \mathbf{QCB}_0 or a term model of Gödel’s T with weak W -types instead of just N).

Van den Berg shows that ΠW -pretoposes are closed under taking (pre)sheaves and under glueing. He uses the latter for showing that in the free ΠW -pretopos \mathcal{E} all finite simple types over N are projective⁹⁷ but not internally projective. For this reason \mathcal{E} is not equivalent to \mathcal{D}_{ex} , the exact completion of the free ML-category, where the finite simple types over N are both internally and externally projective. Since in the category *Setoids* of so called “setoids” (the exact completion of the term model of *intensional* Martin-Löf type theory with W -types) N^N is neither internally nor externally projective it follows that the category *Setoids* is neither equivalent to \mathcal{E} nor to \mathcal{D}_{ex} . This is suprising since naively one might expect all three categories to be equivalent.

If a ΠW -pretopos \mathcal{E} is endowed with a class \mathcal{S} of small maps (satisfying appropriate axioms) with weakly generic family $\pi : E \rightarrow U$ then W_π modulo bisimulation gives rise to a model of Aczel’s CZF. (In particular, he identified a “type-theoretic fullness axiom” (F) for classes of small maps ensuring that the ensuing model of CZF satisfies the fullness axiom of CZF).

⁹⁷It is not known (BUT see section 150) whether this holds also for the free topos with a nno. Friedman and Scedrov [FS83] have shown that N is projective in the free topos with nno satisfying also countable choice. Maybe their result can be used for proving that N is projective in the free topos with nno ? Phil Scott claims to have an unpublished proof of this (together with Scedrov) and there is also an unpublished manuscript by M. Makkai. The logician’s proof employs partial reflection principles (à la Kreisel and Levy). Categorists want to replace this by inner model constructions (internal categories like in Joyal’s arithmetic universes).

Further, he reinvented the so called Lubarsky-Streicher model for CZF which validates full separation but refutes the powerset axiom since it validates the axiom that all sets are subcountable. In 2003/04 I constructed the model as the type $(WA : \nabla(\text{PER}(K_1)))A$ modulo bisimulation within $\mathbf{Asm}(K_1)$. Van den Berg constructed it as W_π modulo bisimulation within the effective topos $\mathcal{E}ff = \mathbf{RC}(K_1)$ where π is a weakly⁹⁸ generic family of modest sets (over K_1) which lives within $\mathbf{Asm}(K_1)$. Exploiting this he showed that the Lubarsky-Streicher model coincides with the *hereditary subcountable sets* in the McCarty model for IZF (which is known to be equivalent to $\mathcal{E}ff$).⁹⁹

Moreover, he considered so called ΠM -pretoposes, i.e. locally cartesian closed pretoposes where for every $f : B \rightarrow A$ there exists an final solution M_f to the recursive type equation $X \cong (\Sigma a:U)E(a)$. In work together with F. de Marchi he showed that for ΠM -pretoposes \mathcal{E} endowed with a class \mathcal{S} of small maps with weakly generic family $\pi : E \rightarrow U$ the object M_π , i.e. $(Ma:U)E(a)$ modulo bisimulation, gives rise to a model of non-wellfounded CZF, i.e. CZF without \in -induction but instead validating the so called antifoundation axiom AFA (originally introduced by P. Aczel). For this purpose it is not needed that W -types exist (though if \mathcal{E} is a topos W_f can be carved out from M_f by an inductively defined predicate expressing wellfoundedness). But unfortunately one does not know of any ΠM -pretopos which is not also ΠW -pretopos. Moreover, it is not investigated what is the proof theoretic strength of non-wellfounded CZF. (But this has probably be done by M. Rathjen already!) If \mathcal{T} is a typed pca (with weak M -types) capturing the provably recursive functions of non-wellfounded CZF then $\mathbf{RC}(\mathcal{T})$ might give rise to a ΠM -topos which is not a ΠW -pretopos.

⁹⁸Does there exist a generic family of modest sets within $\mathcal{E}ff$? Maybe this is a bit easier than the question whether for every strongly inaccessible cardinal κ there exists a generic κ -small family.

⁹⁹This is wrong in general e.g. if ω^ω is not subcountable as e.g. in \mathbf{Set} . Thus, he heavily uses Church's Thesis together with a kind of Shanin's Axiom

$$\forall x \exists y \in \mathcal{P}_{\neg\neg}(\omega \times V) \forall z (z \in x \leftrightarrow \exists n \in \omega \langle n, z \rangle \in y)$$

which was used also by Kouwenhoven-Gentil and van Oosten in their JSL paper showing that the McCarty model in $\mathcal{E}ff$ is the initial fixpoint of \mathcal{P}_s .

149 Partial Reflection Principle for HAS

Let HAS be intuitionistic second order arithmetic formulated as a two sorted first order theory with the *comprehension schema*

$$\exists X \forall n (n \in X \leftrightarrow A(x))$$

for all formulas A and X not free in A .

Theorem 149.1 *For every finite fragment \mathcal{T} of HAS we have*

$$\text{HAS} \vdash \text{Pr}_{\mathcal{T}}(\ulcorner A \urcorner) \rightarrow A$$

for all formulas in the language of HAS.

Proof: Let \mathcal{T} be a finite fragment of HAS and A a formula in the language of HAS. Let n be an upper bound on the logical complexity (number of connectives and quantifiers) of the formulas in $\mathcal{T} \cup \{A\}$. There is a truth predicate T_n in the language of HAS such that for all formulas ϕ of logical complexity $\leq n$ we have $\text{HAS} \vdash T_n(\ulcorner \phi \urcorner) \leftrightarrow \phi$.

First observe that HA proves the cut elimination theorem for intuitionistic predicate logic. Thus, whenever $\mathcal{T} \vdash A$ then HAS proves that there is a cut free proof of the sequent $\mathcal{T} \vdash A$ in intuitionistic predicate logic. Thus w.l.o.g. we may assume that for ϕ of logical complexity $\leq n$ the statement $\text{Pr}_{\mathcal{T}}(\ulcorner \phi \urcorner)$ means that there is a cut free proof of $\mathcal{T} \vdash \phi$ in intuitionistic predicate logic. Since cut free proofs in intuitionistic predicate logic satisfy the subformula property every formula arising in a derivation of $\mathcal{T} \vdash \phi$ is necessarily of logical complexity $\leq n$. Internalizing the correctness proof we have $\text{HAS} \vdash \forall x (\text{Pr}_{\mathcal{T}}(x) \rightarrow T_n(x))$. Thus, we have in particular that $\text{HAS} \vdash \text{Pr}_{\mathcal{T}}(\ulcorner A \urcorner) \rightarrow T_n(\ulcorner A \urcorner)$ from which it follows by $\text{HAS} \vdash T_n(\ulcorner A \urcorner) \leftrightarrow A$ that $\text{HAS} \vdash \text{Pr}_{\mathcal{T}}(\ulcorner A \urcorner) \rightarrow A$ as desired. \square

150 Church's Rule for HAS (as in Troelstra's SLNM 344)

Suppose $\text{HA} \vdash \forall n \exists X A(n, X)$. Then there exists a natural number n_0 which is an upper bound for the logical complexity of formulas arising in this proof. For every natural number n there is a proof of $\exists X A(\underline{n}, X)$ all whose formulas have logical complexity less or equal n_0 . Thus, there exists a total recursive function f for which

$$\text{HAS} \vdash \forall n \text{Pr}^{(n_0)}(\ulcorner A(\underline{n}, S_{f(n)}) \urcorner) \quad (1)$$

where $\text{Pr}^{(n_0)}$ refers to provability in the fragment of HAS where all formulas have logical complexity bounded by n_0 and $(S_k)_{k \in \omega}$ is some Gödelization of set expressions. From (*) it follows that

$$\text{HAS} \vdash \forall n \text{Tr}^{(n_0)}(\ulcorner A(\underline{n}, S_{f(n)}) \urcorner) \quad (2)$$

where $\text{Tr}^{(n_0)}$ is the truth predicate for formulas of logical complexity bounded by n_0 for which $\text{HAS} \vdash \forall k (\text{Pr}^{(n_0)}(k) \rightarrow \text{Tr}^{(n_0)}(k))$. But since for formulas of logical complexity less or equal n_0

$$\text{HAS} \vdash \text{Tr}^{(n_0)}(\ulcorner A \urcorner) \leftrightarrow A \quad (3)$$

we conclude that

$$\text{HAS} \vdash \forall n A(n, Y_n)$$

where $Y = \{\langle n, m \rangle \mid \text{Tr}^{(n_0)}(\ulcorner S_{f(n)}(m) \urcorner)\}$ and Y_n stands for $\{m \mid \langle n, m \rangle \in Y\}$.

More details can be found in 4.5.8 (p.321) of Troelstra's SLNM 344 *Investigations of Intuitionistic Arithmetic and Analysis* and in his corrections to that book (<http://staff.science.uva.nl/~anne/Corr/c344.ps>) incorporating observations from S. Hayashi's paper *On derived rules of intuitionistic second order arithmetic* Commentarii Mathematici Universitatis Sancti Pauli 26 (1977), pp. 77-103.

151 Set Existence Property for HASC

We outline the proof in [FS83].

Let HA be Heyting arithmetic with constants for prim. rec. functions and HAS its second order extension with the *comprehension schema*

$$\exists X \forall n (n \in X \leftrightarrow A(x))$$

for all A without any free occurrence of the species (i.e. set) variable X . We write HASC for the system obtain from HAS by adding the *countable axiom of choice*

$$(CAC) \quad \forall n \exists X A(n, X) \rightarrow \exists Y \forall n \exists X (A(n, X) \wedge \forall k (k \in X \leftrightarrow \pi(n, k) \in Y))$$

where π is a prim. rec. pairing function.

Our aim is to show the *set existence property* for HASC, i.e. whenever $\text{HASC} \vdash \exists X A(X)$ then there exists a predicate $B(n)$ (containing no other free variables besides n) such that $\text{HASC} \vdash A(\{n \mid B(n)\})$ (where the latter is an abbreviation for $\text{HASC} \vdash \exists X (A(X) \wedge \forall n (n \in X \leftrightarrow B(n)))$).

Suppose HASC proves the existential sentence $\exists X A(X)$. Then there exists a finite fragment HASC_0 which proves $\exists X A(X)$.

Working (informally) in $\text{HASC} + \text{CT}$ (where CT is Church's Thesis $\forall n \exists m A(n, m) \rightarrow \exists e \forall n A(n, \{e\}(n))$) we define an extension HASC'_0 of HASC_0 inductively generated by the following two clauses

- (1) if $\exists X \forall n (n \in X \leftrightarrow A(n))$ (with X not free in A) is a sentence in HASC'_0 then C_A is a species (i.e. set) constant of HASC'_0 and $\forall n (n \in C_A \leftrightarrow A(n))$ is a sentence of HASC'_0
- (2) if $\forall n \exists X A(n, X)$ is a sentence of HASC'_0 and $(T_n)_{n \in \mathbb{N}}$ is a sequence of HASC'_0 species constants with index e such that $A(\underline{n}, T_n)$ is a theorem of HASC'_0 for all n then $F_{e,A}$ is a species constant of HASC'_0 and

$$\forall n \exists X (A(n, X) \wedge \forall k (k \in X \leftrightarrow \pi(n, k) \in F_{e,A}))$$

and for all n

$$\forall k (k \in T_n \leftrightarrow \pi(\underline{n}, k) \in F_{e,A})$$

are theorems of HASC'_0 .

Let HASC_0^* be the language obtained from that of HASC_0 by adding species constants of the form (τ, V) where τ is a species constant of HASC'_0 and V is a species. If $T = (\tau, V)$ we write T^- for τ and T^+ for V . For each formula A in the language of HASC_0^* we write A^- for the formula in HASC'_0 obtained by replacing each species term T by T^- .

For each sentence A of HASC'_0 we define $R(A)$ (" A is realized") by induction on the structure of A as follows

$$\begin{aligned}
R(t = s) & \text{ iff } t = s \\
R(t \in (\tau, V)) & \text{ iff } t \in V \\
R(A \wedge B) & \text{ iff } R(A) \text{ and } R(B) \\
R(A \vee B) & \text{ iff } (R(A) \text{ and } \text{HASC}'_0 \vdash A^-) \text{ or } (R(B) \text{ and } \text{HASC}'_0 \vdash B^-) \\
R(A \rightarrow B) & \text{ iff } R(A) \text{ and } \text{HASC}'_0 \vdash A^- \text{ implies } R(B) \\
R(\forall n A(n)) & \text{ iff } R(A(\underline{n})) \text{ for all } n \\
R(\exists n A(n)) & \text{ iff } R(A(\underline{n})) \text{ and } \text{HASC}'_0 \vdash A(\underline{n}) \text{ for all } n \\
R(\forall X A(X)) & \text{ iff } R(A(T)) \text{ for every species term } T \text{ of } \text{HASC}_0^* \\
R(\exists X A(X)) & \text{ iff } R(A(T)) \text{ and } \text{HASC}'_0 \vdash A^-(T^-) \\
& \text{ for some species term } T \text{ of } \text{HASC}_0^*
\end{aligned}$$

Lemma 151.1 *HASC + CT proves that for closed formulas A if $\text{HASC}_0 \vdash A$ then $R(A)$.*

Notice that CT is needed for proving the case of instances of CAC.

Lemma 151.2 *There exists a unique map ϕ from species constants of HASC'_0 to species such that*

(1) *if HASC'_0 proves $\exists X \forall n (n \in X \leftrightarrow A(n, S_1, \dots, S_m))$ then*

$$\phi(C_A) = \{n \mid A(n, \phi(S_1), \dots, \phi(S_m))\}$$

(2) *if HASC'_0 proves $\forall n \exists X A(n, X, S_1, \dots, S_m)$ and e is an index of a recursive sequence of species constants of HASC'_0 such that HASC'_0 proves $A(\underline{n}, \{e\}(n), \phi(S_1), \dots, \phi(S_m))$ for all n then*

$$\phi(F_{e,A}) = \{n \mid \pi_1(n) \in \phi(\{e\}(\pi_0(n)))\}$$

The following lemma requires the *partial reflection principle* of [KL68] for HASC.

Lemma 151.3 *For closed formulas A and species constants T the system HAS proves that if $\text{HASC}'_0 \vdash A(T)$ then $A(\phi(T))$.*

From the assumption $\text{HASC}_0 \vdash \exists X A(X)$ it follows by Lemma 151.1 that $\text{HASC} + \text{CT} \vdash R(\exists X A(X))$. Thus by definition of R it follows that

$$\text{HASC} + \text{CT} \vdash \exists \text{ species constant } T \text{ with } \text{HASC}'_0 \vdash A(T) \quad (\dagger)$$

It is well know that

Lemma 151.4 *If $\text{HASC} + \text{CT} \vdash A$ then $\text{HASC} \vdash \underline{n} \text{ r } A$ for some n .*

Moreover one can show that

Lemma 151.5 *For the inductively defined set I of pairs of Gödel numbers for species constants T and predicates A with $\text{HASC}'_0 \vdash A(T)$ it holds that $\text{HAS} \vdash m \mathbf{r}(n \in I) \rightarrow n \in I$.*

Using these two lemmas it follows from (†) that

$$\text{HASC} \vdash \exists \text{ species constant } T \text{ with } \text{HASC}'_0 \vdash A(T)$$

from which it follows by numerical existence property for HASC and Lemma 151.3 that $\text{HASC} \vdash A(B)$ where B is the species $\phi(T)$.

Countable Choice Rule for HAS

Notice that the proof of Lemma 151.1 can obviously be adapted to a proof of

Lemma 151.6 *HAS + CT proves that for closed formulas A if $\text{HAS}_0 \vdash A$ then $R(A)$.*

In Lemma 151.3 the implication can be proved already in HAS and the proof of Lemma 151.2 also does not require CAC (unlike as claimed in [FS83]) since ϕ can be defined by structural recursion on species constants of HASC'_0 .

Suppose $\text{HAS} \vdash \forall n \exists X A(n, X)$. Then $\text{HASC}'_0 \vdash \exists X \forall n A(x, X_n)$ where $k \in X_n$ iff $\pi(n, k) \in X$. Accordingly, by Lemma 151.6 it follows that HAS + CT proves $R(A)$. Thus, by definition of R the system HAS + CT proves that there exists a species term T with $\text{HASC}'_0 \vdash \forall n A(x, T_n)$. Thus, by Lemmas 151.4 and 151.5 and the numerical existence property for HAS it follows by Lemma 151.3 that $\text{HAS} \vdash \forall n A(n, B_n)$ where B is the species with Gödel number $\phi(T)$.

References

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152 “Neutral” Models for BISH

Most realizability models validate intuitionistic principles like Church’s Thesis, WC-N or Ishihara’s BD-N which are not available in Bishop’s constructive mathematics (BISH).

However, the extensional realizability topos **Ext** (Grayson, van Oosten) refutes all these principles. BISH is known to be compatible with classical logic. However, the topos **Ext** validates propositions which don’t hold in **Set** as e.g.

- (1) there exist no nonrecursive functions from \mathbb{N} to \mathbb{N}
- (2) there exist no discontinuous functionals from $\mathbb{N}^{\mathbb{N}}$ to \mathbb{N} .

This, however, can be remedied by glueing the functor $\nabla : \mathbf{Set} \rightarrow \mathbf{Ext}$ right adjoint to the global sections functor $\Gamma = \mathbf{Ext}(1, -) : \mathbf{Ext} \rightarrow \mathbf{Set}$, i.e. the topos $\mathbf{Ext} \downarrow \nabla$, together with the logical functor $\partial_1 : \mathbf{Ext} \downarrow \nabla \rightarrow \mathbf{Set}$.

Open Question: Does this topos validate the Fan Theorem?

Another example is the domain realizability topos $\mathbf{RC}(\mathcal{D})$ where \mathcal{D} is the category of coherently complete countably algebraic domains and Scott continuous functions between them. It validates the principle that all maps $2^{\mathbb{N}}$ to \mathbb{N} are uniformly continuous but not that all functions from $\mathbb{N}^{\mathbb{N}}$ to \mathbb{N} are continuous. For this reason it cannot validate BD-N (since BD-N together with continuity of all functions from $2^{\mathbb{N}}$ to \mathbb{N} entails continuity of all maps from $\mathbb{N}^{\mathbb{N}}$ to \mathbb{N}). Still $\mathbf{RC}(\mathcal{D})$ validates (1) and (2) in contradiction with classical mathematics. Again this can be remedied by considering the glued topos $\mathbf{RC}(\mathcal{D}) \downarrow \nabla$.

Finally, instead of \mathcal{D} one may consider the category \mathcal{F} of FPC terms modulo observational equivalence and the realizability topos $\mathbf{RC}(\mathcal{F})$ (see A. Rohr’s Thesis). Using $\mathbf{PER}(\mathcal{F})$ instead of $\mathbf{PER}(\mathbb{N})$ one may construct a topos which can be expected to be a synthesis of **Ext** and $\mathbf{RC}(\mathcal{D})$.

153 How good a model is $\mathcal{P}(\mathbb{N})$?

Let $\mathcal{P}(\mathbb{N})$ be the category of modest projectives in $\text{RT}(\mathcal{K}_1)$, i.e. subsets of \mathbb{N} with realizable maps between them. One easily observes that $\mathcal{P}(\mathbb{N})$ has finite cartesian products. As observed by Carboni and Rosolini it is cartesian closed in the very weak sense that for $A, B \in \mathcal{P}(\mathbb{N})$ there is an object B^A in $\mathcal{P}(\mathbb{N})$ together with a map $\text{ev}_{A,B} : B^A \times A \rightarrow B$ such that for every map $f : C \times A \rightarrow B$ there is a map $\Lambda(f) : C \rightarrow B^A$ with $\text{ev}_{A,B} \circ (\Lambda(f) \times A) = f$. One chooses B^A as the set $\{e \in \mathbb{N} \mid \forall n \in A. e \cdot n \in B\}$ where $e \cdot n$ stands for $\{e\}(n)$ (Kleene's notation). The map $\text{ev}_{A,B}$ sends $\langle e, n \rangle$ to $e \cdot n$. The operation of Λ -abstraction is implemented using the *snm*-theorem of basic recursion theory. More precisely, if e realizes $f : C \times A \rightarrow B$ then any Gödel number for $n \mapsto s(e, n)$ realizes $\Lambda(f)$.¹⁰⁰

Categorically speaking this very weak notion of cartesian closedness can be reformulated as the requirement that for all objects A and B the presheaf $\mathcal{P}(\mathbb{N})(- \times A, B)$ can be covered by a representable object $\mathcal{P}(\mathbb{N})(-, B^A)$. The image of id_{B^A} under this covering is $\text{ev}_{A,B}$ and by Yoneda the covering property amounts to the requirement that for every map $f : C \times A \rightarrow B$ there is a map $\Lambda(f) : C \rightarrow B^A$ with $\text{ev}_{A,B} \circ (\Lambda(f) \times A) = f$. Notice that even this very weak notion of cartesian closedness entails the validity of the ξ -rule since $f = g$ implies $\Lambda(f) = \Lambda(g)$. However, for interpreting the simply typed λ_β -calculus we need that for $h : D \rightarrow C$ we have $\Lambda(f \circ (h \times A)) = \Lambda(f) \circ h$ since λ -abstraction is required to commute with substitution. Categorically speaking, this means that Λ is a natural transformation splitting the covering induced by $\text{ev}_{A,B}$. Thus, for interpreting λ_β -calculus in a category \mathcal{C} with finite cartesian products for all objects A and B in \mathcal{C} the presheaf $\mathcal{C}(- \times A, B)$ has to appear as a retract of some representable object $\mathcal{C}(-, B^A)$.

Now if $\mathcal{P}(\mathbb{N})$ could be organized into such a model then we would have a surjective $\phi : F \rightarrow \mathcal{R}$ where \mathcal{R} is the set of unary total recursive functions such that

- (1) there is a partial recursive binary function u such that $\phi_n(m) = u(n, m)$ for all $n \in F$ and $m \in \mathbb{N}$ and
- (2) there is a partial recursive unary function s defined on all of F such that $s(n) = s(m)$ iff $\phi_n = \phi_m$.

Since $\mathcal{P}(\mathbb{N})$ has a natural numbers object given by \mathbb{N} we could interpret Gödel's T in $\mathcal{P}(\mathbb{N})$ which due to (2) would allow us to decide extensional equality of primitive recursive function definitions which, however, is impossible by Gödel's first incompleteness theorem.

¹⁰⁰Here s is a (primitive) recursive function with $\{s(e, n)\}(m) \simeq \{e\}(\langle n, m \rangle)$.

154 Is Church's Thesis consistent with Intensional Martin-Löf Type Theory ?

In section 2 of Ch.11 of [TvD] the authors present a system ML_0^i where propositional and judgemental equality are identified (via an equality reflection rule) *but* where only those conversions are postulated which are needed for computing weak head normal forms but not the ξ -rule allowing one to conclude $\lambda x.t = \lambda x.s$ from $t = s$.

In section 6 of Ch.11 of [TvD] the authors construct a HRO model for ML_0^i where types and contexts are interpreted as subsets of \mathbb{N} (but this would actually work for arbitrary pca's). A term t of type A in context Γ is interpreted as a natural number e such that $e \cdot n \in \llbracket A \rrbracket$ for all $n \in \llbracket \Gamma \rrbracket$ and $\Gamma \vdash t_1 = t_2 \in A$ holds in the model iff $e_1 \cdot n = e_2 \cdot n$ for all $n \in \llbracket \Gamma \rrbracket$ where $e_i = \llbracket \Gamma \vdash t_i \rrbracket$ for $i=1, 2$. Since $\llbracket N \rightarrow N \rrbracket = \{e \in \mathbb{N} \mid \{e\} \text{ total}\}$ Church's Thesis is realized (essentially) by the identity on \mathbb{N} . Notice that this model does not validate the ξ -rule since Λ -abstraction¹⁰¹ does not respect extensional equality. More precisely, from $x : N \vdash t_1 = t_1 \in N$ it does not follow in general that $\vdash (\lambda x:N)t_1 = (\lambda x:N)t_2 \in N \rightarrow N$. Thus, although HRO certainly is a category ($\mathcal{P}(\mathbb{N})$) as in the previous section (153) it is not a categorical model since it does not allow one to interpret λ -abstraction as an operation on morphisms. Any attempt to remedy this by choosing Gödel numbers when interpreting λ -abstraction leads to a conflict with the Beck-Chevalley condition for λ -abstraction (see section 153 above).

Church's Thesis is certainly consistent with Maietti and Sambin's minimal type theory mTT due to its separation of propositions and types which allows one to interpret types as modest sets ($\text{PER}(\mathbb{N})$) and propositions as subsets of \mathbb{N} .¹⁰²

At the end of their paper Maietti and Sambin raise the question whether Martin-Löf's intensional type theory ITT (with ξ -rule!) is consistent with Church's Thesis (CT). One knows that already $HA_\omega + \text{Ext} + \text{AC} + \text{CT}$ is inconsistent. Thus *a fortiori* extensional type theory ETT is consistent with CT since already ITT and thus also ETT prove AC. But the absence of Ext in ITT leaves – in principle – open the possibility that ITT + CT were consistent. The HRO model certainly establishes consistency of $HA_\omega + \text{AC} + \text{CT}$ and – as observed above – also of $ML_0^i + \text{CT}$. *But* HRO cannot be used as a model for ITT + CT since HRO does not validate the ξ -rule of ITT as explained in section 153. Thus it is not even clear whether $HA_\omega + (\xi) + \text{AC} + \text{CT}$ is consistent. The literature on constructive mathematics in the tradition of proof theory does not consider the ξ -rule at all unless when studying E- HA_ω (i.e. HA_ω with full extensionality for functions).

In any case it is still possible that Church's Thesis is inconsistent with in-

¹⁰¹in the sense of Troelstra employing the *snm*-theorem of recursion theory

¹⁰²If one just looks at mTT as presented in their paper one might get the impression that they don't postulate any congruence rules (like the ξ -rule) and thus HRO is a model for their mTT. However, if one reads more carefully the words before the listing of the rules one notices that they do have the congruence rules in mind and just don't list them for the sake of conciseness.

tensional Martin-Löf type theory! Actually, I would prefer if things turned out this way. The reason why I believe it could be inconsistent is that LISP's `quote` construct definitely destroys the validity of the ξ -rule which can be seen as follows. Consider the terms $t_1 \equiv \lambda x.x$ and $t_2 \equiv \lambda x.(\lambda y.y)x$ then due to weak head reduction `quote` t_1 and `quote` t_2 are distinct whereas t_1 and t_2 are convertible using the ξ -rule. (Of course, one might think also of a less separating `quote` construct which takes a term t , normalizes it and gives back the (Gödel number of) this normal form.) However, such a `quote` construct does not behave like a function since it can be applied only to closed terms.

A realizability model for ITT + CT could be constructed if there existed a functional programming language \mathcal{L}_q containing the underlying functional language of ITT and a function `quote` sending objects f of type $N \rightarrow N$ to a Gödel number `quote`(f) of f . It is not clear at all whether factoring such a language by the least congruence \sim (including the crucial ξ -rule!) containing all computation rules is nontrivial in the sense that $t_1 \sim t_2$ does not hold for all terms t_1, t_2 . The literature on programming languages is not very informative in this respect one notable exception being [Wa98] where the author shows that for a simple λ -calculus with a (kind of) quote construct the induced observational equivalence coincides with syntactic equality (modulo α -conversion). I hope that M. Wand's theorem extends to the case where one postulates just an operation `quote` computing a Gödel number for functions of type $N \rightarrow N$.

References

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155 Axioms for the Alternative Set Theory of P. Vopenka

The language of Vopenka's Alternative Set Theory (AST) is the first order language with equality and a binary predicate \in . Capital Latin variables X, Y, Z, \dots range over classes and small Latin variables x, y, z, \dots range over those classes which are sets, i.e. satisfy the predicate $Set(X) \equiv \exists Y (X \in Y)$. Capital Greek letters Φ, Ψ, \dots range over general formulas and small Greek letters φ, ψ, \dots range over so-called *set formulas*, i.e. formulas where all free variables and quantifiers are restricted to set variables.

Axiom 1. Extensionality: $\forall Z (Z \in X \leftrightarrow Z \in Y) \rightarrow X = Y$

Axiom 2. Comprehension schema: $\exists X \forall x (x \in X \leftrightarrow \Phi)$
for all Φ without free occurrence of X

We write $\{x \mid \Phi(x)\}$ for the class of all sets x satisfying $\Phi(x)$. Using comprehension one defines the usual boolean operations on classes $X \cap Y$, $X \cup Y$, $-X$, the empty class \emptyset and the universal class $\mathbf{V} = \{x \mid x = x\}$.

Axiom 3. Existence of sets: $Set(\emptyset) \wedge \forall x \forall y Set(x \cup \{y\})$

Axiom 4. Induction: $\varphi(\emptyset) \wedge \forall x \forall y (\varphi(x) \rightarrow \varphi(x \cup \{y\})) \rightarrow \forall x \varphi(x)$
for all set formulas φ

Using Induction one can show that for every set x and set formula $\varphi(y)$ without any free occurrence of x the class $\{y \mid y \in x \wedge \varphi(y)\}$ is a set which we denote as $\{y \in x \mid \varphi(y)\}$. Moreover, using Induction one can show that sets validate all axioms of ZF_{fin} , i.e. Zermelo Fraenkel set theory without the Infinity axiom.

The class \mathbf{N} of *natural numbers* is defined as the class

$$\{x \mid \bigcup x \subseteq x \wedge \forall y \forall z (y \in z \vee y = z \vee z \in y)\}$$

i.e. the class of all transitive sets linearly ordered by \in . The class \mathbf{FN} of *finite natural numbers* is defined as the class

$$\{x \mid x \in \mathbf{N} \wedge \forall X \subseteq x Set(X)\}$$

i.e. the class of all natural numbers which are so small that every subclass is a set. Subclasses of sets which are not sets themselves are called *semisets*.

We write $X \preceq Y$ iff there exists an injective class function $F : X \rightarrow Y$ and we write $X \approx Y$ iff there exists a bijective class function $F : X \rightarrow Y$, i.e. iff X and Y are equipollent.

Axiom 5. Prolongation: $Fnc(F) \wedge F \approx \mathbf{FN} \rightarrow \exists f (Fnc(f) \wedge F \subseteq f)$
 where $Fnc(F)$ expresses that F is a class function (a functional relation).¹⁰³

Consider the class function $I = \{\langle x, x \rangle \mid x \in \mathbf{FN}\}$. Then by the Prolongation axiom there exists a set function f containing I as a subclass. Then by the remark after Axiom 4 it follows that $dom(f)$ is a set containing \mathbf{FN} as a subclass. But \mathbf{FN} itself is not a set since otherwise $\mathbf{FN} \in \mathbf{N}$ and would thus¹⁰⁴ contain a greatest element contradicting the closure of \mathbf{FN} under the successor operation $S(x) = x \cup \{x\}$.

Thus \mathbf{FN} is a semiset since it is a proper class but subclass of a set. Whereas all sets are *formally finite* by Induction a set may be called *finite* iff all its subclasses are finite. Semisets show how the infinite may appear within the formally finite which is the basic intuition behind AST.

A class is called *real* iff all its intersections with sets are sets themselves. Typical examples of real classes are those of the form $\{x \mid \varphi(x)\}$ where $\varphi(x)$ is a set formula (*c.f.* the classes of GBN set theory). Obviously, a set x is a real class iff it is finite and a real class X which is not a set cannot be a semiset either as otherwise $X = X \cap x$ is a set (where x is some set with $X \subseteq x$).

Axiom 6. Choice: $\exists R WO(\mathbf{V}, R)$
 where $WO(X, R)$ expresses that class X is wellordered by R .

Axiom 7. Cardinalities: $X \preceq \mathbf{FN} \vee X \approx \mathbf{V}$

Axiom 7. Foundation (or \in -induction): for any set formula φ

$$\forall x (\forall y \in x (\varphi(y) \rightarrow \varphi(x))) \rightarrow \forall x \varphi(x)$$

Let \mathcal{F} be a non-principal ultrafilter on the natural numbers. A model $\mathcal{M}_{\mathcal{F}}$ for AST can be constructed from the ultrapower V_{ω}/\mathcal{F} of (V_{ω}, \in) in the following way. For every $x \in \mathcal{M}$ let $\hat{x} = \{y \in \mathcal{M} \mid y \in_{V_{\omega}/\mathcal{F}} x\}$. The underlying set of the model $\mathcal{M}_{\mathcal{F}}$ for AST is given by the disjoint union

$$V_{\omega}/\mathcal{F} \dot{\cup} \{X \in \mathcal{P}(V_{\omega}/\mathcal{F}) \mid \neg \exists x \in V_{\omega}/\mathcal{F} X = \hat{x}\}$$

where elements of the first component are called sets and elements of the second component are called proper classes. The relation \in is interpreted in $\mathcal{M}_{\mathcal{F}}$ in the obvious way: $a \in_{\mathcal{M}_{\mathcal{F}}} b$ iff either b is a set and $a \in \hat{b}$ or b is a class and $a \in b \subseteq V_{\omega}/\mathcal{F}$.

¹⁰³At other places (early papers by A. Sochor) the prolongation axiom was stated in the apparently *weaker* form

$$Fnc(F) \wedge dom(F) = \mathbf{FN} \rightarrow \exists f (Fnc(f) \wedge F \subseteq f)$$

where $dom(R)$ denotes the domain of a class relation R .

¹⁰⁴by Induction one can show that every element of \mathbf{N} contains a greatest element w.r.t. \in

156 Existential Quantification in DN

The *Diller-Nahm tripos* DN is defined as follows.

Let $A \in \Sigma^I$ and $u : I \rightarrow J$ then $\exists_u(A)$ (according to Bodil BIERING) should be

- (1) $\exists_u(A)_j^+ = \bigcup_{i \in u^{-1}(j)} A_i^+$
- (2) $\exists_u(A)_j^- = \mathcal{P}_f(\bigcap_{i \in u^{-1}(j)} [A_i^+ \rightarrow A_i^-])$
- (3) $\exists_u(A)_j(a, b) \equiv \exists i \in u^{-1}(j). \forall c \in b. A_i(a, c \cdot a)$

where $c \in b$ is a shorthand for $c \in e_b$.

For $B \in \Sigma^J$ we have $\exists_u(A) \vdash_J B$ iff

$$\forall j \in J. \forall a \in \exists_u(A)_j^+, b \in B_j^-. (\forall d \in e^-(a, b). \exists i \in u^{-1}(j). \forall c \in d \cdot a. A_i(a, c)) \Rightarrow B_j(e^+ a, b)$$

for some

$$e^+ \in \bigcap_{j \in J} \bigcap_{i \in u^{-1}(j)} [A_i^+ \rightarrow B_{u(i)}^+] \quad \text{and} \quad e^- \in \bigcap_{j \in J} [\exists_u(A)_j^+ \times B_j^- \rightarrow \mathcal{P}_f(\exists_u(A)_j^-)]$$

157 A Unifying Approach to Dialectica Triples and Toposes

Let \mathbf{e}_2 be the **Set**-based tripos giving rise to $\mathcal{E}ff_2$. There is a connected geometric morphism $q : \mathbf{dn} \rightarrow \mathbf{e}_2$ with $q_*(U, X, R) = (\{x \in X \mid \forall y \in Y. R(x, y)\}, X)$ and $q^*(A_0, A_1) = (A_1, \{0\}, A_0 \times \{0\})$. Jaap van Oosten has identified \mathbf{m} and \mathbf{e} as (complemented) subtriposes of \mathbf{e}_2 . One may now consider the pullbacks of the inclusions $\mathbf{m} \hookrightarrow \mathbf{e}_2$ and $\mathbf{e} \hookrightarrow \mathbf{e}_2$ along the geometric morphism $q : \mathbf{dn} \rightarrow \mathbf{e}_2$.

Let $u = (\emptyset, \{0\})$ be a proposition of \mathbf{e}_2 . Then the subtriposes \mathbf{m} and \mathbf{e} of \mathbf{e}_2 are determined by the local operators $u \vee (-)$ and $(u \rightarrow (-))$, respectively. From Lemma C.1.2.10 of PTJ's *Elephant* one knows that the inverse images of \mathbf{m} and \mathbf{e} under q are determined by the local operators $q^*u \vee (-)$ and $q^*u \rightarrow (-)$ on \mathbf{dn} , respectively. Thus, the inverse image of \mathbf{m} under q is the subtripos of \mathbf{dn} consisting of those families of propositions (U, X, R) where $0 \in U \subseteq \mathbb{N}$, $X \subseteq \mathbb{N}$ and $R \subseteq U \times X$.

Similarly, one may consider the same task for \mathbf{d} instead of \mathbf{dn} where \mathbf{d} is the following variant of the original Dialectic tripos. The propositions are triples (U, X, R) where $U, X \subseteq \mathbb{N}$ and $R \subseteq U \times X$ and $(U, X, R) \vdash (V, Y, S)$ iff there exist recursive maps $f : U \rightarrow V$ and $F : U \times Y \rightarrow P(\{0\})$ such that $S(f(u), y)$ whenever $\pi(F(u, y)) = 0$ or $\pi(F(u, y)) = 1$ and $R(u, \pi'(F(u, y)))$. This new \mathbf{d} is an ‘‘affine’’ variant of \mathbf{dn} whereas the old \mathbf{d} (of Birkedal and Rosolini) is a ‘‘linear’’ variant of \mathbf{dn} .

In her Thesis (spring 2008) B. Biering observed that, alas, q is not a geometric morphism since q^* does not preserve finite limits. Nevertheless, the proposition $u = (\{0\}, \{0\}, \emptyset)$ induces appropriate subtriposes of the triposes \mathbf{dn} and \mathbf{d} , respectively.

158 Typed Diller-Nahm Hyperdoctrine

Let \mathcal{A} be a typed partial combinatory algebra (tpca) with a list type $\ell(\sigma)$ for all types σ in \mathcal{A} . A typical example is the HRO like tpca whose types are sets of natural numbers and where $\sigma \rightarrow \tau$ consists of all $e \in \mathbb{N}$ with $e \cdot n \in \tau$ whenever $n \in \sigma$.

For such a tpca \mathcal{A} with list types we want to build a hyperdoctrine $\mathcal{P}_d(\mathcal{A})$ (over **Set** corresponding as close as possible to Gödel's Dialectica interpretation. For $I \in \mathbf{Set}$ the predicates on I are given by triples (A^+, A^-, A) where A^+ and A^- are types of \mathcal{A} and $A \in \mathcal{P}(A^+ \times A^-)^I$, i.e. A is an I -indexed family of relations between A^+ and A^- . On the collection $\mathcal{P}_d(\mathcal{A})^I$ of all predicates on I we may define a notion of entailment in two different ways, namely

- (1) $A \vdash_I B$ iff $\exists e^+ \in A^+ \rightarrow B^+ \exists e^- \in A^+ \times B^- \rightarrow A^- \forall i \in I$
 $\forall a \in A^+ \forall b \in B^- (A_i(a, e^-(a, b)) \supset B_i(e^+a, b))$
- (2) $A \vdash_I B$ iff $\exists e^+ \in A^+ \rightarrow B^+ \exists e^- \in A^+ \times B^- \rightarrow \ell(A^-) \forall i \in I$
 $\forall a \in A^+ \forall b \in B^- ((\forall c \in e^-(a, b) A_i(a, c)) \supset B_i(e^+a, b))$

where $e^-(a, b)$ stands for $e^- \langle a, b \rangle$ and $\forall c \in e^-(a, b)$ quantifies over all elements of the list $e^- \langle a, b \rangle$. Variant (2) is a generalisation of the Diller-Nahm variant of the Dialectica interpretation which has the advantage that it does not require the A_i to be decidable which we cannot guarantee when interpreting quantification over arbitrary sets. Variant (1) is Gödel's original definition of entailment. Birkedal and Rosolini have shown that one may define implication if all types contain 0 as distinguished element and if the A^+ and A^- may vary with the index $i \in I$. As we have decided to stay typed, i.e. that A^+ and A^- are independent of the index, we are forced to stick to the Diller-Nahm variant, i.e. to version (2).¹⁰⁵

If we assume \mathcal{A} to have weak initial and sum types then $\mathcal{P}_d(\mathcal{A})$ is a hyperdoctrine when defining reindexing the obvious way, i.e. for $u : J \rightarrow I$ and $A \in \mathcal{P}_d(\mathcal{A})$ putting $u^*A = (A^+, A^-, (A_u(j))_{j \in J})$. Conjunction and implication are defined as usual in Diller-Nahm.

Next we turn to quantification. Let $A \in \mathcal{P}_d(\mathcal{A})^J$ and $u : J \rightarrow I$. Then universal quantification of A along u is given by

$$\forall_u(A)^+ = A^+ \quad \forall_u(A)^- = A^- \quad \forall_u(A)_i = \bigcup_{j \in u^{-1}(i)} A_j$$

¹⁰⁵This decision to “stay typed” is also possible for Kreisel's modified realizability. Grayson and van Oosten's modified realizability topos deviates from Kreisel's original in the respect that potential realizers may vary with the index of a predicate. A variant closer to Kreisel's original idea is implemented by the realizability category **RC**(HRO) where HRO is the tpca of subsets of \mathbb{N} containing 0. It follows from a result by Lietz and Streicher that **RC**(HRO) is not a topos (because HRO does not admit a universal type!). Notice also that both the modified realizability topos **Mod** as well as **RC**(HRO) validate extensionality and Church's Thesis which is incompatible with AC for simple types over \mathbb{N} (already with $\mathbf{AC}_{1,0}$) which is a distinguishing feature of modified realizability. This problem, however, disappears when considering **RC**(HEO) when putting $\mathbf{HEO} = \mathbf{PER}(K_1)$. The reason for this mismatch in case of HRO is that in Kreisel's modified realizability the type $\iota \rightarrow \iota$ is interpreted as $\Delta(N \rightarrow N)$ and not as $\Delta(N) \rightarrow \Delta(N)$ as in case of **RC**(HRO).

and existential quantification of A along u is given by

$$\exists_u(A)^+ = A^+ \quad \exists_u(A)^- = A^+ \rightarrow \ell(A^-) \quad \exists(A)_i(a, b) \equiv \exists j \in u^{-1}(i). \forall c \in ba. A_j(a, c)$$

Assuming that \mathcal{A} has a terminal type 1 the equality predicate $eq_I = \exists_{\delta_I}(\top_I) \in \mathcal{P}_d(\mathcal{A})^{I \times I}$ is given by $(1, 1, eq_I)$ where $eq_I(i, j) = \{ * \in 1 \mid i = j \}$ for $i, j \in I$. As usual this gives rise to a functor $\Delta : \mathbf{Set} \rightarrow \mathbf{Set}[\mathcal{P}_d[\mathcal{A}]]$ sending I to $\Delta(I) = (I, eq_I)$ and $u : J \rightarrow I$ to $\Delta(u)(j, i) = eq_J(u(j), i)$ representing a functional relation from $\Delta(J)$ to $\Delta(I)$. Following van Oosten we define $\mathbf{Asm}(\mathcal{P}_d(\mathcal{A}))$ as the full subcategory of $\mathbf{Set}[\mathcal{P}_d(\mathcal{A})]$ on subobjects of some $\Delta(I)$.

It is unlikely that $\mathcal{P}_d(\mathcal{A})$ is a tripos since there is no candidate for a generic predicate. One should be able to make this precise using an argument like in Lietz and Streicher's paper. For the same reason it is unlikely that $\mathcal{P}_d(\mathcal{A})$ is a "moral tripos" in the sense of A. Pitts. Thus $\mathbf{Set}[\mathcal{P}_d(\mathcal{A})]$ is logoses but (presumably!) not a topos.

But is $\mathbf{Set}[\mathcal{P}_d(\mathcal{A})]$ at least (locally) cartesian closed? When constructing exponentials $(Y, E_Y)^{(X, E_X)}$ in $\mathbf{Set}[\mathcal{P}]$ for a set based tripos \mathcal{P} it is essential that one has access to $\Sigma_{\mathcal{P}}^{X \times Y}$ since one carves out the exponential as a subquotient of $\Delta(\Sigma_{\mathcal{P}}^{X \times Y})$. But in case of tpca's \mathcal{A} lacking a universal type this is impossible since there does not exist a set $\Sigma_{\mathcal{P}(\mathcal{A})}$ of *all* propositions.

But, of course, this does not *prove* that exponentials do not exist in general. In order to get a handle on the problem we consider the case $\mathcal{A} = \text{HRO}$ and the exponential N^N where N is the nno in $\mathbf{Set}[\mathcal{P}_d(\text{HRO})]$. Notice that $N = (\mathbb{N}, (\mathbb{N}, \{0\}, E_N))$ with $E_N(n, m) = \{(n, 0) \mid n = m\}$. The obvious candidate for N^N is $(\mathcal{R}, (\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}, \mathbb{N}, E_{N^N}))$ where \mathcal{R} is the set of total recursive functions from \mathbb{N} to \mathbb{N} and $E_{N^N}(f, g) = \{((n, m), k) \mid \{n\} = f \wedge \{m\} = g \wedge f(k) = g(k)\}$. Notice that although N is in $\mathbf{Asm}(\mathcal{P}_d(\text{HRO}))$ this is not the case for N^N since E_{N^N} is not equivalent to the predicate $(f, g) \mapsto E_{N^N}(f, f) \wedge eq_{\mathcal{R}}(f, g)$ on $\mathcal{R} \times \mathcal{R}$. Thus $\mathbf{Asm}(\mathcal{P}_d(\text{HRO}))$ is *not closed under exponentiation* already in fairly simple cases.

Notice that the same phenomenon arises for $\mathbf{Set}[\text{dn}_m]$ where dn_m is the modified Diller-Nahm tripos where $E_N(n, m) = \{(n, 0) \mid n = m\}$ and $E_{N^N}(f, g) = (\{(n, m) \mid \{n\} = f \wedge \{m\} = g\}, \mathbb{N}, \{((n, m), k) \mid \{n\} = f \wedge \{m\} = g \wedge f(k) = g(k)\})$. Thus, although in $\mathbf{Set}(\text{dn}_m)$ exponentials exist since it is a topos the category $\mathbf{Asm}(\text{dn}_m)$ of assemblies is not closed under exponentiation since N is an assembly whereas N^N is not.

This trouble with exponentials is not unexpected from the point of view of the usual proof-theoretic treatment of the Dialectica interpretation. As shown by W. A. Howard in 1973 the extensionality principle

$$(E_2) \quad (\forall x: N(f(x) = g(x))) \rightarrow F(f) = F(g)$$

for type 2 functionals has no functional interpretation via Gödel T functionals. Moreover, as also shown by Howard in 1973, the Dialectica interpretability the extensionality principle (E_3) for type 3 functionals is independent from ZF, i.e. ZF does not prove the existence of functionals witnessing the Dialectica interpretation of (E_3) .

However, I believe that due to a density theorem¹⁰⁶ for HEO there are witnesses for the extensionality principles in the whole finite type hierarchy over N .

159 Isbell Conjugacy

Let \mathbb{C} be a small category. Then one may define an adjunction $(-)^* \dashv (-)^\sharp : (\mathbf{Set}^{\mathbb{C}})^{\text{op}} \rightarrow \mathbf{Set}^{\mathbb{C}^{\text{op}}}$ as

$$A^*(I) = \mathbf{Set}^{\mathbb{C}^{\text{op}}}(A, Y_{\mathbb{C}}(I)) \quad \text{and} \quad B^\sharp(I) = \mathbf{Set}^{\mathbb{C}}(B, Y_{\mathbb{C}^{\text{op}}}(I))$$

whose unit and counit are given by

$$((\eta_A)_I(a))_J(\tau) = \tau_I(a) \quad \text{and} \quad ((\varepsilon_B)_I(b))_J(\sigma) = \sigma_I(b)$$

respectively. One can show that it restricts to an adjunction $(-)^\sharp \dashv (-)^* : \mathbf{pro}(\mathbb{C}) \rightarrow \mathbf{ind}(\mathbb{C})$ bearing in mind that $\mathbf{pro}(\mathbb{C}) = (\mathbf{ind}(\mathbb{C}^{\text{op}}))^{\text{op}}$ and $\mathbf{ind}(\mathbb{C})$ is the completion of representable objects under filtered colimits in $\mathbf{Set}^{\mathbb{C}^{\text{op}}}$.¹⁰⁷

For more information see Lawvere's paper *Taking Categories Seriously*.

¹⁰⁶HEO is the total part of the finite type hierarchy over \mathbb{N}_\perp in effectively given Scott domains as shown by U. Berger (or even earlier by Yu. L. Ershov).

¹⁰⁷If \mathbb{C} is finitely cocomplete then $\mathbf{ind}(\mathbb{C})$ consists precisely of the finite limit preserving functors from \mathbb{C}^{op} to \mathbf{Set} and if \mathbb{C} is finitely complete then $\mathbf{pro}(\mathbb{C})$ consists precisely of the finite limits preserving functors from \mathbb{C} to \mathbf{Set} .

160 An Observation of P. Hofstra

An exact category \mathcal{E} is equivalent to an exact completion \mathcal{C}_{ex} of some finite limit category \mathcal{C} iff projectives in \mathcal{E} are closed under finite limits and every object of \mathcal{E} is covered by some projective.

Suppose \mathcal{E} is a balanced category equivalent to \mathcal{C}_{ex} for some finite limit category \mathcal{C} . Then in \mathcal{E} all epis are regular and e is an epi iff $P \perp e$ for all projectives P .

Proof: Of course, this condition is necessary. For the reverse direction suppose $e : A \rightarrow B$ with $P \perp e$ for all projectives. Let $c : P \twoheadrightarrow B$ be an epi with P projective. Since $P \perp e$ there is a map $f : P \rightarrow A$ with $c = ef$ and thus e is epic. \square

Lemma

Let \mathcal{E} and \mathcal{F} be toposes which are exact completions of some finite limit categories and $F \dashv G : \mathcal{E} \rightarrow \mathcal{F}$. Then G preserves (regular) epis iff F preserves projectives.

Proof: Since $F \dashv G$ we have $FP \perp e$ iff $P \perp Ge$.

Suppose F preserves projectives and e is an epi in \mathcal{E} . Then for all projectives P in \mathcal{F} we have $P \perp Ge$ since $FP \perp e$. Thus Ge is epic.

Suppose G preserves epis and P is projective in \mathcal{F} . Then for all epis e in \mathcal{E} we have $FP \perp e$ since $P \perp Ge$. \square

For this reason many global sections functors $\Gamma : \mathcal{E} \rightarrow \mathbf{Set}$ are not regular, i.e. do not preserve regular epis. E.g. if \mathcal{E} is the topos of graphs or $\widehat{G} = \mathbf{Set}^{G^{\text{op}}}$ for some nontrivial group G .

161 Iteration of the Tripos-to-Topos Construction as characterized by A. M. Pitts

Let \mathcal{C} be a finitely complete category, \mathcal{E} a topos and $F : \mathcal{C} \rightarrow \mathcal{E}$ a finite limit preserving functor. Theorem 3.10 of Pitts' Thesis says that $F : \mathcal{C} \rightarrow \mathcal{E}$ is equivalent to $\Delta_{\mathcal{P}} : \mathcal{C} \rightarrow \mathcal{C}[\mathcal{P}]$ for some tripos \mathcal{P} over \mathcal{C} iff

- (\ddagger) for all $A \in \mathcal{E}$ there exists an object $\widehat{A} \in \mathcal{C}$ and an epi $\beta_A : F\widehat{A} \rightarrow \widetilde{A}$ (where $\eta_A : A \rightarrow \widetilde{A}$ is the partial map classifier for A) such that for every $f : FX \rightarrow \widetilde{A}$ there exists a (not necessarily unique) map $g : X \rightarrow \widehat{A}$ with $\beta_A \circ Fg = f$.

The key idea of the proof is to take for \mathcal{P} the tripos $F^*\text{Sub}_{\mathcal{E}}$.

Pitts shows (Lemma 6.1) that for a finitely complete category \mathcal{C} , toposes \mathcal{E} and \mathcal{F} and finite limit preserving functors $F : \mathcal{C} \rightarrow \mathcal{E}$ and $G : \mathcal{E} \rightarrow \mathcal{F}$ with G regular if both F and G satisfy condition (\ddagger) then the composite functor $G \circ F : \mathcal{C} \rightarrow \mathcal{F}$ satisfies (\ddagger) as well.¹⁰⁸ Thus, if $G : \mathcal{E} \rightarrow \mathcal{F}$ is equivalent to

¹⁰⁸Let B be an object of \mathcal{F} . Then we have

$$\begin{array}{ccc}
 \cdot & \xrightarrow{e'} & B \\
 \downarrow m' & \lrcorner & \downarrow \eta_B \\
 G(\widehat{B}) & \xrightarrow{\beta'_B} & \widetilde{B}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \cdot & \xrightarrow{e} & \widehat{B} \\
 \downarrow m & \lrcorner & \downarrow \eta_{\widehat{B}} \\
 F(\widehat{\widehat{B}}) & \xrightarrow{\beta_{\widehat{B}}} & \widetilde{\widehat{B}}
 \end{array}$$

by assumption on F and G . Let $b : G(\widetilde{\widehat{B}}) \rightarrow \widetilde{B}$ classify the partial map $(G(\eta_{\widehat{B}})m', e')$ as depicted in

$$\begin{array}{ccccc}
 \cdot & \xrightarrow{e''} & \cdot & \xrightarrow{e'} & B \\
 \downarrow m'' & \lrcorner & \downarrow m' & \lrcorner & \downarrow \eta_B \\
 \cdot & \xrightarrow{G(e)} & G(\widehat{B}) & & \\
 \downarrow G(m) & \lrcorner & \downarrow G(\eta_{\widehat{B}}) & & \\
 GF(\widehat{\widehat{B}}) & \xrightarrow{G(\beta_{\widehat{B}})} & G(\widetilde{\widehat{B}}) & \xrightarrow{b} & \widetilde{B}
 \end{array}$$

where we have used intrinsically that G preserves pullbacks and epis. Thus, we have also

$$\begin{array}{ccc}
 \cdot & \xrightarrow{e'} & B \\
 \downarrow m' & \lrcorner & \downarrow \eta_B \\
 G(\widehat{B}) & \xrightarrow{G(\eta_{\widehat{B}})} & G(\widetilde{\widehat{B}}) \xrightarrow{b} \widetilde{B}
 \end{array}$$

$\Delta_{\mathcal{E}} : \mathcal{E} \rightarrow \mathcal{E}[G^* \text{Sub}_{\mathcal{F}}]$ and regular¹⁰⁹ then the composite GF is equivalent to $\Delta_{\mathcal{C}[(GF)^* \text{Sub}_{\mathcal{F}}]} : \mathcal{C} \rightarrow \mathcal{C}[(GF)^* \text{Sub}_{\mathcal{F}}]$.

Again following chapter 3 of Pitts' Thesis we make explicit the construction of a generic family for $\mathbf{Set}[(GF)^* \text{Sub}_{\mathcal{F}}]$ in case $\mathcal{C} = \mathbf{Set}$. Let $F = \Delta_{\mathbf{P}} : \mathbf{Set} \rightarrow \mathbf{Set}[\mathbf{P}]$ and $G = \Delta_{\mathbf{Q}} : \mathcal{E} \rightarrow \mathcal{E}[\mathbf{Q}]$ where \mathbf{P} and \mathbf{Q} are canonically presented as $\mathbf{P} = \mathbf{Set}(-, \Sigma)$ and $\mathbf{Q} = \mathcal{E}(-, (\Lambda, \sim))$ respectively. Define $S_{\Lambda} : \Sigma^{\Lambda} \rightarrow \Sigma$ as $S_{\Lambda}(R) = \llbracket \exists x \in \Lambda. \forall x' \in \Lambda. (R(x) \leftrightarrow x \sim x') \rrbracket$. Consider the diagram

$$\begin{array}{ccc} \llbracket S_{\Lambda} \rrbracket & \xrightarrow{b_{\Lambda}} & (\Lambda, \sim) \\ \downarrow i_{\Lambda} & & \\ \Delta(\Sigma^{\Lambda}) & & \end{array}$$

where i_{Λ} is the subobject of $\Delta(\Sigma^{\Lambda})$ determined by S_{Λ} and b_{Λ} is determined by the predicate $b_{\Lambda}(R, x) \equiv \llbracket \exists x' : \Lambda. (S_{\Lambda}(R) \wedge R(x') \wedge x' \sim x) \rrbracket$. A generic family for the tripos $(GF)^* \text{Sub}_{\mathcal{F}}$ is given by $\exists_{i_{\Lambda}}(b_{\Lambda}) : \Delta(\Sigma^{\Lambda}) \rightarrow (\Lambda, \sim)$ where $\exists_{i_{\Lambda}}(b_{\Lambda})(R, x) = \llbracket \exists x' : \Lambda. (S_{\Lambda}(R) \wedge R(x') \wedge x' \sim x) \rrbracket$.

We are interested in contracting the following two steps of construction into one. Let \mathcal{A} be a pca. Put $\mathcal{C} = \mathbf{Set}$, $\mathcal{E} = \text{RT}[\mathcal{A}]$ and $F = \nabla : \mathbf{Set} \rightarrow \text{RT}[\mathcal{A}]$. Let further H be a complete Heyting algebra in \mathcal{E} . Put $\mathcal{F} = \text{Sh}_{\mathcal{E}}(H)$ and $G = \Delta : \mathcal{E} \rightarrow \text{Sh}_{\mathcal{E}}(H)$. Then by Pitts' Lemma 6.2 the topos $\text{Sh}_{\text{RT}[\mathcal{A}]}(H)$ is equivalent to the topos $\mathbf{Set}[(\Delta \nabla)^* \text{Sub}_{\mathcal{F}}]$. Now for a set I the subobjects of $\Delta \nabla I$ correspond to morphisms $\nabla I \rightarrow H$ in \mathcal{E} and for $\varphi, \psi : \nabla I \rightarrow H$ we have $\varphi \vdash_I \psi$ iff $\forall i \in \nabla(I). \varphi_i \leq_H \psi_i$ holds in \mathcal{E} .

from which it follows that $\beta'_B = b \circ G(\eta_{\widehat{B}})$. Thus b is epic since β'_B is epic. Since G preserves epis $G(\beta'_B)$ is epic, too. Thus $\beta''_B = b \circ G(\beta'_B)$ is epic. For showing that β''_B validates the requirement of (\dagger) suppose $f : GF I \rightarrow \widehat{B}$. Then there exists $g : FI \rightarrow \widehat{B}$ with $f = \beta'_B \circ G(g)$ and $h : I \rightarrow \widehat{B}$ with $\beta_{\widehat{B}} \circ F(h) = \eta_{\widehat{B}} \circ g$. Then we have

$$\begin{aligned} \beta''_B \circ GFh &= b \circ G(\beta_{\widehat{B}}) \circ GFh = b \circ G(\beta_{\widehat{B}} \circ Fh) = \\ &= b \circ G(\eta_{\widehat{B}} \circ g) = b \circ G(\eta_{\widehat{B}}) \circ Gg = \beta'_B \circ Gg = \\ &= f \end{aligned}$$

as desired.

¹⁰⁹For a tripos \mathbf{P} the functor $\Delta_{\mathbf{P}}$ is regular iff $\exists_e(\top) \dashv\vdash \top$ for all epis e in the base.

162 A Geometric View of (‘Moral’) Tripuses

In Chapter 3 of Pitts’ Thesis *Tripus Theory* (1981) (around Prop. 3.8) one finds a “geometric” characterization of tripuses over a base topos \mathcal{S} . The main fact is that for a finite limit preserving functor $F : \mathcal{S} \rightarrow \mathcal{E}$ between toposes we have

$$\begin{array}{ccc}
 \mathcal{S} & \xrightarrow{\Delta} & \mathcal{S}[P] \\
 & \searrow F & \downarrow K \text{ full and faithful} \\
 & & \mathcal{E}
 \end{array}$$

where P is the posetal hyperdoctrine $F^*\text{Sub}_{\mathcal{E}}$, $\mathcal{S}[P]$ is obtained from P by adding subquotients, Δ is the corresponding “constant objects” functor and K is the functor which “takes subquotients”. Obviously, the functor K is an equivalence iff 1 is a bound for F , i.e. every object A of \mathcal{E} appears as a subquotient of some FI . In Pitts’ *Tripus Theory in Retrospect* it has been shown that for a posetal hyperdoctrine P over \mathcal{S} the category $\mathcal{S}[P]$ is a topos iff for every $I \in \mathcal{S}$ there is an object $P(I)$ in \mathcal{S} and a predicate $\in_I \in P(I \times P(I))$ such that for all $J \in \mathcal{S}$ and $R \in P(I \times J)$ the statement

$$\forall j \in J. \exists p \in P(I). \forall i \in I. R(i, j) \leftrightarrow i \in_I p$$

holds in the logic of P . Such hyperdoctrines are called “moral” tripuses (following a suggestion of PTJ). Thus, moral tripuses over \mathcal{S} correspond up to equivalence to finite limit preserving functors from \mathcal{S} to a topos \mathcal{E} for which 1 is a bound, i.e. “localic geometric morphisms to \mathcal{S} without local smallness” (in the sense of Moens’s Theorem).

If $F : \mathcal{S} \rightarrow \mathcal{E}$ is a functor giving rise to a moral tripus then $F^*\text{Sub}_{\mathcal{E}}$ is a proper tripus iff there is a $\Sigma \in \mathcal{S}$ together with a $t : F(\Sigma) \rightarrow \Omega_{\mathcal{E}}$ such that for all $p : F(I) \rightarrow \Omega_{\mathcal{E}}$ there exists an $f : I \rightarrow \Sigma$ with $p = t \circ F(f)$, i.e. the fibration P_F has a generic family of subterminals, namely the subobject of $F(\Sigma)$ classified by t , which, however, in general need not be classifying.

It is an open question whether finite limit preserving functors between toposes with bound 1 are closed under composition. Well, they are if they are regular, i.e. preserve (regular) epis. Thus, regular constant objects functors are closed under composition. This is no restriction in practice since the constant objects functor corresponding to a tripus is regular if and only if the tripus has “fibrewise quantification”.

Examples and Counterexamples

For every Grothendieck topos \mathcal{E} the functor $\Gamma = \mathcal{E}(1, -) : \mathcal{E} \rightarrow \mathbf{Set}$ gives rise to a moral tripus since it preserves finite limits and is bounded by 1 .

Let **Graph** be the topos of (non-reflexive) graphs and G be the representable graph with two nodes connected by one edge. The global elements functor

$F = \mathbf{Graph}(1, -) : \mathbf{Graph} \rightarrow \mathbf{Set}$ is not regular since the epi $G \rightarrow 1$ is sent by F to $0 \rightarrow 1$ in \mathbf{Set} . Moreover F is a ‘constant objects’ functor corresponding to a proper tripos which can be seen as follows. For a set S let $L(S)$ be the graph with one node and edge set S and $\beta_S : F(L(S)) \rightarrow S$ be the obvious bijection. For every $f : F(A) \rightarrow S$ there is a graph morphism $g : A \rightarrow L(S)$ with $f = \beta_S \circ F(g)$. Such a g operates on loops as prescribed by f and is arbitrary for non-loops (and thus in general not unique!).

The functor $\Gamma : \mathbf{Set} \times \mathbf{Set} \rightarrow \mathbf{Set} : (A, B) \mapsto A \times B$ preserves regular epis but does not give rise to a tripos since $\mathbf{P} = \Gamma^* \mathbf{Sub}_{\mathbf{Set}}$ has no truth predicate. Suppose on the contrary that there is a $t : A \times B \rightarrow 2$ such that for every $p : C \times D \rightarrow 2$ there exist $f : C \rightarrow A$ and $g : D \rightarrow B$ with $p = t \circ (f \times g)$. Now instantiate C by 2^{2^B} , D by 2^B and p by the evaluation map. Then there exist $f : 2^{2^B} \rightarrow A$ and $g : 2^B \rightarrow B$ such that for every $\mathcal{X} \in 2^{2^B}$ and $X \in 2^B$ we have $\mathcal{X}(X) = t(f(\mathcal{X}), g(X))$. The map $i = \lambda \mathcal{X} : 2^{2^B} . \lambda y : B . t(f(\mathcal{X}), y) : 2^{2^B} \rightarrow 2^B$ is one-to-one since $i(\mathcal{X}) \circ g = \mathcal{X}$ and thus $|2^{2^B}| \leq |2^B|$ which is impossible.

163 A moral tripos \mathcal{P} over \mathbf{Set} such that $\mathbf{Set}[\mathcal{P}]$ is a non-localic Grothendieck topos

Let \mathcal{E} be the topos of reflexive graphs which is Grothendieck but not localic. The functor $\nabla : \mathbf{Set} \rightarrow \mathcal{E}$ right adjoint to $\Gamma = \mathcal{E}(1, -)$ preserves finite limits and every object X in \mathcal{E} appears as subquotient of some $\nabla(I)$. Thus $\mathcal{P} = \nabla^* \mathbf{Sub}_{\mathcal{E}}$ is a moral tripos over \mathbf{Set} whose associated topos $\mathbf{Set}[\mathcal{P}]$ is the non-localic Grothendieck topos \mathcal{E} .

Note, however, that \mathcal{P} is not a proper tripos since P_{∇} does not admit a generic mono. Notice also that \mathcal{E} is not localic over the Schanuel topos since objects X in \mathcal{E} with decidable equality are those in the image of Δ , i.e. have no nontrivial edges.

164 Generalizing Bounded Geometric Morphisms?

One might consider an analogous generalisation of bounded geometric morphisms, namely finite limits preserving functors $F : \mathcal{S} \rightarrow \mathcal{E}$ bounded by some $B \in \mathcal{E}$, i.e. all $X \in \mathcal{E}$ appear as subquotients of some $B \times FI$. For such F one may consider the full subfibration \mathbf{P} of $P_F = \partial_1 : \mathcal{E} \downarrow F \rightarrow \mathcal{S}$ on families of subobjects of B which is not posetal unless B is subterminal and try to construct analogues of $\mathcal{S}[\mathbf{P}]$, $\Delta : \mathcal{S} \rightarrow \mathcal{S}[\mathbf{P}]$ and $K : \mathcal{S}[\mathbf{P}] \rightarrow \mathcal{E}$.

For such $F : \mathcal{S} \rightarrow \mathcal{E}$ one might also consider the following reformulation of condition (‡) of section 161 : there is an object $B \in \mathcal{E}$ such that for every $A \in \mathcal{E}$ there is an epimorphism $\beta_A : B \times F\hat{A} \rightarrow \tilde{A}$ such that for every map $f : B \times FI \rightarrow \tilde{A}$ there is some map $g : I \rightarrow \hat{A}$ with $f = \beta_A \circ (B \times F(g))$.

Actually, the following approach based on a direct generalisation of the case where 1 is a bound seems to be more appropriate. Namely, a functor $F : \mathcal{S} \rightarrow \mathcal{E}$

together with an object $B \in \mathcal{E}$ such that

- (1) F preserves finite limits
- (2) B is a bound, i.e. every $A \in \mathcal{E}$ appears as subquotient of some $B \times FI$
- (3) there is an object $S_B \in \mathcal{S}$ and a mono $r_B : R_B \rightarrow B \times FS_B$ such that for every subobject $r : R \rightarrow B \times FI$ there exists some (not necessarily unique) map $\rho : I \rightarrow S_B$ such that

$$\begin{array}{ccccc}
 R & \longrightarrow & R_B & & \\
 r \downarrow & \lrcorner & \downarrow r_B & & \\
 B \times FI & \longrightarrow & B \times FS_B & \longrightarrow & B \\
 \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
 FI & \xrightarrow{F\rho} & FS_B & \longrightarrow & F1
 \end{array}$$

- (4) $B \times B$ appears as subobject of B and B has a point.

Notice that by (3) the map $g_B = \pi_2 \circ r_B : R_B \rightarrow FS_B$ is generic for the full subfibration of $P_F = F^*P_{\mathcal{E}}$ on objects of the form $R \rightarrow B \times FI \xrightarrow{\pi_2} FI$. This subfibration is obviously closed under vertical subobjects and by condition (4) it is also closed under fibrewise finite products. Moreover, from (2) it follows that g_B is a generating family for P_F (see footnote 15 on p.78 of my notes on Fibred Categories).

In case $B = 1$ condition (4) holds for trivial reasons and condition (2) just says that $F^*\text{Sub}_{\mathcal{E}}$ has a generic family $T \rightarrow F\Sigma$, i.e. for every $m : P \rightarrow FI$ there exists a (not necessarily unique) map $p : I \rightarrow \Sigma$ with

$$\begin{array}{ccc}
 P & \longrightarrow & T \\
 m \downarrow & \lrcorner & \downarrow \\
 FI & \xrightarrow{Fp} & F\Sigma
 \end{array}$$

If one wants to show that regular functors F satisfying conditions (1-4) are closed under composition one runs into problems with (3). For this reason I suggest to replace conditions (2) and (3) by the condition

- (2+3) there is an object $B \in \mathcal{E}$ such that for every $A \in \mathcal{E}$ there is an epimorphism $\beta_A : B \times F\hat{A} \rightarrow \tilde{A}$ such that for every map $f : B \times FI \rightarrow \tilde{A}$ there is some map $g : I \rightarrow \hat{A}$ with $f = \beta_A \circ (B \times F(g))$

we have considered already above.¹¹⁰ Notice that this condition entails condition (2) since we have

$$\begin{array}{ccc} C & \longrightarrow & A \\ \downarrow & \lrcorner & \downarrow \eta_A \\ B \times F\hat{A} & \xrightarrow{\beta_A} & \tilde{A} \end{array}$$

and it entails condition (3) when instantiating A by 1 since $\Omega_{\mathcal{E}} = \tilde{1}$ and thus $t = \beta_1 : B \times F\hat{1} \rightarrow \Omega_{\mathcal{E}}$ is an epimorphism such that for every $r : B \times FI \rightarrow \Omega_{\mathcal{E}}$ there is a $\rho : I \rightarrow \hat{1}$ with $r = t \circ (B \times F\rho)$.

Notice that given a B satisfying condition (2+3) we can extend β_A to a morphism $\tilde{\beta}_A : B \times F\tilde{A} \rightarrow \tilde{A}$ such that $\beta_A = \tilde{\beta}_A \circ (B \times F(\eta_{\hat{A}}))$ where $\eta_{\hat{A}} : \hat{A} \rightarrow \tilde{A}$ is the partial map classifier for \hat{A} . Obviously $\tilde{\beta}_A$ is epic since β_A is. Suppose $f : B \times FI \rightarrow \tilde{A}$. Then there exists a $g : I \rightarrow \tilde{A}$ with $f = \beta_A \circ (B \times F(g))$. Thus for $\tilde{g} = \eta_{\hat{A}} \circ g$ we have $f = \beta_A \circ (B \times F(g)) = \tilde{\beta}_A \circ (B \times F(\eta_{\hat{A}})) \circ (B \times F(g)) = \tilde{\beta}_A \circ (B \times F(\tilde{g}))$.

Composition Lemma

Suppose $F : \mathcal{S} \rightarrow \mathcal{E}$ and $G : \mathcal{E} \rightarrow \mathcal{F}$ are regular functors between toposes satisfying conditions (2+3) and (4) then so does their composition $GF : \mathcal{S} \rightarrow \mathcal{F}$.

Proof. Since regular functors are closed under composition the functor GF is regular. Suppose B is a bound for F and C is a bound for G . We show now that $C \times G(B)$ is a bound for GF . Let $A \in \mathcal{F}$. Then by condition (2+3) and the subsequent remark there exist a $D \in \mathcal{E}$ and an epimorphism $\gamma_A : C \times G\tilde{D} \twoheadrightarrow \tilde{A}$ witnessing condition (2+3). Furthermore there exists an epimorphism $\beta_D : B \times F\tilde{D} \twoheadrightarrow \tilde{D}$ witnessing condition (2+3) for B w.r.t. F . Consider the morphism $\delta_A = \gamma_A \circ (C \times G(\beta_D)) : C \times G(B) \times GF\tilde{D}$ (where we exploit that G preserves finite products). Obviously δ_A is epic since G preserves epis. Let $f : C \times G(B) \times GF I \rightarrow \tilde{A}$. By assumption on γ_A there exists a map $h : B \times FI \rightarrow \tilde{D}$ with $f = \gamma_A \circ (C \times G(h))$. By assumption on β_D there exists a map $g : I \rightarrow \tilde{D}$ with $h = \beta_D \circ (B \times F(g))$. Thus we have

$$f = \gamma_A \circ (C \times G(h)) = \gamma_A \circ (C \times G(\beta_D)) \circ (C \times G(B \times F(g))) = \delta_A \circ (C \times G(B) \times GF(g))$$

as desired. Finally we show that $C \times G(B)$ satisfies condition (4). Since C and B have a point so has $G(B)$ and thus $C \times G(B)$. By assumption we have $B \times B \twoheadrightarrow B$ and $C \times C \twoheadrightarrow C$ and thus also $G(B) \times G(B) \twoheadrightarrow G(B)$ from which it follows that $C \times G(B) \times C \times G(B) \cong C \times C \times G(B) \times G(B) \twoheadrightarrow C \times G(B)$ as desired. \square

¹¹⁰Condition (2+3) is an obvious weakening of condition (iii) in Lemma 4.42 of Johnstone's *Topos Theory* (1977) which is one of three equivalent conditions for B being a bound for a geometric morphism $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$.

165 Frey's Generalisation of Tripases (Summer 2010)

The intention is to generalize tripases in such a way that they encompass the construction of Grothendieck toposes. He starts from the following observation. Let \mathcal{E} be a Grothendieck topos over \mathbf{Set} with a small generating family $(G_i)_{i \in I}$ and \mathbb{C} the full subcategory of \mathcal{E} on subobjects of finite products of generators. Let $P : \mathbf{Fam}(\mathbb{C}) \rightarrow \mathbf{Set}$ be \mathbb{C} fibred over \mathbf{Set} and $Q = \text{sub}(P) : \text{Sub}(\mathbf{Fam}(\mathbb{C})) \rightarrow \mathbf{Fam}(\mathbb{C})$ be the fibration of vertical subobjects of $\mathbf{Fam}(P)$. The posetal fibration Q has universal quantification along arbitrary maps satisfying BCC. Moreover Q has *parametric power objects*, i.e. weak power objects (in the sense of tripases) such that for all vertical $U \rightarrow A \times B$ there is a cartesian $\chi_U : B \rightarrow PA$ with $(A \times \chi_U)^* \in_A \cong U$.¹¹¹ It can be shown that $\mathbf{Fam}(\mathbb{C})[Q]$ is equivalent to \mathcal{E} .

Definition 1

An xyz-fibration is a fibration $P : \mathbb{X} \rightarrow \mathbb{C}$ of finite limit categories over a finite limit category \mathbb{C} such that P has a generic family¹¹² and its vertical subobject fibration $\text{sub}(P)$ has universal quantification and parametric power objects.

Notice that every tripos is an xyz-fibration.

Definition 2

An abc-functor is a functor $F : \mathcal{S} \rightarrow \mathcal{E}$ between toposes such that

- (1) F preserves finite limits
- (2) $P_F = \partial_1 : \mathcal{E} \downarrow \mathcal{S} \rightarrow \mathcal{S}$ is weakly well powered (i.e. well powered without uniqueness condition), i.e. for every $a : A \rightarrow FI$ there is an arrow $\sigma_a : S_a \rightarrow I$ in \mathcal{S} together with a subobject $m_a : E_a \rightarrow F(S_a) \times_{FI} A$ such that for all $u : J \rightarrow I$ in \mathcal{S} and subobjects $m : P \rightarrow FJ \times_{FI} A$ we have

$$\begin{array}{ccccc}
 P & \longrightarrow & E_a & & \\
 m \downarrow \lrcorner & & \downarrow m_a & & \\
 FJ \times_{FI} A & \longrightarrow & F(S_a) \times_{FI} A & \longrightarrow & A \\
 u^* a \downarrow \lrcorner & & \downarrow \sigma_a^* a \lrcorner & & \downarrow a \\
 FJ & \xrightarrow{Fv} & F(S_a) & \xrightarrow{F\sigma_a} & FI
 \end{array}$$

for some (generally not unique) arrow $v : J \rightarrow S_a$ with $\sigma_a \circ v = u$.

- (3) there is a map $c : C_0 \rightarrow FI_0$ in \mathcal{E} such that the full subfibration P_c of P_F generated by c is closed under finite products and vertical subobjects and

¹¹¹For $A = (A_i)_{i \in I}$ construct PA as $(PA)_{k \in K}$ with $K = \{(C, R) \mid C \in \mathbb{C} \text{ and } R_i \rightarrow A_i \times C\}$ and $PA_{(C, R)} = C$ and $(\in_A)_{(i, (C, R))} = R_i \rightarrow A_i \times C$.

¹¹²i.e. there is a $G \in \mathbb{X}$ such that for every $X \in \mathbb{X}$ there is a cartesian arrow from X to G

furthermore all $A \in \mathcal{E}$ fit into a diagram of the form

$$\begin{array}{ccccc}
 A & \longleftarrow & C & \longrightarrow & C_0 \\
 & & c \downarrow & \lrcorner & \downarrow c_0 \\
 & & FI & \xrightarrow{Fu} & FI_0
 \end{array}$$

Possibly this notion should be replaced by the slightly weaker one I discussed in section 164 because it is a more immediate generalisation of triposes. In January 2011 Jonas Frey came up with the following fusion of the tentative definition above and my suggestions of section 164.

Definition 2

An abc-functor is a functor $F : \mathcal{S} \rightarrow \mathcal{E}$ between toposes such that

- (1) F preserves finite limits
- (2) $P_F = \partial_1 : \mathcal{E} \downarrow F \rightarrow \mathcal{S}$ is weakly well powered (i.e. well powered without uniqueness condition)
- (3) there is a map $c : C_0 \rightarrow FI_0$ in \mathcal{E} such that the full subfibration P_c of P_F generated by c is closed under finite products and vertical subobjects and for all $X \in \mathcal{E}$ there is an $a : A \rightarrow FI$ in P_c and an epimorphism $\beta_X : A \rightarrow X$ in \mathcal{E} such that all $b : B \rightarrow FJ$ in P_c and $f : B \rightarrow X$ in \mathcal{E} fit into a diagram

$$\begin{array}{ccccc}
 & & & & X \\
 & & & & \uparrow \beta_X \\
 B & \xrightarrow{g} & A & & \\
 b \downarrow & \lrcorner & \downarrow a & & \\
 FJ & \xrightarrow{Fu} & FI & &
 \end{array}$$

for some $u : J \rightarrow I$ in \mathcal{S} and g in \mathcal{E} .

Suppose $P : \mathbb{X} \rightarrow \mathcal{S}$ is an xyz-fibration over a topos \mathcal{S} . Then the posetal fibration $Q = \text{sub}(P)$ over \mathbb{X} is a tripos and the functor F in

$$\begin{array}{ccc}
 \mathbb{X} & \xrightarrow{\Delta} & \mathbb{X}[Q] \\
 1 \uparrow & \nearrow \hat{F} & \\
 \mathcal{S} & &
 \end{array}$$

is an abc-functor. Notice that $FI = \coprod_I 1_I$.

Suppose $F : \mathcal{S} \rightarrow \mathcal{E}$ is an abc-functor. Then the full subfibration P_c of P_F generated by $c : C_0 \rightarrow FI_0$ is an xyz-fibration.

It can be shown that these two processes are inverse to each other.

166 Characterization of Realizability Toposes

(J. Frey, February 2012)

For a topos \mathcal{S} functors of the form $\Delta : \mathcal{S} \rightarrow \mathbf{RT}(\mathcal{A})$ for some pca \mathcal{A} in \mathcal{S} can be characterized as functors $\Delta : \mathcal{S} \rightarrow \mathcal{E}$ such that

- (1) \mathcal{E} is locally cartesian closed and exact
- (2) Δ preserves regular epis
- (3) Δ is full and faithful and has a finite limit preserving left adjoint Π s.t.
 - (3a) for every $X \in \mathcal{E}$ the map $\Pi_{1,X} : \mathcal{E}(1, X) \rightarrow \mathcal{S}(\Pi 1, \Pi X)$ is bijective
- (4) there is a map $\phi : M \rightarrow \Delta(A)$ such that
 - ϕ is indecomposable projective in P_Δ
 - M is modest w.r.t. Δ
 - the subfibration of P_Δ generated by ϕ is closed under finite meets
 - every $X \in \mathcal{E}$ fits into a diagram

$$\begin{array}{ccccc}
 X & \xleftarrow{e} & N & \longrightarrow & M \\
 & & \psi \downarrow & \lrcorner & \downarrow \phi \\
 & & \Delta B & \xrightarrow{Fu} & \Delta A
 \end{array}$$

In case \mathcal{S} is **Set** condition (2) is automatic and condition (3) can be reformulated as follows

- (3) Δ is full and faithful and right(sic!) adjoint to $\Gamma = \mathcal{E}(1, -)$

in which case Δ is determined by \mathcal{E} (which determines its left adjoint $\Gamma = \mathcal{E}(1, -)$). Thus, for base **Set** this gives rise to a characterization of categories \mathcal{E} (and not of functors $\Delta : \mathbf{Set} \rightarrow \mathcal{E}$) that are equivalent to a realizability topos.

167 Characterization of Realizability Toposes

(J. Frey, April 2014)

Let \mathcal{E} be a locally small topos and $\Gamma = \mathcal{E}(1, -) : \mathcal{E} \rightarrow \mathbf{Set}$ be the global sections functor from \mathcal{E} to \mathbf{Set} . If ∇ is a right adjoint of Γ then ∇ is necessarily full and faithful since $\text{Id}_{\mathbf{Set}}$ and $\Gamma\nabla$ are naturally isomorphic via $\Gamma\nabla(I) = \mathcal{E}(1_{\mathcal{E}}, \nabla(I)) \cong \mathbf{Set}(\Gamma(1_{\mathcal{E}}), I) \cong I$.

Suppose now that $\Gamma \dashv \nabla : \mathbf{Set} \hookrightarrow \mathcal{E}$. An object X of \mathcal{E} is called *separated* iff $\eta_X : X \rightarrow \nabla\Gamma X$ is a monomorphism. A morphism $f : Y \rightarrow X$ in \mathcal{E} is called a *family of sets* iff

$$\begin{array}{ccc}
 Y & \xrightarrow{f} & X \\
 \eta_Y \downarrow & \lrcorner & \downarrow \eta_X \\
 \nabla\Gamma Y & \xrightarrow{\nabla\Gamma f} & \nabla\Gamma X
 \end{array}$$

is a pullback. A *family of non-empty sets* is a (regular) epimorphisms e which, moreover, is a family of sets. An object X of \mathcal{E} is called *discrete* iff $\mathcal{E}(e, X)$ is a bijection for every family e of non-empty sets.

In a paper from April 2014 Jonas Frey has shown that up to equivalence a realizability topos is a locally small topos \mathcal{E} such that

- (1) \mathcal{E} is the exact completion of its projective objects¹¹³
- (2) the global sections functor $\Gamma : \mathcal{E} \rightarrow \mathbf{Set}$ has a right adjoint ∇ factoring through the inclusion of the full subcategory of projective objects in \mathcal{E}
- (3) there is a separated discrete projective object A in \mathcal{E} such that for every projective object P in \mathcal{E} there is a family of sets $P \rightarrow A$.

One may weaken the assumption that \mathcal{E} is a topos to the assumption that \mathcal{E} is exact (regular) and locally cartesian closed. The separated discrete objects are the modest sets. The category $\mathcal{P}(\mathcal{E})$ of modest projectives can be thought of as the category of subsets of the underlying pca and maps between them which are realized by elements of this pca. From $\mathcal{P}(\mathcal{E})$ one can construct an underlying pca which, however, is not uniquely determined.¹¹⁴

One obtains a characterization of categories of assemblies if one replaces condition (1) by the requirement the category is the *regular* completion of its projective objects.

¹¹³i.e. the projective objects of \mathcal{E} are closed under finite limits in \mathcal{E} and every object of \mathcal{E} is covered by a projective object

¹¹⁴Even in case of Hyland's effective topos $\mathcal{E}ff$ the underlying pca is not uniquely determined since there are different admissible Gödel numberings of the partial recursive functions.

Characterization of Categories of Partitioned Assemblies

(J. Frey, March 2017)

In his revised version of his paper on characterization of realizability toposes J. Frey has come up with the following characterization of categories of partitioned assemblies, namely as locally small categories \mathcal{C} with finite limits such that

- (1) the global elements functor $\Gamma = \mathcal{C}(1, -) : \mathcal{C} \rightarrow \mathbf{Set}$ is faithful and has a right adjoint ∇
- (2) there is a discrete object A in \mathcal{C} such that for every X in \mathcal{C} there is a family of sets $X \rightarrow A$
- (3) \mathcal{C} is weakly locally cartesian closed

where “family of sets” is defined as above but “family of non-empty sets” is defined as family of sets which, moreover, is split epic (equivalently mapped to an epi in \mathbf{Set} by Γ).¹¹⁵ Discreteness of objects is defined as above w.r.t. this new definition (of family of non-empty sets). All objects are separated since by assumption Γ is faithful, i.e. all η_X are monic.

168 A characterization of epis in elementary toposes

Let \mathcal{E} be an elementary topos. Then $f : Y \rightarrow X$ is epic iff for every subobject $m : P \rightarrow X$ from f^*m iso it follows that m is an iso.

Proof. Suppose f is epic and f^*m is an iso. Let $\chi : X \rightarrow \Omega$ be the characteristic map of m . Then χf is the characteristic map of the iso f^*m and thus $\chi f = \top_Y = \top_X f$. Since f is epic it follows that $\chi = \top_X$ and thus m is an iso.

Suppose m is an iso whenever f^*m is an iso. Let $f = me$ with m monic and e epic. Since in

$$\begin{array}{ccccc}
 Y & \xrightarrow{e} & I & \xlongequal{\quad} & I \\
 \parallel & & \parallel & & \downarrow m \\
 Y & \xrightarrow{e} & I & \xrightarrow{m} & X
 \end{array}$$

both squares are pullbacks it follows that f^*m is an iso. By assumption on f the mono m is an isomorphism and thus f is epic. \square

¹¹⁵This ammendment is necessary since categories of partitioned assemblies are typically not regular categories. However, even in the above case this condition works!

169 Triposes which are Prestacks

Let \mathcal{S} be a topos and \mathcal{P} a tripos over \mathcal{S} . From A. Pitts Thesis *Tripes Theory* we know (see also section 168) that $\Delta : \mathcal{S} \rightarrow \mathcal{S}[\mathcal{P}]$ is regular, i.e. preserves regular epis, if and only if for every regular epi $e : J \rightarrow I$ in \mathcal{S} from $\top_J \vdash_J e^* \phi$ it follows that $\top_I \vdash_I \phi$, i.e. e^* reflects truth. Since \mathcal{P} is a fibration of cartesian closed categories (actually preorders) this is equivalent to the requirement that reindexing along (regular) epis reflects entailment, i.e. for every (regular) epi $e : J \rightarrow I$ from $e^* \phi \vdash_J e^* \psi$ it follows that $\phi \vdash_I \psi$. In more abstract terms this means that \mathcal{P} is a *prestack* w.r.t. the *regular cover topology* on \mathcal{S} .¹¹⁶

If \mathcal{S} validates AC, i.e. all epis split in \mathcal{S} , e.g. in case \mathcal{S} is **Set**, then every tripos over \mathcal{S} is a stack w.r.t. the regular cover topology. First of all it is a prestack w.r.t. this topology since if $e : J \rightarrow I$ is a regular epi and $\top_J \vdash e^* \phi$ then by AC there is a section s of e and we have $\top_I \cong s^* \top_J \vdash_I s^* e^* \phi \cong \phi$. For showing that \mathcal{P} is also a stack suppose $e : J \rightarrow I$ is a regular epi and $\psi \in \mathcal{P}^J$ with $k_1^* \psi \cong k_2^* \psi$ where $k_1, k_2 : K \rightarrow J$ is a kernel pair of e . We have to show that $\psi \cong e^* \phi$ for some $\phi \in \mathcal{P}^I$. Since \mathcal{S} validates AC there is map $s : I \rightarrow J$ with $es = \text{id}_I$. We put $\phi = s^* \psi$ and show that $\psi \cong e^* \phi$. Let $u : J \rightarrow K$ be the unique map with $k_1 u = se$ and $k_2 u = \text{id}_J$. Then $e^* \phi \cong e^* s^* \psi \cong u^* k_1^* \psi \cong u^* k_2^* \psi \cong \text{id}_J^* \psi \cong \psi$ as desired.

J. Frey has observed that for triposes \mathcal{P} over \mathcal{S} that are prestacks w.r.t. the regular cover topology the entailment relation is fully determined by the collection \mathcal{R} of all subobjects $r = \langle r_1, r_2 \rangle$ of $\Sigma \times \Sigma$ such that $r_1^* \top \vdash r_2^* \top$ (where \top over Σ is the generic family for \mathcal{P}).¹¹⁷ Obviously, the collection \mathcal{R} contains a greatest element \leq_Σ (thought of as the entailment relation on Σ) iff the tripos \mathcal{P} is a locally small fibration (of preorders). Thus \mathcal{P} is locally small iff it arises as the externalisation of the cHa (Σ, \leq_Σ) internal to \mathcal{S} . This observation makes precise in which sense triposes over \mathcal{S} are a generalisation of localic toposes over \mathcal{S} , one simply drops the assumption of local smallness (see section 162).

The observation that triposes which are prestacks (w.r.t. the regular cover topology on the base) are determined by (Σ, \mathcal{R}) has led J. Frey to his notion of *basic relational object*. For a regular category \mathcal{E} fibered posets over \mathcal{E} which are prestacks w.r.t. the regular cover topology on \mathcal{E} and admit a generic family \top over Σ are uniquely determined by a *basic relational object* (BRO) (Σ, \mathcal{R}) , i.e. Σ is an object of \mathcal{E} and \mathcal{R} is a downclosed subset of $\text{Sub}_{\mathcal{E}}(\Sigma \times \Sigma)$ which contains the diagonal δ_Σ and is closed under composition of relations.

¹¹⁶This explains why it is natural to assume that $\Delta : \mathcal{S} \rightarrow \mathcal{S}[\mathcal{P}]$ is regular as required for Pitts' Iteration Theorem.

¹¹⁷Suppose $\phi_1, \phi_2 : I \rightarrow \Sigma$. Let $\langle \phi_1, \phi_2 \rangle = r \circ e$ with r monic and e a (regular) epi. Then $\phi_1 \vdash_I \phi_2$ iff $e^* r_1 \vdash_I e^* r_2$ iff $r_1 \vdash_R r_2$ (where R is the source of r) iff $r \in \mathcal{R}$.

170 Iterated realizability as a comma construction (after P.J.W.Hofstra)

A *basic combinatorial object* (bco) is a triple $\Sigma = (\Sigma, \leq, \mathcal{F}_\Sigma)$ where \leq is a partial order on Σ and \mathcal{F}_Σ is a set of monotone partial endofunctions on Σ with downward closed domain of definition satisfying the requirements

- $\exists i \in \mathcal{F}_\Sigma \forall x \in \Sigma i(x) \preceq x$
- $\forall f, g \in \mathcal{F}_\Sigma \exists h \in \mathcal{F}_\Sigma \forall x \in \Sigma h(x) \preceq g(f(x))$

where $t \preceq s$ stands for t defined and $t \leq s$ whenever s is defined.

A *morphism of bco's* $\phi : \Sigma \rightarrow \Theta$ is a function between the underlying posets such that

- $\exists t \in \mathcal{F}_\Theta \forall a \leq a' t(\phi(a)) \preceq \phi(a')$
- $\forall f \in \mathcal{F}_\Sigma \exists g \in \mathcal{F}_\Theta \forall a \in \Sigma g(\phi(a)) \preceq \phi(f(a))$.

we write **BCO** for the ensuing category.

If $\phi, \psi : \Sigma \rightarrow \Theta$ then $\phi \vdash \psi$ iff there exists $g \in \mathcal{F}_\Theta$ such that $g(\phi(x)) \preceq \psi(x)$ for all $x \in \Sigma$. This makes the category **BCO** preorder enriched.

A bco Σ has binary meets if the diagonal map $\Sigma \rightarrow \Sigma \times \Sigma$ has a right adjoint (w.r.t. the enrichment \vdash) and Σ has top element if $\Sigma \rightarrow 1$ has a right adjoint (w.r.t. the enrichment \vdash). A bco is cartesian iff it has binary meets and a top element (NB this does not mean that Σ has finite infima w.r.t. \leq).

Let Σ and Θ be cartesian bco's. Then a morphism $\phi : \Sigma \rightarrow \Theta$ is cartesian iff there exists $g \in \mathcal{F}_\Theta$ with $g(\phi(a) \wedge \phi(b)) \preceq \phi(a \wedge b)$. This defines a sub-2-category **CartBCO** on cartesian objects and morphisms and all 2-cells.

If $\phi : \Sigma \rightarrow \Theta$ is a morphism in **CartBCO** then we can define a new cartesian BCO $\Sigma \times_\phi \Theta$ whose underlying poset is the product of the underlying posets of Σ and Θ , resp., and a partial endomap h is in $\mathcal{F}_{\Sigma \times_\phi \Theta}$ iff there exists $f \in \mathcal{F}_\Sigma$ and $g \in \mathcal{F}_\Theta$ such that

$$(f(a), g(\phi(a) \wedge b)) \preceq h(a, b)$$

for all $a \in \Sigma$ and $b \in \Theta$. We write π_Σ and π_Θ for first and second projection, respectively. One can show that

$$\begin{array}{ccc} \Sigma \times_\phi \Theta & \xrightarrow{\pi_\Sigma} & \Sigma \\ \pi_\Theta \downarrow & \vdash & \downarrow \phi \\ \Theta & \xlongequal{\quad} & \Theta \end{array}$$

is a *lax comma object* in **CartBCO** (and in **BCO**).

One can define the above comma construction also for morphisms of ordered partial combinatory algebras (pca's) as introduced by Hofstra and van Oosten.

Recall that an opca is a triple (Σ, \leq, \bullet) where \leq is a partial order on Σ and $\bullet : \Sigma \times \Sigma \rightarrow \Sigma$ satisfying the following requirements¹¹⁸

1. if $ab \downarrow$ and $a' \leq a, b' \leq b$ then $a'b' \downarrow$ and $a'b' \leq ab$
2. there are elements k and s of A such that
 - $kab \downarrow$ and $kab \leq a$
 - $sab \downarrow$ and if $ac(bc) \downarrow$ then $sabc \downarrow$ and $sabc \leq ac(bc)$.

A map $\phi : \Sigma \rightarrow \Theta$ is a morphism of opca's iff there exist $u, v \in \Theta$ such that for all $a, b \in \Sigma$

1. $u\phi(a) \preceq \phi(b)$ whenever $a \leq b$
2. $v\phi(a)\phi(b) \preceq \phi(ab)$ whenever $ab \downarrow$.

Obviously, every opca (Σ, \leq, \bullet) can be considered as a bco when putting $\mathcal{F}_\Sigma = \{a \bullet (-) \mid a \in \Sigma\}$. An opca is called cartesian iff its associated bco is cartesian. If Σ and Θ are cartesian opca's then a morphism $\phi : \Sigma \rightarrow \Theta$ of opca's is called cartesian iff it is cartesian as a morphism between cartesian bco's, i.e. if there is a $d \in \Theta$ with $d \bullet (\phi(a) \wedge \phi(b)) \preceq \phi(a \wedge b)$ for all $a, b \in \Sigma$.

For a cartesian morphism $\phi : \Sigma \rightarrow \Theta$ between cartesian opca's the opca $\Sigma \times_\phi \Theta$ has underlying set $\Sigma \times \Theta$ ordered componentwise and its application is defined as

$$(a, b) \bullet (a', b') \simeq (a \bullet a', b \bullet (\phi(a') \wedge b'))$$

for $a, a' \in \Sigma$ and $b, b' \in \Theta$.

¹¹⁸we write ab as an abbreviation for $a \bullet b$

171 Unifying pca's and cHa's (à la P. Hofstra)

Definition 171.1 (Basic Combinatorial Objects)

A basic combinatorial object (BCO for short) is a triple $(\Sigma, \leq, \mathcal{F}_\Sigma)$ iff (Σ, \leq) is a poset and \mathcal{F}_Σ is a class of partial endofunctions on Σ such that

1. for $f \in \mathcal{F}_\Sigma$ and $a \in \text{dom}(f)$ from $b \leq a$ it follows that $b \in \text{dom}(f)$ and $f(b) \leq f(a)$
2. there is a “weak identity” $i \in \mathcal{F}_\Sigma$ with $\text{dom}(i) = \Sigma$ and $i(a) \leq a$ for all $a \in \Sigma$
3. we have “weak composition” in the sense that for $f, g \in \mathcal{F}_\Sigma$ there exists $h \in \mathcal{F}_\Sigma$ such that from $f(a) \in \text{dom}(g)$ it follows that $a \in \text{dom}(h)$ and $h(a) \leq g(f(a))$.

Given a BCO $\Sigma = (\Sigma, \leq, \mathcal{F}_\Sigma)$ we can construct another BCO $\mathcal{D}\Sigma$ as follows: the underlying set of $\mathcal{D}\Sigma$ is $\{A \subseteq \Sigma \mid A \text{ downward closed}\}$ and ordered by \subseteq and $\mathcal{F}_{\mathcal{D}\Sigma}$ is the set of all $F : \mathcal{D}\Sigma \rightarrow \mathcal{D}\Sigma$ whose domain of definition is downward closed and on which F preserves \subseteq and which, moreover, is *uniformly realizable* in the sense that there exists an $f \in \mathcal{F}_\Sigma$ such that for all $A \in \text{dom}(F)$ it holds that $\forall a \in A. f(a) \in F(A)$. It can be shown that \mathcal{D} gives rise to a monad on the category of BCOs.

Definition 171.2 Every BCO $(\Sigma, \leq, \mathcal{F}_\Sigma)$ induces a **Set**-indexed preorder $[-, \Sigma]$ as follows: for $I \in \mathbf{Set}$ let $[I, \Sigma] = \Sigma^I$, the set of all functions from I to Σ , preordered by

$$\phi \vdash_I \psi \quad \text{iff} \quad \exists f \in \mathcal{F}_\Sigma. \forall i \in I. f(\phi(i)) \leq \psi(i)$$

On morphisms $[-, \Sigma]$ acts by precomposition.

Typical examples of BCOs arise from ordered pca's.

Definition 171.3 An ordered pca (OPCA for short) is a triple (Σ, \leq, \bullet) where \leq is a partial order on Σ and $\bullet : \Sigma \times \Sigma \rightarrow \Sigma$ satisfying the following conditions¹¹⁹

1. if $ab \downarrow$ and $a' \leq a, b' \leq b$ then $a'b' \downarrow$ and $a'b' \leq ab$
2. there are elements k and s of A such that
 - $kab \downarrow$ and $kab \leq a$
 - $sab \downarrow$ and if $ac(bc) \downarrow$ then $sabc \downarrow$ and $sabc \leq ac(bc)$.

An OPCA (Σ, \leq, \bullet) gives rise to the BCO $(A, \leq, \mathcal{F}_\Sigma)$ where \mathcal{F}_Σ consists of all $f : A \rightarrow A$ such that there exists $a \in A$ with $f(b) \simeq a \bullet b$ for all $b \in A$.

Notice that every pca (Σ, \bullet) gives rise to the (discrete) OPCA $(\Sigma, =, \bullet)$ (also denoted by Σ). If (Σ, \leq, \bullet) is an OPCA then one may construct a new OPCA

¹¹⁹we write ab as shorthand for $a \bullet b$

$\mathcal{D}\Sigma$ whose underlying set consists of the non-empty downward closed subsets of Σ , which is ordered by \subseteq and on which \bullet is defined as follows: $\alpha \bullet \beta \downarrow$ iff $a \bullet b \downarrow$ for all $a \in \alpha$ and $b \in \beta$ and if $\alpha \bullet \beta \downarrow$ then $\alpha \bullet \beta = \{a \bullet b \mid a \in \alpha, b \in \beta\}$. If Σ is a pca then the indexed poset associate with $\mathcal{D}\Sigma$ coincides with the realizability tripos induced by the pca Σ .

Another class of examples of BCOs is given by complete Heyting algebras (cHa's) : if (Σ, \leq) is a cHa then $(\Sigma, \leq, \{\text{id}_\Sigma\})$ is a BCO.

These two classes of examples can be unified by considering OPCAs with filters.

Definition 171.4 *Let (Σ, \leq, \bullet) be an OPCA. A filter on this OPCA is a subset Φ of Σ such that Φ is closed under \bullet and contains (some choice of) k and s .*

Any such pair (Σ, Φ) where Σ is an OPCA and Φ is a filter on Σ gives rise to the BCO $(\Sigma, \leq, \mathcal{F}_\Sigma)$ where $\mathcal{F}_\Sigma = \{a \bullet (-) \mid a \in \Phi\}$.

Obviously, it does not change the induced **Set**-indexed poset $[-, \Sigma]$ if we assume Φ to be upward closed which justifies the name “filter”.

If Σ is an OPCA then $\Phi = \Sigma$ is a filter and the associate OPCA coincides with the one associated with Σ . If (Σ, \leq) is a cHa then (Σ, \leq, \bullet) with $a \bullet b = a \wedge b$ is an OPCA and $\Phi = \{\top\}$ is a filter on (Σ, \leq, \bullet) . Then the BCO associated with $((\Sigma, \leq, \bullet), \{\top\})$ is $(\Sigma, \leq, \{\text{id}_\Sigma\})$.

Thus OPCAs with filters generalize both cHa's and order pca's.

172 Strong AKS's as total OPCA's

Suppose we are given an abstract Krivine structure (see section 139) where $\perp \subseteq L \times P$ satisfies the following strengthening of (S1)

$$(SS1) \quad ts * \pi \in \perp \quad \text{iff} \quad t * s.\pi \in \perp$$

Such structures we call *strong abstract Krivine structures* (SAKS's).

With every SAKS we may associate the total OPCA (see section 171) whose underlying set is $\mathcal{P}_{\perp}(L) = \{x^{\perp\perp} \mid x \in \mathcal{P}(L)\}$ ordered by \subseteq and with application defined as

$$a \bullet b = (a \cdot b)^{\perp\perp}$$

where $a \cdot b = \{ts \mid t \in a, s \in b\}$.

Obviously, the operation \bullet is monotonic since it preserves \subseteq . The required elements k and s are given by $\{K\}^{\perp\perp}$ and $\{S\}^{\perp\perp}$, respectively. For seeing that the so defined s and k satisfy the requirements of a total OPCA we need the following considerations.

For $x, u \in \mathcal{P}(L)$ let $x \cdot y = \{ts \mid t \in x, s \in y\}$.

Lemma 172.1 *For $x, y \in \mathcal{P}(L)$ we have $(x^{\perp\perp} \cdot y)^{\perp\perp} = (x \cdot y)^{\perp\perp}$.*

Proof: It suffices to show that $(x^{\perp\perp} \cdot y)^{\perp} = (x \cdot y)^{\perp}$. That $(x^{\perp\perp} \cdot y)^{\perp} \subseteq (x \cdot y)^{\perp}$ is immediate from $x \cdot y \subseteq x^{\perp\perp} \cdot y$.

For the reverse inclusion suppose $\pi \in (x \cdot y)^{\perp}$, i.e. $\forall t \in x, s \in y \quad ts * \pi \in \perp$. Thus, by (SS1) we have $\forall s \in y, t \in x \quad t * s.\pi \in \perp$, i.e. $\forall s \in y \quad x \subseteq \{s.\pi \mid s \in y\}^{\perp}$, from which it follows $\forall s \in y \quad x^{\perp\perp} \subseteq \{s.\pi \mid s \in y\}^{\perp}$. Thus, we have $\forall t \in x^{\perp\perp}, s \in y \quad t * s.\pi \in \perp$ from which it follows by (S1) since \perp is saturated that $\forall t \in x^{\perp\perp}, s \in y \quad ts * \pi \in \perp$, i.e. $\pi \in (x^{\perp\perp} \cdot y)^{\perp}$ as desired. \square

The following corollary allows one to simplify reasoning about \bullet .

Corollary 172.1 *For $n \geq 1$ we have*

$$a_1 \bullet \dots \bullet a_n = (a_1 \cdot \dots \cdot a_n)^{\perp\perp}$$

for all $a_1, \dots, a_n \in \mathcal{P}_{\perp}(L)$.

Proof: For $n = 1$ the claim is obvious since $a_1 = a_1^{\perp\perp}$.

Suppose as induction hypothesis that $a_1 \bullet \dots \bullet a_n = (a_1 \cdot \dots \cdot a_n)^{\perp\perp}$. Then for $a_{n+1} \in \mathcal{P}_{\perp}(L)$ we have

$$\begin{aligned} a_1 \bullet \dots \bullet a_n \bullet a_{n+1} &= ((a_1 \bullet \dots \bullet a_n)^{\perp\perp} \cdot a_{n+1})^{\perp\perp} = && \text{(by i.h.)} \\ &= ((a_1 \cdot \dots \cdot a_n)^{\perp\perp} \cdot a_{n+1})^{\perp\perp} && \text{(by Lemma 172.1)} \\ &= (a_1 \cdot \dots \cdot a_n \cdot a_{n+1})^{\perp\perp} \end{aligned}$$

as desired. \square

Suppose $a, b, c \in \mathcal{P}_{\perp}(L)$. Then we have

$$(1') \quad a^\perp \subseteq (\{\mathbf{K}\} \cdot a \cdot b)^\perp$$

$$(2') \quad (a \cdot c \cdot (b \cdot c))^\perp \subseteq (\{\mathbf{S}\} \cdot a \cdot b \cdot c)^\perp$$

as follows from (S2) and (S3), respectively, using (S1) and the assumption that \perp is saturated. But then using (1') and (2') we get

$$(1) \quad k \bullet a \bullet b = (\{\mathbf{K}\} \cdot a \cdot b)^{\perp\perp} \subseteq a^{\perp\perp} = a$$

$$(2) \quad s \bullet a \bullet b \bullet c = (\{\mathbf{S}\} \cdot a \cdot b \cdot c)^{\perp\perp} \subseteq$$

$$\subseteq (a \cdot c \cdot (b \cdot c))^{\perp\perp} = a \bullet c \bullet (b \bullet c)$$

as desired.

This concludes our argument that $(\mathcal{P}_\perp, \subseteq, \bullet)$ is actually a total OPCA.

Since non-empty subsets are closed under \cdot and $a \cdot b \subseteq (a \cdot b)^{\perp\perp} = a \bullet b$ the non-empty elements of $\mathcal{P}_\perp(L)$ are close under \bullet . Since $k = \{\mathbf{K}\}^{\perp\perp}$ and $s = \{\mathbf{S}\}^{\perp\perp}$ are non-empty elements of $\mathcal{P}_\perp(L)$ it follows that the non-empty elements of $\mathcal{P}_\perp(L)$ form a nontrivial filter Φ in the total OPCA $\mathcal{P}_\perp(L)$.

173 AKS's as total OPCA's

There are interesting $\perp \subseteq \Lambda \times \Pi$ which are saturated but not closed under reduction. For example Krivine considers a system where for every quasi-proof¹²⁰ t there is a stack π_t but no reduction rule for $t * \pi_t$. Let \perp be the set of all processes which do not expand to a process of the form $t * \pi_t$. Of course, this set is closed under expansion but not under reduction. Thus for every quasi-proof t we have $t \notin |\perp|$ since $t * \pi_t \notin \perp$. Thus, no quasi-proof realizes the false proposition \perp .

Thus saturated sets \perp which are not reduction closed are important, i.e. we cannot assume in general that (SS1) holds. We investigate now whether one can associate an opca also with general non-strong aks's. For this purpose we have to replace condition (S3) by the requirement that

$$S * a.b.c.\pi \in \perp^c \text{ implies } ac(bc) * \pi \in \perp^c$$

which entails the old (S3) by (S1).

The underlying set is again $\mathcal{P}_{\perp}(L) = \{x^{\perp\perp} \mid x \in \mathcal{P}(L)\}$. But for $x, y \in \mathcal{P}(L)$ we now define

$$x \bullet y = \{\pi \in P \mid \forall a \in x \forall b \in y \ a * b.\pi \in \perp\}^{\perp}$$

which is necessarily in $\mathcal{P}_{\perp}(L)$. We order the set $\mathcal{P}_{\perp}(L)$ by \subseteq which is obviously preserved by \bullet . Again we define k and s as $\{K\}^{\perp\perp}$ and $\{S\}^{\perp\perp}$, respectively. We have to show that

- (1) $k \bullet x \bullet y \subseteq x$ and
- (2) $s \bullet x \bullet y \bullet z \subseteq x \bullet z \bullet (y \bullet z)$

Notice that for showing $x \bullet y \subseteq z$ it suffices to show that if $c * \pi \in \perp$ for all $c \in z$ then $a * b.\pi \in \perp$ for all $a \in x$ and $b \in y$.

Moreover, we have $x^{\perp\perp} \bullet y = x \bullet y$ which can be seen as follows. Suppose $a * b.\pi \in \perp$ for all $a \in x$ and $b \in y$. Then also $a * b.\pi \in \perp$ for all $a \in x^{\perp\perp}$ and $b \in y$. But this condition entails that in particular also $a * b.\pi \in \perp$ for all $a \in x$ and $b \in y$.

ad (1) : Suppose $a * \pi \in \perp$ for all $a \in x$. We will show that $c * b.\pi \in \perp$ for all $c \in k \bullet x$ and $b \in y$. Since $c \in k \bullet x$ we have $c * \pi' \in \perp$ for all π' with $K * a.\pi' \in \perp$ for all $a \in x$. Thus, since $K * a.b.\pi \succ a * \pi \in \perp$ for all $a \in x$ we have $c * b.\pi \in \perp$ as desired.

ad (2) : Suppose for all $u \in x \bullet z$ and $v \in y \bullet z$ it holds that $u * v.\pi \in \perp$. We will show that for all $u \in s \bullet x \bullet y$ and $c \in z$ it holds that $u * c.\pi \in \perp$.

Suppose $c \in z$. For showing that $u * c.\pi \in \perp$ holds for all $u \in s \bullet x \bullet y$ it suffices to show that $v * b.c.\pi \in \perp$ for all $v \in s \bullet x$ and $b \in y$.

Suppose $b \in y$. For showing that $v * b.c.\pi \in \perp$ for all $v \in s \bullet x$ it suffices to show that $S * a.b.c \in \perp$ for all $a \in x$.

Suppose $a \in x$. Since $S * a.b.c.\pi \succ ac * bc.\pi$ it suffices to show that $ac * bc.\pi \in \perp$. But this follows from the assumption on π since $ac \in x \bullet z$ and $bc \in y \bullet z$.

¹²⁰i.e. a term without subterms of the form k_π

A more elegant proof of (1) and (2) suggested by Benno van den Berg

We define propositions as subsets of P . If x is a proposition then $|x| = \{t \in L \mid \forall \pi \in x \ t * \pi \in \perp\}$. We write $x \leq y$ for $|x| \subseteq |y|$. As usual we have $x \rightarrow y = \{t.\pi \mid t \in |x|, \pi \in y\}$. We (re)define application as

$$x \bullet y = \{\pi \in P \mid \forall t \in |a| \forall |s| \in |b| \ t * s.\pi \in \perp\}$$

Before showing (1) and (2) we prove two auxiliary lemmas.

Lemma 1

From $x \leq y \rightarrow z$ it follows that $x \bullet y \leq z$.

Proof. Suppose $x \leq y \rightarrow z$, i.e. $\forall t \in |x| \forall s \in |y| \forall \pi \in z \ t * s.\pi \in \perp$, i.e. $z \subseteq x \bullet y$, from which it follows that $x \bullet y \leq z$ since $(-)^{\perp}$ is antitonic. \square

Lemma 2

If $t \in |x|$ and $s \in |y|$ then $ts \in |x \bullet y|$.

Proof. Suppose $t \in |x|$ and $s \in |y|$. Let $\pi \in x \bullet y$. Then $t * s.\pi \in \perp$. Thus $ts * \pi \in \perp$. \square

Now we can show that (1) and (2) hold for $k = \{K\}^{\perp}$ and $s = \{S\}^{\perp}$.

ad (1) : For showing that $k \bullet x \bullet y \leq x$ it suffices by Lemma 1 (applied twice) to show that $k \leq x \rightarrow y \rightarrow x$. But, obviously, we have $K \in |x \rightarrow y \rightarrow x|$ and thus $k = \{K\}^{\perp\perp} \subseteq |x \rightarrow y \rightarrow x|$.

ad (2) : For showing that $s \bullet x \bullet y \bullet z \leq x \bullet z \bullet (y \bullet z)$ it suffices by (multiple applications of) Lemma 1 to show that $s \subseteq x \rightarrow y \rightarrow z \rightarrow (x \bullet z \bullet (y \bullet z))$. Thus it suffices to show that $S \in |x \rightarrow y \rightarrow z \rightarrow (x \bullet z \bullet (y \bullet z))|$. For this purpose suppose $t \in |x|$, $s \in |y|$, $u \in |z|$ and $\pi \in x \bullet z \bullet (y \bullet z)$. Applying Lemma 2 iteratively we have $tu(su) \in |x \bullet z \bullet (y \bullet z)|$ and thus $tu(su) * \pi \in \perp$. Since \perp is closed under expansion it follows that $S * t.s.u.\pi \in \perp$ as desired.

174 Hoare triples for continuation semantics

(after L. Birkedal and H. Yang)

Let S be a type/domain of states and Σ^S the type of continuations. For $P, Q \subseteq S$ and $f : S \rightarrow S$ we write $\{P\}f\{Q\}$ as abbreviation for $\forall s \in p \ f(s) \in q$. Define $P^\perp = \{U \in \Sigma^S \mid P \subseteq U\}$. Let $\widehat{Q} = \{s \in S \mid \eta(s) \in Q^{\perp\perp}\}$. Then we have

$$\{P\}f\{\widehat{Q}\} \iff \{Q^\perp\}\Sigma^f\{P^\perp\}$$

which can be seen as follows

$$\begin{aligned} & \{Q^\perp\}\Sigma^f\{P^\perp\} \\ \text{iff } & \forall U (U \in Q^\perp \rightarrow \Sigma^f(U) \in P^\perp) \\ \text{iff } & \forall U \forall x \in P (U \in Q^\perp \rightarrow x \in \Sigma^f(U)) \\ \text{iff } & \forall x \in P \forall U (U \in Q^\perp \rightarrow f(x) \in U) \\ \text{iff } & \forall x \in P \forall U (U \in Q^\perp \rightarrow U \in \eta(f(x))) \\ \text{iff } & \forall x \in P \ Q^\perp \subseteq \eta(f(x)) \\ \text{iff } & \forall x \in P \ \eta(f(x)) \in Q^{\perp\perp} \\ \text{iff } & \forall x \in P \ f(x) \in \widehat{Q} \\ \text{iff } & \{p\}f\{\widehat{Q}\} \end{aligned}$$

where U ranges over Σ^S .

Notice that for discrete S all $\eta(s)$ are maximal in $\Sigma^2(S)$ and thus in this case we have $Q = \widehat{Q}$.

Partial correctness means that as pre- and postconditions we allow just (Scott) closed subsets of S . To obtain an equivalence $\{A\}f\{\widehat{B}\} \iff \{B^\perp\}\Sigma^f\{A^\perp\}$ we have to redefine $(-)^{\perp}$ as $A^\perp = \{U \in \Sigma^S \mid U \cap A = \emptyset\}$. (Notice that A^\perp is Scott closed for arbitrary $A \subseteq S$.) Then for Scott closed subsets A and B of S we have (where U ranges over Σ^S)

$$\begin{aligned} & \{B^\perp\}\Sigma^f\{A^\perp\} \\ \text{iff } & \forall U (U \in B^\perp \rightarrow \Sigma^f(U) \in A^\perp) \\ \text{iff } & \forall U \forall x \in A (U \in B^\perp \rightarrow x \notin \Sigma^f(U)) \\ \text{iff } & \forall x \in A \forall U (U \in B^\perp \rightarrow f(x) \notin U) \\ \text{iff } & \forall x \in A \forall U (U \in B^\perp \rightarrow U \not\subseteq \eta(f(x))) \\ \text{iff } & \forall x \in A \ B^\perp \cap \eta(f(x)) = \emptyset \\ \text{iff } & \forall x \in A \ \eta(f(x)) \in B^{\perp\perp} \\ \text{iff } & \forall x \in A \ f(x) \in \widehat{B} \\ \text{iff } & \{A\}f\{\widehat{B}\} \end{aligned}$$

175 Hereditarily majorizable functionals do not arise from realizability models over pca's \mathcal{A} with $|\mathcal{A}| < \beth_\omega$

The hereditarily majorizable functionals form an extensional type structure over \mathbb{N} which is defined as follows. For base type 0 we put $\mathcal{M}_0 = \mathbb{N}$ and $n \text{ maj}_0 m \equiv n \geq m$. For $F^*, F : \mathcal{M}_\sigma \rightarrow \mathcal{M}_\tau$ we define $F^* \text{ maj}_{\sigma \rightarrow \tau} F$ as

$$\forall G^*, G \in \mathcal{M}_\sigma (G^* \text{ maj}_\sigma G \rightarrow F^* G^* \text{ maj}_\tau F^* G \wedge F^* G^* \text{ maj}_\tau FG)$$

and $\mathcal{M}_{\sigma \rightarrow \tau} = \{F : \mathcal{M}_\sigma \rightarrow \mathcal{M}_\tau \mid \exists F^* \mathcal{M}_\sigma \rightarrow \mathcal{M}_\tau. F^* \text{ maj } F\}$.

Obviously, for all types σ all maps $\mathcal{M}_\sigma \rightarrow \{0, 1\}$ are majorizable (by the constant 1 functional) and thus $|\mathcal{M}_{\sigma \rightarrow 0}| \geq 2^{|\mathcal{M}_\sigma|}$. Thus for pure types n we have $|\mathcal{M}_n| \geq \beth_n$. Now if \mathcal{M} were isomorphic to the finite type hierarchy over \mathbb{N} in $\mathbf{Mod}(\mathcal{A})$ for some pca \mathcal{A} then for all n we have $|\mathcal{A}| \geq |\mathcal{M}_n| \geq \beth_n$ from which it follows that $|\mathcal{A}| \geq \beth_\omega$.

This observation renders it unlikely that \mathcal{M} appears as finite type hierarchy over \mathbb{N} in some (realizability) model of system T .

176 The Classifying Topos for Boolean Algebras and the Object Classifier

Let \mathbf{FinSet} be the category of finite cardinals and arbitrary functions between them and \mathbb{S} the full subcategory on nonempty finite cardinals.

The presheaf topos $\mathbf{Set}^{\mathbf{FinSet}^{\text{op}}}$ is the classifying topos for Boolean algebras since by Stone duality \mathbf{FinSet} is equivalent to the opposite of the category of finitely presentable boolean algebras which are of the form 2^n for n a finite cardinal. The generic boolean algebra is given by $\mathbf{Y}_{\mathbf{FinSet}}(2) = \mathbf{FinSet}(-, 2)$.

The presheaf topos $\mathbf{Set}^{\mathbb{S}^{\text{op}}}$ is the classifying topos for *nontrivial* boolean algebras since by Stone duality \mathbb{S} is equivalent to the category of finitely presentable nontrivial Boolean algebras which are of the form 2^n for n a finite cardinal different from 0. The generic nontrivial boolean algebra is given by $\mathbf{Y}_{\mathbb{S}}(2) = \mathbb{S}(-, 2)$

Notice that boolean algebras correspond to finite limit preserving functors from \mathbf{FinSet} to \mathbf{Set} . A functor from \mathbb{S} to \mathbf{Set} is flat iff it preserve finite products, i.e. it appears as the restriction to \mathbb{S} of a finite limit preserving functor from \mathbf{FinSet} to \mathbf{Set} which, moreover, preserves the empty set.

The presheaf topos $\mathbf{Set}^{\mathbf{FinSet}}$ is known as the **object classifier** since $\mathbf{FinSet}^{\text{op}}$ is the single sorted algebraic theory with no constants and no equations. Geometric morphisms from \mathcal{E} to $\mathbf{Set}^{\mathbf{FinSet}}$ correspond to finite limit preserving functors from $\mathbf{FinSet}^{\text{op}}$ to \mathcal{E} which up to isomorphism are of the form $A^{F(-)}$ where A is an object of \mathcal{E} and $F : \mathbf{FinSet} \rightarrow \mathcal{E}$ preserves finite sums and the terminal object. The generic model of the algebraic theory $\mathbf{FinSet}^{\text{op}}$ is given by $\mathbf{Y}_{\mathbf{FinSet}^{\text{op}}}(1) = \mathbf{FinSet}^{\text{op}}(-, 1) \cong \mathbf{FinSet}(1, -)$ isomorphic to the inclusion of \mathbf{FinSet} into \mathbf{Set} .

177 Gaeta topos (of Lawvere)

The classifying topos for rings is $\mathbf{Set}^{\mathcal{R}}$ where \mathcal{R} is the category of finitely presented rings, i.e. rings of the form $\mathbb{Z}[X_1, \dots, X_n]/(P_1, \dots, P_m)$ where $P_1, \dots, P_m \in \mathbb{Z}[X_1, \dots, X_n]$. The generic ring object in $\mathbf{Set}^{\mathcal{R}}$ is given by the forgetful functor $U : \mathcal{R} \rightarrow \mathbf{Set}$. The *Gaeta topos* is the subtopos \mathcal{G} of $\mathbf{Set}^{\mathcal{R}}$ consisting of the finite product preserving functors from \mathcal{R} to \mathbf{Set} .¹²¹ It can be shown that this topos is classifying for rings satisfying the geometric axiom $x^2 = x \vdash x = 0 \vee x = 1$ where again the forgetful functor U is the generic ring object.

Notice that \mathcal{R}^{op} is a category with finite stable sums. For any such category \mathcal{C} one may consider the category of finite product preserving functors from \mathcal{C}^{op} to \mathbf{Set} which is the topos of sheaves on \mathcal{C} w.r.t. the topology on \mathcal{C} given by finite disjoint covering families of subobjects. Lawvere calls this topos the *Gaeta topos* for \mathcal{C} .

Notice that the topology on \mathcal{R}^{op} generated by finite disjoint covering families of subobjects is contained in the *Zariski* topology on \mathcal{R}^{op} which consists of sieves containing a family $(A \rightarrow A[s_i^{-1}])_{i=1, \dots, n}$ where $\{s_1, \dots, s_n\}$ is not contained in a proper ideal of A , i.e. $\sum_{i=1}^n a_i s_i = 1$ for some $a_1, \dots, a_n \in A$. Sheaves on \mathcal{R}^{op} w.r.t. the Zariski topology are the classifying topos for *local rings*, i.e. rings satisfying the geometric axiom $(x \text{ invertible}) \vee (1-x \text{ invertible})$.

For background information on \mathcal{R} , the category of finitely presented rings, see the book M. Demazure & P. Gabriel *Groupes algébriques* I (1970).

For example one can show that \mathcal{R} is closed under finite limits taken in the category of commutative rings with 1. Let A and B be elements of \mathcal{R} thought of as $\mathcal{O}(X)$ and $\mathcal{O}(Y)$, respectively. Then $A \times B$ is thought of as $\mathcal{O}(X + Y)$ containing the characteristic functions Z_1 and Z_2 of X and Y , respectively. These validate the equations $Z_i^2 = Z_i$, $Z_1 Z_2 = 0$ and $Z_1 + Z_2 = 1$. Moreover, for $P \in A$ we have $Z_1 P = P$ and $Z_2 P = 0$ and similarly for $Q \in B$ (here we think of A as the observable on $X + Y$ which behaves on X as prescribed by X and is constantly 0 on Y). Thus, for finitely presented rings $A = \mathbb{Z}[X_1, \dots, X_n]/I$ and $B = \mathbb{Z}[Y_1, \dots, Y_m]/J$ their product $A \times B$ is isomorphic to $\mathbb{Z}[X_1, \dots, X_n, Y_1, \dots, Y_m]/K$ where K is obtained from $I \cup J$ by adding the equations¹²²

$$Z_i^2 = Z_i \quad Z_1 Z_2 = 0 \quad Z_1 + Z_2 = 1 \quad Z_1 X_i = X_i \quad Z_2 Y_j = Y_j \quad Z_1 Y_j = 0 = Z_2 X_i$$

having the effect that they equate every polynomial in $\mathbb{Z}[X_1, \dots, X_n, Y_1, \dots, Y_m]$ with a unique polynomial of the form $Z_1 P_1 + Z_2 P_2$ with $P_1 \in \mathbb{Z}[X_1, \dots, X_n]$ and $P_2 \in \mathbb{Z}[Y_1, \dots, Y_m]$.

Notice that the Z_i are examples of so-called *projections* which make sense for any ring: a projection in a (commutative) ring A (with 1) is an element $p \in A$ with $p^2 = p$. If p is a projection then $1 - p$ is a projection, too, because

¹²¹F. Gaeta took notes of Grothendieck's Buffalo lectures in 1973 on *A Functorial Approach to Algebraic Geometry*. There it was emphasized that \mathcal{R}^{op} is an extensive category.

¹²²If A and B are thought of as $\mathcal{O}(X)$ and $\mathcal{O}(Y)$, respectively, then $A \times B$ is thought of as $\mathcal{O}(X + Y)$ and the elements Z_1 and Z_2 are thought of as the characteristic functions of X and Y , respectively, which motivates the equations postulated for them!

$(1 - p)^2 = 1 - p - p + p^2 = 1 - p$. Moreover, we obviously have $p + (1 - p) = 1$ and $p(1 - p) = p - p^2 = p - p = 0$. Thinking of A as $\mathcal{O}(X)$ projections are thought of as those observables on X which are characteristic functions. If p is the characteristic function of $X_1 \subseteq X$ then $1 - p$ is the characteristic function of $X_2 = X - X_1$. Thus X being indecomposable can be expressed by the requirement that every projection $p \in \mathcal{O}(X)$ is either 0 or 1, i.e. every part of X is empty or X itself. The ring $A[p^{-1}]$ obtained by freely inverting the projection p , i.e. adding the equation $p = 1$ or equivalently $1 - p = 0$, corresponds to the ring $\mathcal{O}(X_1)$ and the canonical homomorphism $A \rightarrow A[p^{-1}]$ corresponds to the projection $\mathcal{O}(X) \rightarrow \mathcal{O}(X_1)$ obtained by precomposing with inclusion $X_1 \rightarrow X_1 + X_2 = X$.

I think it is these considerations which motivated Lawvere's definition of Gaeta topos on a category with disjoint stable finite sums since the classifying topos for $x^2 = x \vdash x = 0 \vee x = 1$ is obtained by applying the Gaeta topos construction to the category \mathcal{R}^{op} of *loci*, i.e. *formal varieties* and polynomial functions between them.

Models of SDG are obtained by considering instead of rings so-called C^∞ -rings, i.e. algebras for the theory of smooth functions on \mathbb{R} . This was suggested by Lawvere as a generalisation of the account in the book of Demazure & Gabriel (which he seems to have studied very carefully) and worked out in detail by E. Dubuc. A comprehensive account can be found in the book by Moerdijk & Reyes on *Models of Smooth Infinitesimal Analysis*.

178 Some constructions of frames and preframes by generators and relations

Let S be a meet-semilattice. A *precoverage* is a set R of pairs (C, a) with $a \in S$ and $C \subseteq \downarrow a$. A *coverage* is a precoverage which is also pullback stable in the sense that for every $(C, a) \in R$ and $b \leq a$ the pair $(\{x \wedge b \mid x \in C\}, b)$ is also in R . Of course, for every precoverage R there is a least coverage containing R (which may be obtained predicatively by taking the downward closure).

If R is a coverage of S an R -ideal in S is a downclosed subset I of S such that for every $(C, a) \in R$ with $C \subseteq I$ it holds that $a \in I$, too. In Johnstone's *Stone Spaces* (1982) it has been shown that R -ideals ordered by \subseteq form the free frame over S *qua* meet-semilattice satisfying all relations in R . We denote this free frame by $\text{Fr}\langle S \text{ qua } \wedge\text{-semilattice} \mid R \rangle$.

In his "quantales" paper with Samson Abramsky Vickers shows that the same construction via R -ideals gives the free \vee -lattice over S *qua* poset satisfying the relations in R . It's shown there also that this construction goes through without assuming S to be a meet-semilattice.

A *preframe* is a dcpo with finite meets where directed suprema distribute over \wedge , i.e. $y \wedge \bigvee^\uparrow x_i = \bigvee^\uparrow y \wedge x_i$ whenever $(x_i)_{i \in I}$ is directed. For meet-semilattices S we call R a *directed* (pre)coverage iff for all $(C, a) \in R$ the set C is directed. As shown by Johnstone and Vickers (1990) the free preframe generated by S and a directed coverage R is given by the least sub-preframe of $\langle S \mid R \rangle$.

The lower power locale $\mathcal{P}_L(A)$ for a locale A has been defined (by Vickers¹²³) as the free frame over ΩA *qua* sup-lattice. For this purpose one needs a left adjoint F to the forgetful functor U from frames to sup-lattice. Given a sup-lattice L one may consider L^* , the set of finitely generated upward closed subsets $[x_1, \dots, x_n] = \uparrow\{x_1, \dots, x_n\}$ of L ordered by reverse subset inclusion. Obviously L^* is a meet-semilattice. On L^* we may consider the least coverage R_L containing all pairs of the form $(\{[x_1, \dots, x_n, y] \mid y \in X\}, [x_1, \dots, x_n, \bigvee X])$ for some $x_1, \dots, x_n \in L$ and $X \subseteq L$. Then the free frame over L is given by $\langle L^* \mid R_L \rangle$ as shown by R. Börger in an unpublished report and recalled in Bunge and Funk's *Constructive Points of the Lower Power Locale*.

The upper power locale $\mathcal{P}_U(A)$ is the free frame over ΩA *qua* preframe, i.e. set of all Scott closed subsets of A ordered by \subseteq .

Vickers and Johnstone in their 1990 paper have shown that $\mathcal{P}_L \mathcal{P}_U \cong \mathcal{P}_U \mathcal{P}_L$. The ensuing functor \mathbb{P} is called the *double power locale*. One can show that $\mathbb{P}(A)$ can be obtained as $\langle A \mid R \rangle$ where R is the coverage generated by all pairs $(D, \bigvee D)$ where D is a directed subset of A . Moreover, as shown by Townsend and Vickers $\mathbb{P}(A)$ is isomorphic to \mathbb{S}^A in the category of presheaves over \mathbf{Loc} , the category of locales. Notice that \mathbb{S}^A need not be representable unless A is exponentiable, i.e. a continuous lattice.

¹²³actually, he observed how to decompose Johnstone's "Vietoris construction" into the lower and upper powerdomain construction

179 Funayama's Theorem

Let A be a cHa. Let $\tilde{A} = A^{|A|}$ be the $|A|$ -fold product of A which is again a cHa. Let $B \subseteq \tilde{A}$ consist of those families $\phi \in \tilde{A}$ where $(\phi_r \rightarrow r) \rightarrow r \leq \phi_r$ for all $r \in |A|$. Obviously, B is a complete boolean algebra since $B = \{\phi \in \tilde{A} \mid (\phi \rightarrow \text{id}) \rightarrow \text{id} \leq \phi\}$. There is an embedding $\iota : A \rightarrow B : a \mapsto ((a \rightarrow r) \rightarrow r)_{r \in |A|}$ which has a right adjoint $\rho : B \rightarrow A : \phi \mapsto \bigwedge_{r \in |A|} \phi_r$. Obviously, we have $\rho \circ \iota = \text{id}_A$ and thus $\square = \iota \circ \rho : B \rightarrow B$ is a modal operator on B since one can show that ι preserves \wedge (and ρ as a right adjoint preserves \wedge anyway).

Flagg (and H. Friedman) have considered a version of Funayama's Theorem internal to $\mathcal{E}ff$ (or $\mathbf{Asm}(K_1)$) giving rise to a realizability semantics of *epistemic arithmetic*.

There is also a topos variant of Funayama's Theorem (Prop. A.4.5.23 of PTJ's *Elephant*). Let \mathcal{E} be an elementary topos and the topology j on \mathcal{E}/Ω be given by $j(p) = \lambda u : \Omega. (p(u) \rightarrow u) \rightarrow u$. Then $\text{Sh}_j(\mathcal{E}/\Omega)$ is boolean and the geometric morphism

$$\text{Sh}_j(\mathcal{E}/\Omega) \hookrightarrow \mathcal{E}/\Omega \longrightarrow \mathcal{E}$$

is surjective, i.e. its inverse image part reflects isomorphisms.

180 Combining Continuation with Algebraic Effects - A Definitional Approach (Plotkin '07)

Let \mathbb{C} be a cartesian closed category with 0 and T a strong monad on \mathbb{C} representing an algebraic effect and $\alpha : P \times TR \rightarrow R$ a P -indexed family of T -algebra structures on $R \in \mathbb{C}$. For $X \in \mathbb{C}$ we write $\neg X$ for R^X and $\sim X$ for $P \times R^X$. Then $X \mapsto \neg \sim X$ is a monad on \mathbb{C} with unit $\eta_X : X \rightarrow \neg \sim X : x \mapsto \lambda(p, k). k(x)$ and for $f : X \rightarrow \neg \sim Y$ its lifting $f^\# : \neg \sim X \rightarrow \neg \sim Y$ is defined as $R^{f^\#}$ where $f^\# = \langle \pi, \lambda k. \lambda x. f(x)(k) \rangle : \neg \sim Y \rightarrow \neg \sim X$, i.e. $f^\#(\gamma)(p, k) = \gamma(p, \lambda x. f(x)(p, k))$. Given $f : X \rightarrow \neg \sim Y$ and $k \in \sim Y$ put $(k)f = f^\#(k)$. Define $\cdot : \sim X \times R^{\sim X} \rightarrow R^{\sim 0}$ as $(p, k) \cdot \gamma = \lambda - . f(p, k)$ (notice that $\sim 0 \cong P$). The control operator $\mathcal{C} : (R^{\sim 0})^{\sim X} \rightarrow R^{\sim X}$ is defined as $\mathcal{C}(\delta)(p, k) = \delta(p, k)(p, i_R)$ where $i : 0 \rightarrow R$. Define $d : R^{\sim X} \rightarrow (R^{\sim 0})^{\sim X}$ as $d(\gamma)(p, k)(p', i_R) = \gamma(p, k)$. Then, obviously $\mathcal{C} \circ d = \text{id}_{R^{\sim X}}$.

A P -indexed family $\alpha : P \times TR \rightarrow R$ of algebra structures on R induces a monad morphism $m : T \Rightarrow \neg \sim$ where $m_X(\gamma)(p, k) = \alpha(p, T(k)(\gamma))$ and a monad morphism $m : T \Rightarrow \neg \sim$ induces a P -indexed family $\alpha : P \times TR \rightarrow R$ of T -algebra structures on R where $\alpha(p, c) = m_R(c)(p, \text{id}_R)$

Given a monad morphism $m : T \Rightarrow \neg \sim$ elements $c \in TX$ gives rise to P -indexed families of X -ary operations $m_X(c) : P \times R^X \rightarrow R$ and an $o : Y \rightarrow TX$ gives rise to a P -indexed family of maps $R^X \rightarrow R^Y$ whose p -th item is given by $\lambda k. \lambda y. m_X(o(y))(p, k)$ or equivalently to a P -indexed family of maps $Y \rightarrow R^{R^X}$ whose p -th item is given by $\lambda y. \lambda k. m_X(o(y))(p, k)$

181 Synthetic Account of Control as an Effect

R. Møgelberg and A. Simpson in their paper at MFPS'07 have presented a categorical account of *control as an effect* using ideas from SDT, namely R -replete objects. They work in (a model of) IZF postulating a small complete full subcategory \mathcal{C} of **Set** and a small complete internal category \mathcal{A} together with a forgetful functor $U : \mathcal{A} \rightarrow \mathcal{C}$ which preserves limits and reflects iso. Thus for all $\underline{A}, \underline{B} \in \mathcal{A}$ the collection $\mathcal{A}(\underline{A}, \underline{B})$ of algebra morphisms is a set $\underline{A} \multimap \underline{B}$. Concretely, they take for \mathcal{C} the full subcategory of **Set** on R -replete sets (where R is an object of some small complete full subcategory $\tilde{\mathcal{C}}$ of **Set** as e.g. the modest sets in a realizability model for IZF), for \mathcal{A} the small category \mathcal{C}^{op} which is internally complete since \mathcal{C} is also internally cocomplete and for $U : \mathcal{A} \rightarrow \mathcal{C}$ the functor $R^{(-)} : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$ which is left adjoint to $U^{\text{op}} = R^{(-)} : \mathcal{C} \rightarrow \mathcal{C}^{\text{op}}$.

Notice that for $\underline{A}, \underline{B}$ in \mathcal{A} their homset $\mathcal{A}(\underline{A}, \underline{B}) = \mathcal{C}(\underline{B}, \underline{A})$ is canonically isomorphic to the subset $\underline{A} \multimap \underline{B}$ of $\mathcal{C}(U(\underline{A}), U(\underline{B})) = \mathcal{C}(R^{\underline{A}}, R^{\underline{B}})$ consisting of those maps $f \in (R^{\underline{B}})^{(R^{\underline{A}})}$ such that its transpose $\tilde{f} : \underline{B} \rightarrow R^2(\underline{A})$ factors through $\eta_{\underline{A}} : \underline{A} \rightarrow R^2 \underline{A}$, i.e. $\forall y \in \underline{B}. \exists^1 x \in \underline{A}. \forall k \in R^{\underline{A}}. f(k)(y) = k(x)$. Since $\eta_{\underline{A}}$ is monic the $g : \underline{B} \rightarrow \underline{A}$ with $f = R^g$ is uniquely determined by f .

Using this framework Møgelberg und Simpson interpret their¹²⁴ calculus PE (Polymorphic Effect calculus) as follows. With every value type A they associate an element $\mathcal{C}[\underline{A}]$ of \mathcal{C} and with every computational type \underline{A} they associate an object $\mathcal{A}[\underline{A}]$ of \mathcal{A} in such a way that $\mathcal{C}[\underline{A}] = U(\mathcal{A}[\underline{A}])$.

Now sequents of the form $x_1:A_1, \dots, x_n:A_n \vdash t : C$ are interpreted as functions $\mathcal{C}[\underline{A}_1] \times \dots \times \mathcal{C}[\underline{A}_n] \rightarrow \mathcal{C}[\underline{C}]$ and sequents of the form $x_1:A_1, \dots, x_n:A_n \mid y:\underline{B} \vdash t : \underline{C}$ are interpreted as functions $f : \mathcal{C}[\underline{A}_1] \times \dots \times \mathcal{C}[\underline{A}_n] \times \mathcal{C}[\underline{B}] \rightarrow \mathcal{C}[\underline{C}]$ such that for all $\gamma \in \mathcal{C}[\underline{A}_1] \times \dots \times \mathcal{C}[\underline{A}_n]$ the function $f(\gamma, -) \in \mathcal{C}[\underline{B} \multimap \underline{C}]$, i.e. is an algebra morphism from $\mathcal{A}[\underline{B}]$ to $\mathcal{A}[\underline{C}]$.

Since $\mathcal{A}[\underline{R}] = 1$ we have $\mathcal{C}[\underline{A} \multimap \underline{R}] = \mathcal{A}(\mathcal{A}[\underline{A}], 1) \cong \mathcal{C}(1, \mathcal{A}[\underline{A}])$ and thus also $\mathcal{C}[(\underline{A} \multimap \underline{R}) \rightarrow \underline{R}] \cong R^{\mathcal{A}[\underline{A}]} \cong \mathcal{C}[\underline{A}]$ from which it follows that the interpretation of $\eta_{\underline{A}} \equiv \lambda x:\underline{A}. \lambda f:\underline{A} \multimap \underline{R}. f(x) : \underline{A} \rightarrow (\underline{A} \multimap \underline{R}) \rightarrow \underline{R}$ is an isomorphism whose inverse we may denote by $\epsilon_{\underline{A}} : ((\underline{A} \multimap \underline{R}) \rightarrow \underline{R}) \rightarrow \underline{A}$ where ϵ is a constant of type $\forall \underline{X}. ((\underline{X} \multimap \underline{R}) \rightarrow \underline{R}) \rightarrow \underline{X}$.

For constructing relationally parametric models of PE we first have to define an appropriate notion of *admissible* subobjects of objects in \mathcal{C} and \mathcal{A} , respectively. For $A \in \mathcal{C}$ let $\text{Sub}_{\mathcal{C}}(A) = \{C \subseteq A \mid C \in \mathcal{C}\}$ and for $\underline{A} \in \mathcal{A}$ let $\text{Sub}_{\mathcal{A}}(\underline{A}) = \{R^e \mid e : \underline{A} \rightarrow \underline{P} \text{ an } R\text{-epi}\}$ where e is an R -epi if R^e is an iso in \mathcal{C} , i.e. e is epic in \mathcal{C} . For a $Q \in \text{Sub}_{\mathcal{C}}(\underline{A} \times \underline{B})$ the map $\eta_{\underline{A}} \times \eta_{\underline{B}}$ preserves and reflects Q to $(\underline{Q} \multimap \Delta_R) \rightarrow \Delta_R$ iff $Q = (\eta_{\underline{A}} \times \eta_{\underline{B}})^{-1}((\underline{Q} \multimap \Delta_R) \rightarrow \Delta_R) = Q^{\top\top}$ where

$$Q^{\top} = \{(f, g) \in (\underline{A} \multimap \underline{R}) \times (\underline{B} \multimap \underline{R}) \mid \forall (x, y) \in Q. f(x) = g(y)\}$$

$$Q^{\top\top} = \{(x, y) \in U(\underline{A}) \times U(\underline{B}) \mid \forall (f, g) \in Q^{\top}. f(x) = g(y)\}$$

and thus $\epsilon_{\underline{A}} \times \epsilon_{\underline{B}}$ preserves $(\underline{Q} \multimap \Delta_R) \rightarrow \Delta_R$ to Q .

¹²⁴based on P. B. Levy's Call-by-Push-Value calculus but considering *computational* types as a subset of *value* types

182 Continuation semantics for Girard's LC

We sketch how to interpret Girard's **LC** in a distributive category \mathcal{C} with a baseable object $R \in \mathcal{C}$.¹²⁵

Positive formulas P of **LC** are interpreted as objects $\llbracket P \rrbracket$ of \mathcal{C} and negative formulas $N = \sim P$ are interpreted as $\sim \llbracket P \rrbracket$ where $\sim X$ stands for R^X . If $\Gamma \equiv A_1, \dots, A_n$ then $\llbracket \Gamma \rrbracket$ stands for $\llbracket A_1 \rrbracket \times \dots \times \llbracket A_n \rrbracket$. A sequent $\Gamma \vdash$ denotes a morphism $\llbracket \Gamma \rrbracket \rightarrow R$ in \mathcal{C} and a sequent $\Gamma \vdash P$ denotes a morphism $\llbracket \Gamma \rrbracket \rightarrow \llbracket P \rrbracket$ in \mathcal{C} . The meaning of dereliction is to introduce $\sim \sim$ since if $\Gamma \vdash P$ denotes $f : \llbracket \Gamma \rrbracket \rightarrow \llbracket P \rrbracket$ then the sequent $\Gamma, \neg P \vdash$ obtained by dereliction denotes the function $\tilde{f} : \llbracket \Gamma \rrbracket \times \sim \llbracket P \rrbracket \rightarrow R$ defined as $\tilde{f}(\gamma, k) = k(f(\gamma))$.

Despite sequences being interpreted as certain morphism in \mathcal{C} it is not the case that cut can be simply interpreted as composition in \mathcal{C} . Moreover, there is not just one cut but *two* different kinds of cuts, namely positive and negative ones.

Interpretation of positive cuts : if $\Gamma \vdash P$ is interpreted by $f : \llbracket \Gamma \rrbracket \rightarrow \llbracket P \rrbracket$ and $\Gamma, P \vdash \Pi$ is interpreted as $g : \llbracket \Gamma \rrbracket \times \llbracket P \rrbracket \rightarrow \llbracket \Pi \rrbracket$ then $\Gamma \vdash \Pi$ is interpreted by the function $h : \llbracket \Gamma \rrbracket \rightarrow \llbracket \Pi \rrbracket$ where $h(\gamma) = g(\gamma, f(\gamma))$.

Interpretation of negative cuts : if $\Gamma, P \vdash$ is interpreted as $f : \llbracket \Gamma \rrbracket \times \llbracket P \rrbracket \rightarrow R$ and $\Gamma, \neg P \vdash \Pi$ is interpreted as $g : \llbracket \Gamma \rrbracket \times \sim \llbracket P \rrbracket \rightarrow \llbracket \Pi \rrbracket$ then $\Gamma \vdash \Pi$ is interpreted as the function $h : \llbracket \Gamma \rrbracket \rightarrow \llbracket \Pi \rrbracket$ where $h(\gamma) = g(\gamma, \lambda x. f(\gamma, x))$.

Interpreting **LC** in control categories is only superficially more abstract since every control category \mathcal{K} is equivalent to one of the form $R^{\mathcal{C}}$ where \mathcal{C} is distributive and R is a baseable object in \mathcal{K} . Recall that $R^{\mathcal{C}}$ is the opposite of the Kleisli category for the double negation monad $R^{R^{(-)}}$ on \mathcal{C} . If $\eta_X : X \rightarrow R^{R^X} : x \mapsto \lambda k. k(x)$ is always monic then \mathcal{C} arises as subcategory of the Kleisli category consisting of those morphisms $f : X \rightarrow R^{R^Y}$ factoring through η_Y . Every control category \mathcal{K} arises this way where \mathcal{C} is the opposite of the *center* of \mathcal{K} . Notice, however, that from $\mathcal{K} \simeq R^{\mathcal{C}}$ it does not follow that \mathcal{C} is equivalent to the center of \mathcal{K} .¹²⁶

Girard's interpretation of **LC** in coherence spaces amounts to the interpretation of **LC** in $R^{\mathcal{C}}$ where \mathcal{C} is the category of commutative \otimes -comonoids in the category **Coh** of coherence spaces and linear maps and $R = !\perp$ where $!X$ is the free commutative \otimes -monoid on X in **Coh**, i.e. $\mathcal{C}(M, !X) \cong \mathbf{Coh}(U(M), X)$ where $U(M)$ is the underlying coherence space of M . In the following we will omit U for sake of readability. Notice that, in particular, we have $\mathcal{C}(M, R) = \mathcal{C}(M, !\perp) \cong \mathbf{Coh}(M, \perp)$. Binary products and sums in \mathcal{C} are given by \otimes and \oplus , respectively. Since $\mathcal{C}(M, !N^\perp) \cong \mathbf{Coh}(M, N^\perp) \cong \mathbf{Coh}(M \otimes N, \perp) \cong \mathcal{C}(M \otimes N, !\perp) = \mathcal{C}(M \otimes N, R)$ the exponential R^N in \mathcal{C} is given by $!N^\perp$ and R^h by $!h^\perp$ for

¹²⁵ R baseable means that all presheaves $\mathcal{C}((-) \times X, R)$ over \mathcal{C} are representable by an object R^X in \mathcal{C}

¹²⁶See e.g. P. Taylor's Abstract Stone Duality where he replaces \mathcal{C} by the opposite of the category of Eilenberg-Moore algebras for the $R^{R^{(-)}}$ monad on \mathcal{C} . In a recent paper by Grunenage and Streicher *QCB₀ is not closed under sobrification* (2006) it has been shown that when taking for \mathcal{C} the Σ -replete objects in $\mathbf{ExPer}_\Sigma(\mathcal{P}\omega)$ then this construction gives something different from \mathcal{C} .

$h : M \rightarrow N$ in \mathcal{C} .

Accordingly, the control category $R^{\mathcal{C}}$ has the same objects as the category \mathcal{C} and $R^{\mathcal{C}}(M, N) = \mathcal{C}(R^M, R^N) = \mathcal{C}(!M^\perp, !N^\perp) = \text{Coh}(!M^\perp, !N^\perp) = \text{Coh}_!(M^\perp, N^\perp)$ where $\text{Coh}_!$ is the Kleisli category for the comonad $!$ on Coh . Notice that M^\perp and N^\perp are commutative \wp -monoids and thus our construction of $R^{\mathcal{C}}$ is in accordance with that of O. Laurent and L. Regnier in their paper *About translations of classical logic in polarized linear logic* from LICS'03 where they define a control category \mathcal{K} whose objects are commutative \wp -monoids and $\mathcal{K}(M, N) = \text{Coh}(!M, N)$.

For convenience we explicitate the binoidal functor $\wp : R^{\mathcal{C}} \times R^{\mathcal{C}} \rightarrow R^{\mathcal{C}}$. We write $\varepsilon_X : !X \rightarrow X$ for the counit of the monad $!$ at X . For commutative \otimes -comonoids M and N let $\text{ev}_{M,N} : R^{M \otimes N} \otimes N \rightarrow R^M$ be the \otimes -comonoid homomorphism $h : !(M \otimes N)^\perp \otimes N \rightarrow !M^\perp$ such that $\varepsilon_M \circ h : !(M^\perp \wp N^\perp) \otimes N \rightarrow M^\perp$ is the transpose of $\varepsilon_{M^\perp \otimes N^\perp} : !(M^\perp \wp N^\perp) \rightarrow M^\perp \wp N^\perp$. Now for $f \in \mathcal{C}(!M_1^\perp, !M_2^\perp)$ and N a commutative \otimes -comonoid the map $f \wp N : R^{M_1 \otimes N} \rightarrow R^{M_2 \otimes N}$ is the unique \otimes -comonoid homomorphism $h : !(M_1 \otimes N)^\perp \rightarrow !(M_2 \otimes N)^\perp$ such that $\varepsilon_{(M_2 \otimes N)^\perp} \circ h : !(M_1 \otimes N)^\perp \rightarrow (M_2 \otimes N)^\perp = M_2^\perp \wp N^\perp$ is the transpose of $\varepsilon_{M_2^\perp} \circ f \circ \text{ev}_{M_1, N}$. The map $N \wp f : R^{N \otimes M_1} \rightarrow R^{N \otimes M_2}$ is defined symmetrically.

183 Translating J_0 to LL_{pol}

(after observations by Curien and Munch-Maccagnoni 2010)

Let J_0 be the fragment of intuitionistic logic based on negation and conjunction and LL_{pol} be *polarized linear logic*, i.e. the fragment of linear logic consisting of formulas of positive and negative polarity which are given by the following BNF syntax

$$P ::= X \mid P \otimes P \mid P \oplus P \mid !N \quad N ::= X^\perp \mid N \wp N \mid N \& N \mid ?P$$

For positive P we put $\neg^+ P = !P^\perp$ and for negative N we put $\neg^- N = (!N)^\perp$.

There are two translations of J_0 to LL_{pol} , a positive one $(-)^+$ and a negative one $(-)^-$. The positive translation is given by

$$X^+ = !X^\perp \quad (\neg A)^+ = \neg^+ A^+ \quad (A \wedge B)^+ = A^+ \otimes B^+$$

and the negative one by

$$X^- = X^\perp \quad (\neg A)^- = \neg^- A^- \quad (A \wedge B)^- = A^- \& B^-$$

By induction one easily shows that

$$A^+ = !A^-$$

holds for all formulas A in J_0 . By recursion on derivations in J_0 one can show that whenever $\Gamma \vdash_{J_0} [A]$ then

$$\Gamma^+ \vdash_{LL_{\text{pol}}} [A^+]$$

and

$$!\Gamma^- \vdash_{LL_{\text{pol}}} [A^-] \quad \text{and thus also} \quad !\Gamma^- \vdash_{LL_{\text{pol}}} [!A^-]$$

Since $A^+ = !A^-$ holds for all formulas in J_0 we conclude that the positive and the negative translation coincide on J_0 .

When adding disjunction to J_0 the positive and negative translation can be extended as well by putting¹²⁷

$$(A \vee B)^+ = A^+ \oplus B^+ \quad \text{and} \quad (A \vee B)^- = ?!A^- \wp ?!B^-$$

¹²⁷In order to make the negative translation work one has to formulate the left rule for disjunction as

$$\frac{\Gamma, A \vdash \quad \Gamma, B \vdash}{\Gamma, A \vee B \vdash} \quad \text{and not as} \quad \frac{\Gamma, A \vdash \quad \Gamma, B \vdash \Pi}{\Gamma, A \vee B \vdash \Pi}$$

because from $!\Gamma^-, !A^- \vdash$ and $!\Gamma^-, !B^- \vdash$ we can derive $!\Gamma^-, !A^- \oplus !B^- \vdash$ and thus $!\Gamma^-, ?(!A^- \oplus !B^-) \vdash$, i.e. $!\Gamma^-, ?!A^- \wp ?!B^- \vdash$ as desired. Here we have used in a crucial way that $\frac{!\Gamma, C \vdash}{!\Gamma, ?C \vdash}$ which would be impossible in presence of a nonempty Π right of the turnstile.

This, however, violates $A^+ = !A^-$ which can only be remedied by putting

$$(A \vee B)^+ = !(A^+ \oplus B^+)$$

in the positive translation.¹²⁸

We briefly recall the Lafont-Reus-Streicher cps-translation $(-)^*$ from classical logic to J_0 defined as follows

$$X^* = X \quad (A \rightarrow B)^* = \neg A^* \wedge B^* \quad \perp^* = 1$$

One can show that if $\Gamma \vdash [A]$ is derivable in classical logic then

$$\neg \Gamma^* [A^*] \vdash$$

is derivable in J_0 . Then in \mathbf{LL}_{pol} one can derive the sequents

$$\begin{aligned} (1a) \quad & (\neg \Gamma^*)^+ [A^*]^+ \vdash \\ (2a) \quad & !(\neg \Gamma^*)^- [A^*]^- \vdash \end{aligned}$$

which are actually equal. Thus, one can derive the sequents

$$\begin{aligned} (1b) \quad & (\neg \Gamma^*)^+ \vdash [((A^*)^+)^{\perp}] \\ (2b) \quad & !(\neg \Gamma^*)^- \vdash [!(A^*)^-]^{\perp} \end{aligned}$$

which are again equal. Thus one can derive the sequents

$$\begin{aligned} (1) \quad & !(\Gamma^*)^+ \vdash [((A^*)^+)^{\perp}] \\ (2) \quad & !(\Gamma^*)^- \vdash [!(A^*)^-]^{\perp} \end{aligned}$$

which are again equal.

Curien and Munch-Maccagnoni have observed that for $\tilde{A} = ((A^*)^+)^{\perp}$ we have

$$\tilde{X} = (X^+)^{\perp} = (!X^{\perp})^{\perp} = ?X$$

and

$$(\widetilde{A \rightarrow B}) = ((\neg A^* \wedge B^*)^+)^{\perp} = (!((A^*)^+)^{\perp} \otimes (B^*)^+)^{\perp} = !\tilde{A} \multimap \tilde{B}$$

and, therefore, the translation $(\widetilde{-})$ coincides with the *Girard translation* of the Krivine translation¹²⁹, i.e.

$$\begin{array}{ccc} \text{CL} & \xrightarrow{\text{Krivine}} & \text{IL} \\ \text{LRS} \downarrow & & \downarrow \text{Girard} \\ J_0 & \longrightarrow & \mathbf{LL}_{\text{pol}} \end{array}$$

where the bottom line is $(-)^+ = !(-)^-$.

¹²⁸since $?(A^+ \oplus B^+) = ?A^+ \wp ?B^+ = ?!A^- \wp ?!B^- = (A \vee B)^-$
¹²⁹For the translation $\tilde{A} = (\neg A^*)^- = !(A^*)^-)^{\perp}$ we have $\tilde{X} = !(X^*)^-)^{\perp} = !(X^{\perp})^{\perp} = ?X$ and $(\widetilde{A \rightarrow B}) = (!((A \rightarrow B)^*)^-)^{\perp} = (!(\neg A^* \wedge B^*)^-)^{\perp} = (!((\neg A^*)^- \& (B^*)^-))^{\perp} = !(\neg A^*)^- \otimes !(B^*)^-)^{\perp} = !(\neg A^*)^- \multimap !(B^*)^-)^{\perp} = !\tilde{A} \multimap \tilde{B}$ and, therefore, the translation $(\widetilde{-})$ coincides with the *Girard translation* of the Krivine translation.

However, this can be seen more easily by observing that $\tilde{A} = \tilde{\tilde{A}}$ since $!(A^*)^-)^{\perp} = (A^*)^+$.

184 Continuation Semantics for System **L** of Curien and Munch-Compagnoni

Our aim is to provide a continuation semantics for Curien and Munch's system **L**, a proof term assignment for a polarized system of (propositional) logic.

The types of system **L** are given by the following grammar

$$P ::= X \mid A \otimes B \mid \sim N \qquad N ::= \perp \mid A \rightarrow B$$

where A and B range over the union of P and Q .

Let \mathcal{C} be a category with finite products and a distinguished object R in \mathcal{C} which is *baseable*, i.e. R^X exists for all $X \in \mathcal{C}$.

Formulas A will be interpreted as objects $\llbracket A \rrbracket$ of \mathcal{C} where

$$\llbracket P \otimes Q \rrbracket = \llbracket P \rrbracket \times \llbracket Q \rrbracket \quad \llbracket M \otimes Q \rrbracket = R^{\llbracket M \rrbracket} \times \llbracket Q \rrbracket \quad \llbracket P \otimes N \rrbracket = \llbracket P \rrbracket \times R^{\llbracket N \rrbracket} \quad \llbracket M \otimes N \rrbracket = R^{\llbracket M \rrbracket} \times R^{\llbracket N \rrbracket}$$

and

$$\llbracket \sim N \rrbracket = \llbracket N \rrbracket$$

and

$$\llbracket \perp \rrbracket = 1$$

and

$$\llbracket P \rightarrow Q \rrbracket = \llbracket P \rrbracket \times R^{\llbracket Q \rrbracket} \quad \llbracket P \rightarrow N \rrbracket = \llbracket P \rrbracket \times \llbracket N \rrbracket \quad \llbracket M \rightarrow Q \rrbracket = R^{\llbracket M \rrbracket} \times R^{\llbracket Q \rrbracket} \quad \llbracket M \rightarrow N \rrbracket = R^{\llbracket M \rrbracket} \times \llbracket N \rrbracket$$

Terms of type P will be interpreted as elements of $R^{R^{\llbracket P \rrbracket}}$ and terms of type M as elements of $R^{\llbracket M \rrbracket}$. Values of type P will be interpreted as elements of $\llbracket P \rrbracket$.

Environments of type P will be interpreted as elements of $R^{\llbracket P \rrbracket}$ and environments of type M will be interpreted as elements of $R^{R^{\llbracket M \rrbracket}}$. Stacks of type M will be interpreted as elements of $\llbracket M \rrbracket$.

Variables of type P range over $\llbracket P \rrbracket$ and variables of type M range over $R^{\llbracket M \rrbracket}$. Covariables of type P range over $R^{\llbracket P \rrbracket}$ and covariables of type M range over $\llbracket M \rrbracket$.

184.1 Verification of the Correctness of the Rules of **L**

We will discuss for every rule of **L** how to interpret its conclusions given the interpretation of its premisses. For sake of readability we will neglect the ambient contexts Γ and Δ .

184.1.1 Variable and Covariable Introduction

The introduction rules for variables and covariables are

$$\frac{}{\Gamma, x : A \vdash x : A \mid \Delta} (\text{ax} \vdash) \qquad \frac{}{\Gamma \mid \alpha : A \vdash \alpha : A, \Delta} (\text{ax} \vdash)$$

If A is positive then $(\text{ax} \vdash)$ is interpreted as $\eta_{\llbracket A \rrbracket} : \llbracket A \rrbracket \rightarrow R^{R^{\llbracket A \rrbracket}} : x \mapsto \lambda p. p(x)$ and if A is negative then $(\text{ax} \vdash)$ is interpreted as the identity on $R^{\llbracket A \rrbracket}$.

If A is positive then $(ax \vdash)$ is interpreted as the identity on $R^{[A]}$ and if A is negative then $(ax \vdash)$ is interpreted as $\eta_{[A]}$.

184.1.2 Focalisation Rules

Using the binding constructs $\tilde{\mu}/\mu$ one can put a formula into focus left/right of the \vdash via the rules

$$\frac{c : (\Gamma, x : A \vdash \Delta)}{\Gamma \mid \tilde{\mu}x:A.c \vdash \Delta} (\tilde{\mu} \vdash) \qquad \frac{c : (\Gamma \vdash \alpha : A, \Delta)}{\Gamma \vdash \mu\alpha:A.c : A\Delta} (\vdash \mu)$$

which are interpreted by λ -abstraction in \mathcal{C} .

184.1.3 Cut Rule

The cut rule

$$\frac{\Gamma \vdash t : A \mid \Delta \quad \Gamma \mid e : A \vdash \Delta}{\langle t \mid e \rangle : (\Gamma \vdash \Delta)} (\text{cut})$$

is interpreted by application as follows. If A is positive then

$$\llbracket \langle t \mid e \rangle \rrbracket = \llbracket t \rrbracket (\llbracket e \rrbracket)$$

i.e. t is applied to e . If A is negative then

$$\llbracket \langle t \mid e \rangle \rrbracket = \llbracket e \rrbracket (\llbracket t \rrbracket)$$

i.e. e is applied to t .

184.1.4 Implication

The right and left introduction rules for implication are

$$\frac{\Gamma, x : A \vdash t : B \mid \Delta}{\Gamma \vdash \lambda x.t : A \rightarrow B \mid \Delta} (\vdash \rightarrow) \qquad \frac{\Gamma \vdash t : A \mid \Delta \quad \Gamma \mid e : B \vdash \Delta}{\Gamma \mid t.e : A \rightarrow B \vdash \Delta} (\rightarrow \vdash)$$

The interpretation of $\lambda x.t$ is obtained by currying $\llbracket x : A \vdash t \rrbracket$ in \mathcal{C} .

For interpreting $t.e$ we proceed by the following case analysis. If $A \equiv P$ and $B \equiv Q$ then¹³⁰

$$\llbracket t.e \rrbracket k = \llbracket t \rrbracket \lambda x : \llbracket P \rrbracket . \llbracket e \rrbracket \lambda h : R^{[Q]} . k(x, h)$$

and if $A \equiv P$ and $B \equiv N$ then

$$\llbracket t.e \rrbracket k = \llbracket t \rrbracket \lambda x : \llbracket P \rrbracket . \llbracket e \rrbracket \lambda h : \llbracket N \rrbracket . k(x, h)$$

and if $A \equiv M$ and $B \equiv Q$ then

$$\llbracket t.e \rrbracket k = k(\llbracket t \rrbracket, \llbracket e \rrbracket)$$

and if $A \equiv M$ and $B \equiv N$ then

$$\llbracket t.e \rrbracket k = \llbracket e \rrbracket \lambda h : \llbracket N \rrbracket . k(\llbracket t \rrbracket, h)$$

¹³⁰alternatively, one could evaluate e before t

184.1.5 Conjunction

The right and left introduction rules for conjunction are

$$\frac{\Gamma \vdash t : A \mid \Delta \quad \Gamma \vdash s : B \mid \Delta}{\Gamma \vdash (t, s) : A \otimes B \mid \Delta} (\rightarrow \otimes) \quad \frac{c : (\Gamma, x : A, y : B \vdash \Delta)}{\Gamma \mid \tilde{\mu}(x, y).c \vdash \Delta} (\otimes \vdash)$$

The interpretation of $\tilde{\mu}(x, y).c$ is obtained by currying $\llbracket x : A, y : B \vdash t \rrbracket$ in \mathcal{C} .

For the interpretation of pairing we proceed by the following case analysis. If $A \equiv P$ and $B \equiv Q$ then¹³¹

$$\llbracket (t, s) \rrbracket k = \llbracket t \rrbracket \lambda x : \llbracket P \rrbracket . \llbracket s \rrbracket \lambda y : \llbracket Q \rrbracket . k(x, y)$$

and if $A \equiv P$ and $B \equiv N$ then

$$\llbracket (t, s) \rrbracket k = \llbracket t \rrbracket \lambda x : \llbracket P \rrbracket . k(x, \llbracket s \rrbracket)$$

and if $A \equiv M$ and $B \equiv Q$ then

$$\llbracket (t, s) \rrbracket k = \llbracket s \rrbracket \lambda y : \llbracket Q \rrbracket . k(\llbracket t \rrbracket, y)$$

and if $A \equiv M$ and $B \equiv N$ then

$$\llbracket (t, s) \rrbracket k = k(\llbracket t \rrbracket, \llbracket s \rrbracket)$$

184.1.6 Negation of Negative Formulas

The right and left introduction rules for implication are

$$\frac{\Gamma \mid e : N \vdash \Delta}{\Gamma \vdash [e] : \sim N \mid \Delta} (\vdash \sim) \quad \frac{\Gamma \vdash t : N \mid \Delta}{\Gamma \mid [t] : \sim N \vdash \Delta} (\sim \vdash)$$

Notice that $\llbracket \sim N \rrbracket = \llbracket N \rrbracket$. Thus $\llbracket [e] \rrbracket \in R^{R^{\llbracket N \rrbracket}} = R^{R^{\llbracket \sim N \rrbracket}}$ and we put $\llbracket [[e]] \rrbracket = \llbracket [e] \rrbracket$. We also have $\llbracket [t] \rrbracket \in R^{\llbracket N \rrbracket} = R^{\llbracket \sim N \rrbracket}$ and we put $\llbracket \llbracket [t] \rrbracket \rrbracket = \llbracket [t] \rrbracket$.

184.1.7 Falsity

The right and left introduction rules for falsity are

$$\frac{c : (\Gamma \vdash \Delta)}{\Gamma \vdash \mu \hat{\mathbf{tp}}.c \mid \Delta} (\vdash \perp) \quad \frac{}{\Gamma \mid \hat{\mathbf{tp}} : \perp \vdash \Delta} (\sim \vdash)$$

The interpretation of $\mu \hat{\mathbf{tp}}.c$ is $\lambda x : 1. \llbracket [c] \rrbracket$ and $\hat{\mathbf{tp}}$ is interpreted as $\lambda f : R^1. f(*)$ (where $*$ is the unique element of 1).

We have considered here $\hat{\mathbf{tp}}$ as a constant denoting the unique element of 1. However, in Munch's work dealing with an untyped variant of the calculus \mathbf{P}

¹³¹alternatively one could evaluate s before t

the expression $\widehat{\text{tp}}$ is considered as a variable ranging over all possible top level continuations. In other words he considers $\mu\widehat{\text{tp}}.c$ as a genuine binding construct.

This untyped version of \mathbf{L} can be interpreted in any type U which at the same time is isomorphic to $U \times U$ and to U^U . In domains, using a “hack” due to Dana Scott, such a U can be constructed as follows. Let R be some domain with \perp . Then the domain $V = R^\omega$ is isomorphic to $V \times V$ (since ω is isomorphic to $\omega + \omega$). Now define U as the bifree solution of $U \cong V^U$. Then we have

$$U \times U \cong V^U \times V^U \cong (V \times V)^U \cong V^U \cong U$$

and thus also

$$U^U \cong (V^U)^U \cong V^{U \times U} \cong V^U \cong U$$

as desired. Notice that since $(R^\omega)^U \cong (R^U)^\omega$ we have $U \cong (R^U)^\omega$ and thus may be constructed as the bifree solution of the domain equation $U \cong (R^U)^\omega$. This has to be seen in contrast with the domain equation $D \cong R^{D^\omega}$ whose bifree solution is isomorphic to R_∞ . Both domain equations are obtained from $X \cong R^X$ by inserting $(-)^{\omega}$ at different places. Whereas from $X \cong R^X$ we cannot derive $X \cong X^X$ whereas from $D \cong R^{D^\omega}$ we can derive $D \cong D^D$. From $D \cong R^{D^\omega}$ we cannot derive $D \cong D \times D$ whereas from $U \cong (R^U)^\omega$ we can derive both $U \cong U \times U$ and $U \cong U^U$.

185 A. Joyal's Arithmetic Universes (around 1973)

An *arithmetic universe* is a pretopos \mathcal{A} with *parametrized list objects*, i.e. for every object A in \mathcal{A} there is an object L_A and morphisms $e_A : 1 \rightarrow A$ and $c_A : A \times L_A \rightarrow L_A$ such that for every $f : X \rightarrow Y$ and $g : A \times Y \rightarrow Y$ there exists a unique map $h : L_A \times X \rightarrow Y$ making the diagram

$$\begin{array}{ccc}
 1 \times X & \xrightarrow{e_A \times X} & L_A \times X \\
 \cong \downarrow & & \downarrow h \\
 X & \xrightarrow{f} & Y
 \end{array}
 \qquad
 \begin{array}{ccc}
 A \times L_A \times X & \xrightarrow{c_A \times X} & L_A \times X \\
 A \times h \downarrow & & \downarrow h \\
 A \times Y & \xrightarrow{g} & Y
 \end{array}$$

commute. In particular, one gets a parametrized natural numbers object $N = L_1$ and for a graph $s, t : G_1 \rightarrow G_0$ internal to \mathcal{A} one may construct the free internal category C over G where $C_0 = G_0$ and C_1 is the subobject of L_G consisting of all lists $e_1 \dots e_n$ with $t(e_i) = s(e_{i+1})$ for $i < n$.

The logic of \mathcal{A} is sufficiently strong for constructing the initial arithmetic universe \mathcal{A}_0 . This can be done in particular inside the free arithmetic universe \mathcal{A}_0 and then we get $\mathcal{A}(1, \mathcal{A}_0) \cong \mathcal{A}_0$ which fact A. Joyal used for proving (his variant of) Gödel's First Incompleteness Theorem.

186 Quasicategories and simplicial homotopy

Let Δ be the category of finite nonempty ordinals and monotone maps between them. We write $[n]$ for the ordinal $n+1 = \{0, 1, \dots, n\}$. There is an embedding I of Δ into \mathbf{Sp} , the category of topological spaces and continuous maps, sending $[n]$ to the n -dimensional simplex $\Delta(n) = \{\vec{x} \in [0, 1]^{n+1} \mid \sum_{i=0}^n x_i = 1\}$ and $f : n \rightarrow m$ to the continuous map $\Delta(f) : \Delta(n) \rightarrow \Delta(m)$ with $\Delta(f)(\vec{x})_j = \sum_{i \in f^{-1}(j)} x_i$. We write \mathbf{sSet} for the (presheaf) topos $\widehat{\Delta} = \mathbf{Set}^{\Delta^{\text{op}}}$ of *simplicial sets*. The functor $\mathcal{S} : \mathbf{Sp} \rightarrow \mathbf{sSet}$ obtained by currying the functor $\mathbf{Sp}(I(-_1), -_2) : \Delta^{\text{op}} \times \mathbf{Sp} \rightarrow \mathbf{Set}$ has a left adjoint $\mathcal{R} : \mathbf{sSet} \rightarrow \mathbf{Sp}$ where $\mathcal{R}(X)$ is the colimit of $\mathbf{Elt}_s(X) \rightarrow \Delta \xrightarrow{I} \mathbf{Sp}$. For a space X the counit $\varepsilon_X : \mathcal{R}\mathcal{S}X \rightarrow X$ gives rise to a simplicial approximation of X via the space $\mathcal{R}\mathcal{S}X$ which is obtained by glueing together simplices (a so called CW-complex). We call X a CW-complex iff ε_X is a homeomorphism. One can show that ε_X is always a weak equivalence (see section 187). A map $\phi : K \rightarrow L$ in \mathbf{sSet} is called a weak equivalence iff $\mathcal{R}(\phi)$ is a homotopy equivalence in \mathbf{Sp} . Thus factoring¹³² \mathbf{Sp} and \mathbf{sSet} modulo weak equivalences gives rise to the same *homotopy category* \mathcal{H} .

Now one wants to characterize the simplicial sets in the image of \mathcal{S} within \mathbf{sSet} . For this purpose we need some technical preliminaries about simplicial sets. First of all we identify $[n] \in \Delta$ via Yoneda with the simplicial set $\Delta[n] =$

¹³²in the sense of calculus of fractions

$\Delta(-, [n])$. For $i \in [n]$ let $\partial_i \Delta[n]$, the i -th face of $\Delta[n]$, be the subobject of $\Delta[n]$ consisting of all maps $\phi : m \rightarrow n$ s.t. $i \notin \text{im}(\phi)$ (geometrically speaking $\partial_i \Delta[n]$ is the face opposite to corner i). For $0 \leq k \leq n$ let $\Lambda^k[n] = \bigcup_{i \neq k} \partial_i \Delta[n] \subseteq \Delta[n]$. Simplicial sets of the form $\Lambda^k[n]$ are called *horns* and *inner horns* iff $0 < k < n$. A simplicial set K is called a *Kan complex* iff for every horn $\phi : \Lambda^k[n] \rightarrow K$ there exists a (typically not unique) map $\bar{\phi} : \Delta[n] \rightarrow K$ such that $\bar{\phi} \upharpoonright \Lambda^k[n] = \phi$. All simplicial sets of the form $\mathcal{S}(X)$ are Kan complexes and every Kan complex is weakly equivalent to one of this particular form (via $\eta_K : K \rightarrow \mathcal{S}\mathcal{R}X$).

Kan complexes may be understood as *weak higher dimensional groupoids*. A notion of *weak higher dimensional categories* which are groupoids for dimension $n \geq 1$ is given by *weak Kan complexes* or *quasi-categories*, i.e. simplicial sets K such that all inner horns $\phi : \Lambda^k[n] \rightarrow K$ can be extended to maps $\bar{\phi} : \Delta[n] \rightarrow K$ whose restriction to $\Lambda^k[n]$ coincides with ϕ . A. Joyal prepares a 2 volumes treatise on quasi-categories investigating their properties as weak higher dimensional categories.

n -cells in quasi-categories

Suppose X is a quasi-category. The n -cells of X will be certain elements of $X(n)$. All elements of $X(0)$ are 0-cells and an $f \in X(1)$ is a 1-cell from fd_1 to fd_0 . Suppose $a, b \in X(n)$ are n -cells and $f, g \in X(n+1)$ are $(n+1)$ -cells from a to b then $\alpha \in X(n+2)$ is an $(n+2)$ -cell from f to g iff the following two conditions hold

- 1) $\alpha d_1 = f$ and $\alpha d_0 = g$
- 2) $\alpha d_2 = a s_0$.

187 Higher Homotopy Groups

Let $I = [0, 1]$ with the Euclidean topology. Let X and Y be topological spaces with distinguished subsets $X_0 \subseteq X$ and $Y_0 \subseteq Y$, respectively. We write $f : (X, X_0) \rightarrow (Y, Y_0)$ for the claim that f is a continuous function $f : X \rightarrow Y$ with $f[X_0] \subseteq Y_0$. Functions $f, g : (X, X_0) \rightarrow (Y, Y_0)$ are *homotopy equivalent*, for which we write $f \sim g$, iff there exists a continuous function $h : I \times X \rightarrow Y$ such that $h(0, x) = f(x)$ and $h(1, x) = g(x)$ for all $x \in X$ and $h(t, -) : (X, X_0) \rightarrow (Y, Y_0)$ for all $t \in I$. Obviously, the relation \sim is an equivalence relation on the set $[(X, X_0) \rightarrow (Y, Y_0)]$.

For a pointed space (X, x_0) and $n \in \mathbb{N}$ we write $\pi_n(X, x_0)$ for the equivalence classes of $[(I^n, \partial I^n) \rightarrow (X, \{x_0\})]$ modulo \sim . For $n \geq 1$ on $[(I^n, \partial I^n) \rightarrow (X, \{x_0\})]$ we may define *composition* as follows

$$(gf)(s_1, \dots, s_n) = \begin{cases} f(2s_1, s_2, \dots, s_n) & \text{if } s_1 \in [0, \frac{1}{2}] \\ g(2s_1 - 1, s_2, \dots, s_n) & \text{if } s_1 \in [\frac{1}{2}, 1] \end{cases}$$

Apparently composition is compatible with homotopy equivalence in the sense that $gf \sim g'f'$ whenever $f \sim f'$ and $g \sim g'$. The inverse of f is defined as $f^{-1}(s_1, s_2, \dots, s_n) = f(1 - s_1, s_2, \dots, s_n)$ which operation again respects \sim . Thus $\pi_n(X, x_0)$ forms a group w.r.t. composition (modulo homotopy equivalence \sim) and is called the *n-th homotopy group* of (X, x_0) . For $n = 0$ we have $\partial I^0 = \emptyset$ and thus $\pi_0(X, x_0)$ consists of path-connected components of X . Since $\pi_0(X, x_0)$ does not depend on the base point x_0 we write simply $\pi_0(X)$.

A continuous function $f : X \rightarrow Y$ is called a *weak equivalence* iff for all $x \in X$ and $n \in \mathbb{N}$ the map $f_* : \pi_n(X, x) \rightarrow \pi_n(Y, f(x))$ sending $[g]$ to $[f \circ g]$ is a bijection (and thus a group isomorphism since f_* is always a group homomorphism).

188 Group Extension Problem

is discussed in E.Weiss *Cohomology of Groups* and much shorter also in Ch. Weibel's book *Homological Algebra*. Let G be a group and A an abelian group. Then an *extension of G with coefficients in A* is an exact sequence

$$0 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} G \longrightarrow 1$$

i.e. p is a surjective group homomorphism and A is the kernel of p . Every $g \in E$ induces a group homomorphism $\phi(g) : A \rightarrow A : x \mapsto gxg^{-1}$. Suppose $p(g) = p(h)$, i.e. $h^{-1}g \in A$, then for $x \in A$ we have (since A is commutative)

$$gx = hh^{-1}gx = h x h^{-1}g$$

and thus

$$gxg^{-1} = h x h^{-1}$$

i.e. $\phi(g) = \phi(h)$. Thus $\phi : E \rightarrow \mathbf{Aut}(A)$ factors along p which we again denote by $\phi : G \rightarrow \mathbf{Aut}(A)$.

If G acts on A via a $\phi : G \rightarrow \mathbf{Aut}(A)$ we call A a G -module. For such a ϕ we obtain a *split extension*

$$0 \longrightarrow A \xrightarrow{i} G \times A \xrightarrow{p} G \longrightarrow 1$$

where $p : G \times A \rightarrow G$ is obtained from ϕ via the Grothendieck construction. The surjective homomorphism (fibration) p is split by the homomorphism $s : G \rightarrow G \times A : g \mapsto (g, 0)$ where 0 is the neutral element of A . But there may be other group extensions of G by A inducing the same ϕ . How many there are (up to isomorphism) is the so-called *group extension problem*.

Consider e.g. the constant $\phi : \mathbb{Z}_2 \rightarrow \mathbf{Aut}(\mathbb{Z})$ with value $\text{id}_{\mathbb{Z}}$. This \mathbb{Z}_2 -module is e.g. induced by the paradigmatic non-split fibration $p : \mathbb{Z} \rightarrow \mathbb{Z}_2 : n \mapsto n \text{ mod } 2$.

189 Group Cohomology (Old Style)

As described in E.Weiss' book *Cohomology of Groups* Eilenberg and MacLane in their 1945 paper (introducing category theory) formulated group cohomology as follows. Let G be a group and $A : G \rightarrow \mathbf{Ab}$, i.e. A is an abelian group acted on by G from the left. Let C_n be the set of all maps $\Phi : G^{n+1} \rightarrow A$ satisfying $\Phi(gg_0, gg_1, \dots, gg_n) = g\Phi(g_0, g_1, \dots, g_n)$. Obviously, all C_n are abelian groups under pointwise addition. For every natural number n let the group homomorphism $d_n : C_{n-1} \rightarrow C_n$ be defined as

$$d_n(\Phi)(g_0, \dots, g_n) = \sum_{k=0}^n (-1)^k \Phi(g_0, \dots, \widehat{g}_k, \dots, g_n)$$

where \widehat{g}_k means that the argument g_k is omitted. We define n -cocycles as the elements of C_n which are in the kernel of d_{n+1} .

For computational purposes it is better to replace C_n by the isomorphic set $\widetilde{C}_n = A^{G^n}$. One might be tempted to associate with $\Phi \in C_n$ the element $\phi \in \widetilde{C}_n$ with $\phi(g_1, \dots, g_n) = \Phi(1, g_1, \dots, g_n)$. But the following at first sight less intuitive choice turns out as more appropriate

$$\phi(g_1, \dots, g_n) = \Phi(1, g_1, g_1g_2, \dots, g_1g_2 \dots g_n)$$

The inverse of this correspondence is given by

$$\Phi(1, h_1, \dots, h_n) = \phi(h_1, h_1^{-1}h_2, h_2^{-1}h_3, \dots, h_{n-1}^{-1}h_n)$$

since $g_k = (g_1 \dots g_{k-1})^{-1}g_1g_2 \dots g_k$. Thus, from ϕ we get Φ back as follows

$$\Phi(h_0, h_1, \dots, h_n) = h_0\Phi(1, h_0^{-1}h_1, \dots, h_0^{-1}h_n) = h_0\phi(h_0^{-1}h_1, h_1^{-1}h_2, \dots, h_{n-1}^{-1}h_n)$$

We write $\widetilde{(\cdot)} : C_n \rightarrow \widetilde{C}_n$ for the function mapping Φ to ϕ as defined above. Now we look for an explicitation of the unique map \widetilde{d}_n making the following diagram commute

$$\begin{array}{ccc} \widetilde{C}_{n-1} & \xrightarrow{\widetilde{d}_n} & \widetilde{C}_n \\ \widetilde{(\cdot)} \uparrow & & \uparrow \widetilde{(\cdot)} \\ C_{n-1} & \xrightarrow{d_n} & C_n \end{array}$$

[small]

$$\begin{aligned} \widetilde{d}(\phi)(g_1, \dots, g_n) &= \widetilde{d}(\widetilde{\Phi})(g_1, \dots, g_n) \\ &= d(\Phi)(1, g_1, g_1g_2, \dots, g_1g_2 \dots g_n) \\ &= \sum_{k=0}^n (-1)^k \Phi(1, g_1, \dots, g_1g_2 \dots g_{k-1}, g_1g_2 \dots g_{k-1}g_kg_{k+1}, \dots, g_1g_2 \dots g_n) \\ &= \Phi(g_1, g_1g_2, \dots, g_1g_2 \dots g_n) + \\ &\quad \sum_{k=1}^{n-1} (-1)^k \Phi(1, g_1, \dots, g_1g_2 \dots g_{k-1}, g_1g_2 \dots g_{k-1}g_kg_{k+1}, \dots, g_1g_2 \dots g_n) + \\ &\quad (-1)^n \Phi(1, g_1, \dots, g_1g_2 \dots g_{n-1}) \\ &= g_1\phi(g_2, \dots, g_n) + \sum_{k=1}^{n-1} (-1)^k \phi(g_1, \dots, g_{k-1}, g_kg_{k+1}, g_{k+2}, \dots, g_n) + \phi(g_1, \dots, g_{n-1}) \end{aligned}$$

Thus, in particular, for $n = 3$ we have

$$\widetilde{d}_3(\phi)(g_1, g_2, g_3) = g_1\phi(g_2, g_3) - \phi(g_1g_2, g_3) + \phi(g_1, g_2g_3) - \phi(g_1, g_2)$$

and thus ϕ is a 2-cocycle iff

$$g_1\phi(g_2, g_3) + \phi(g_1, g_2g_3) = \phi(g_1, g_2) + \phi(g_1g_2, g_3)$$

This latter equation characterizing 2-cocycles is analogous to the following coherence condition for pseudo-functors $\mathbb{B} \rightarrow \mathbf{Cat}$ (where we write g_i for the value of the pseudo-functor on g)

$$\begin{array}{ccc}
 (g_1 g_2 g_3)! & \xrightarrow{\phi_{g_1 g_2, g_3}} & (g_1 g_1)! (g_3)! \\
 \downarrow \phi_{g_1, g_2 g_3} & & \downarrow \phi_{g_1, g_2}(g_3)! \\
 (g_1)! (g_2 g_3)! & \xrightarrow{(g_1)! \phi(g_2, g_3)} & (g_1)! (g_2)! (g_3)!
 \end{array}$$

(Notice that in the case of groups or monoids the $(g_3)!$ in $\phi_{g_1, g_2}(g_3)!$ can be omitted since its object part is trivial.) That is the reason why in some places this coherence condition is referred to as “cocycle condition” (not only for pseudo-functors but also for “descent data”).

190 Stability of Universes in AST

Let \mathcal{C} be a Heyting category with stable and disjoint sums and \mathcal{S} a class of small maps satisfying some basic axioms. Suppose further that there exists a covariant power class functor \mathcal{P}_s . For any object X of \mathcal{C} we can extend \mathcal{P}_s to the slice \mathcal{C}/X as follows: if $f : Y \rightarrow X$ then $\mathcal{P}_s^X(f)$ is the first projection of

$$\{(x, \alpha) \in X \times \mathcal{P}_s(Y) \mid \forall y \in \alpha f(y) = x\}$$

on the first component. The subobject $\in_f \mapsto f \times_X \mathcal{P}_s(f)$ is given by the projection of

$$\{(x, y, \alpha) \in X \times Y \times \mathcal{P}_s(Y) \mid f(y) = x \wedge \forall y \in \alpha f(y) = x \wedge y \in \alpha\}$$

on X .

One can show that $i : \mathcal{P}_s(U) \rightarrow U$ is an initial \mathcal{P}_s -algebra iff every subalgebra $m : P \rightarrow U$ is an iso.¹³³ Notice that quantification is here understood in the external sense.¹³⁴

Suppose $i : \mathcal{P}_s(U) \rightarrow U$ is an initial \mathcal{P}_s -algebra and X is an object of \mathcal{C} . Then X^*i is given by $X \times i : X \times \mathcal{P}_s(U) \rightarrow X \times U$ over X using that $X^*\mathcal{P}_s(U)$ is isomorphic to $\mathcal{P}_s^X(X^*U)$. For showing that X^*i is an initial \mathcal{P}_s -algebra suppose $m : L \rightarrow X \times U$ is a subalgebra of X^*i , i.e.

$$\begin{array}{ccc} \mathcal{P}_s^X(L) & \xleftarrow{\mathcal{P}_s^X(m)} & X \times \mathcal{P}_s(U) \\ \downarrow i_L & & \downarrow X \times i \\ L & \xrightarrow{m} & X \times U \end{array}$$

The subobject $\forall_X(m) : M \rightarrow U$ is given by $M = \{u \in U \mid \forall x : X (x, u) \in L\}$. If $\alpha \in \mathcal{P}_s(M)$ then for all $x \in X$ we have $(x, i(\alpha)) \in L$ and thus $i(\alpha) \in M$. Thus M is a sub- \mathcal{P}_s -algebra of i from which it follows that $\forall_X(m)$ is an iso, i.e. $\forall u : U \forall x : X (x, u) \in L$, i.e. m is an iso. Thus X^*i is an initial \mathcal{P}_s^X -algebra in \mathcal{C}/X .

¹³³The functor \mathcal{P}_s preserves monos and m being a subalgebra of i means that

$$\begin{array}{ccc} \mathcal{P}_s(U) & \xrightarrow{i} & U \\ \mathcal{P}_s(m) \uparrow & & \uparrow m \\ \mathcal{P}_s(P) & \xrightarrow{i_P} & P \end{array}$$

commutes for some unique map i_P .

¹³⁴This follows from a proof (of their Th.1) in the JSL paper of van Oosten and Kouwenhoven-Gentil. Their proof makes sense also in a predicative setting and \in -induction is used only in the empty context, i.e. with $Z = 1$. Thus it suffices to state their principle of \in -induction only for the empty context. Obviously, the principle of \in -induction for $Z = 1$ is equivalent to having no proper \mathcal{P}_s -subalgebra.

191 Categorical Distributors

A *categorical distributor* from \mathbb{A} to \mathbb{B} is a span $\mathbb{A} \xleftarrow{P} \mathbb{X} \xrightarrow{Q} \mathbb{B}$ where P is a cofibration and Q is a fibration such that

- (1) P -cocartesian liftings can be chosen as Q -vertical
- (2) Q -cartesian liftings can be chosen as P -vertical
- (3) for every commuting diagram

$$\begin{array}{ccc} v^*X & \xrightarrow{\psi} & X \\ \varphi' \downarrow & & \downarrow \varphi \\ Y & \xrightarrow{\psi'} & u_!X \end{array}$$

if φ and φ' are Q -vertical, ψ and ψ' are P -vertical, φ is P -cocartesian and ψ is Q -cartesian then φ' is P -cocartesian if and only if ψ' is Q -cartesian.

One can show (as usual) that categorical distributors from \mathbb{A} to \mathbb{B} correspond to pseudo-functors $\mathbb{A} \times \mathbb{B}^{\text{op}} \rightarrow \mathbf{Cat}$.

Composition of categorical distributors works as expected. Suppose (P_1, Q_1) is a categorical distributor from \mathbb{A}_1 to \mathbb{A}_2 and (P_2, Q_2) is a categorical distributor from \mathbb{A}_2 to \mathbb{A}_3 then its composite is given by (P_1P, Q_2Q) where

$$\begin{array}{ccccc} \mathbb{X}_3 & \xrightarrow{Q} & \mathbb{X}_2 & \xrightarrow{Q_2} & \mathbb{A}_3 \\ P \downarrow & \lrcorner & \downarrow P_2 & & \\ \mathbb{X}_1 & \xrightarrow{Q_1} & \mathbb{A}_2 & & \\ P_1 \downarrow & & & & \\ \mathbb{A}_1 & & & & \end{array}$$

taken in \mathbf{Cat} . It is clear that P_1P and Q_2Q are cofibrations and fibrations, respectively. It is also easy to show that (P_1P, Q_2Q) satisfy conditions (1) and (2). For (1) suppose $u : I \rightarrow J$ and X is an object of \mathbb{X}_3 with $P_1P(X) = I$. By assumption there exists a P_1 -cocartesian arrow $\psi : P(X) \rightarrow Z$ above u which is Q_1 -vertical. Due to the construction of pullbacks in \mathbf{Cat} there is a cocartesian arrow $\varphi : X \rightarrow Y$ above ψ with $Q(\varphi) = \text{id}$. Thus, the arrow $\varphi : X \rightarrow Y$ is a P_1P -cocartesian arrow above u with $Q_2Q(\varphi) = \text{id}$. The proof of (2) is analogous by symmetry.

192 Delimited Continuations

In an unpublished paper from 1989 Danvy and Filinski came up with a denotational semantics for *delimited continuations* using the predomains

$$\begin{array}{ll}
 Val & = BVal + Fun & \text{(Values)} \\
 Sctx & = Val \rightarrow Ans & \text{(Surrounding Contexts)} \\
 Ctn & = Sctx \rightarrow Sctx & \text{(Continuations)} \\
 Fun & = Ctn \rightarrow Ctn & \text{(Functions)}
 \end{array}$$

where $BVal$ is a predomain of basic values and Ans is some domain of responses, e.g. Val_{\perp} . Notice that $Fun \cong Val \rightarrow Ctn \rightarrow Sctx \rightarrow Ans$. Meanings of terms will be elements of $Den = Ctn \rightarrow Sctx \rightarrow Ans$, so-called *denotations*, and accordingly $Fun \cong Val \rightarrow Den$. An *environment* is a function ρ from variables to Val . We let v range over Val , f over Fun , γ over $Sctx$ and κ over Ctn .

Syntactic values are variables or functional abstractions. The meaning of a syntactic value V in environment ρ is an element of Val and denoted as $\llbracket V \rrbracket_v \rho$. We put

$$\llbracket x \rrbracket_v \rho = \rho(x) \quad \llbracket \lambda x. M \rrbracket_v \rho \kappa \gamma v = \llbracket M \rrbracket \rho[v/x] \kappa \gamma$$

The semantic equations for general terms are

$$\begin{array}{l}
 \llbracket V \rrbracket \rho \kappa \gamma = \kappa \gamma (\llbracket V \rrbracket_v \rho) \\
 \llbracket MN \rrbracket \rho \kappa \gamma = \llbracket M \rrbracket \rho (\lambda \gamma. \lambda v. \mathbf{let} \text{ Fun } f = v \mathbf{in} \llbracket N \rrbracket \rho (f \kappa) \gamma) \gamma \\
 \llbracket \mathbf{shift} \ x \ \mathbf{in} \ M \rrbracket \rho \kappa \gamma = \llbracket M \rrbracket \rho [\lambda \kappa'. \lambda \gamma. \kappa (\kappa' \gamma) / x] \text{id}_{Sctx} \gamma \\
 \llbracket \mathbf{reset} \ M \rrbracket \rho \kappa \gamma = \llbracket M \rrbracket \rho \text{id}_{Sctx} (\kappa \gamma)
 \end{array}$$

In a paper by Danvy and Filinski from 1990 the authors rearranged this semantics in such a way that it looks more like ordinary continuation semantics. It is based on

$$\begin{array}{ll}
 Val & = BVal + Fun & \text{(Values)} \\
 MCtn & = Val \rightarrow Ans & \text{(Metacontinuations)} \\
 Ctn & = Val \rightarrow Ctn \rightarrow Ans & \text{(Continuations)} \\
 Fun & = Val \rightarrow Ctn \rightarrow MCtn \rightarrow Ans & \text{(Functions)}
 \end{array}$$

where the semantics of syntactic values is given by

$$\llbracket x \rrbracket_v \rho = \rho(x) \quad \llbracket \lambda x. M \rrbracket_v \rho = \lambda v. \llbracket M \rrbracket \rho[v/x]$$

and of general terms by

$$\begin{array}{l}
 \llbracket V \rrbracket \rho \kappa = \kappa (\llbracket V \rrbracket_v \rho) \\
 \llbracket MN \rrbracket \rho \kappa = \llbracket M \rrbracket \rho (\lambda v. \mathbf{let} \text{ Fun } f = v \mathbf{in} \llbracket N \rrbracket \rho (\lambda v. f v \kappa)) \\
 \llbracket \mathbf{shift} \ x \ \mathbf{in} \ M \rrbracket \rho \kappa = \llbracket M \rrbracket \rho [\lambda v. \lambda \kappa'. \lambda \gamma. \kappa v (\lambda v. \kappa' v \gamma) / x] (\lambda v. \lambda \gamma. \gamma v) \\
 \llbracket \mathbf{reset} \ M \rrbracket \rho \kappa \gamma = \llbracket M \rrbracket \rho (\lambda v. \lambda \gamma. \gamma v) (\lambda v. \kappa v \gamma)
 \end{array}$$

193 Can one embed N^N into N , constructively?

Even constructively, one can show that there is no surjective function from N to N^N . Suppose $u : N \rightarrow N^N$ is surjective. Then there exists an $n \in N$ with $u(n) = \lambda m. u(m)(m) + 1$. But then $u(n)(n) = u(n)(n) + 1$ which clearly is impossible. Thus, we have shown that

$$(1) \quad \forall u : N \rightarrow N^N. \exists f : N^N. \neg \exists n : N. f = u(n)$$

Classically, from this one concludes that

$$(2) \quad \forall F : N^N \rightarrow N. \forall f, g : N^N. F(f) \neq F(g) \rightarrow \exists n : N. f(n) \neq g(n)$$

since if $F : N^N \rightarrow N$ is injective then one can find a surjective function $u : N \rightarrow N^N$ which is impossible by (1). However, for obtaining a surjective u from an injective F one uses AUC and decidability of the image of F . Alternatively, one could use countable choice and the assumption that the image of F is countable.

In June 2011, however, A. Bauer has constructed a realizability topos validating countable choice where there exists an injective function $F : N^N \rightarrow N$. His construction is based on a pca structure on \mathbb{N} based on Hamkins' Infinite Time Turing Machines (ITTM) which is different from any notion of relative computability as given by an oracle.

Notice, however, that a constructive proof of (2) has been obtained by P. Oliva by unwinding the classical proof of (2) using Gödel's Dialectica interpretation and bar recursion. Thus, A. Bauer's model does not validate bar recursion.

194 Scott Model of Linear Logic (T. Ehrhard)

For every poset P let $\mathcal{D}(P)$ be the set of downwards closed subsets of P ordered by \subseteq and $\mathcal{D}_0(P)$ the subset of compact elements of $\mathcal{D}(P)$, i.e. subsets of P of the form $\bigcup_{i < n} \downarrow p_i$ with $p_0, \dots, p_{n-1} \in P$. Up to isomorphism posets of the form $\mathcal{D}(P)$ are precisely the *complete prime algebraic lattices*.

On the other hand with a poset P one may associate the set $\mathcal{I}(P)$ of directed downward closed subsets of P again ordered by subset inclusion. Up to isomorphism posets of the form $\mathcal{I}(P)$ are precisely the algebraic predomains.

Lemma

Up to isomorphism complete prime algebraic lattices are of the form Σ^A for some algebraic lattice A , i.e. *negated predomains* in the sense of Reus and Streicher.

Proof. For every algebraic predomain A the cpo Σ^A is isomorphic to $\mathcal{D}(\mathcal{K}(A)^{\text{op}})$ and thus a complete prime algebraic lattice. For every poset P we have $\Sigma^{\mathcal{I}(P)} \cong \mathcal{D}(P^{\text{op}})$. Thus, for every poset P the complete prime algebraic lattice $\mathcal{D}(P)$ is isomorphic to $\Sigma^{\mathcal{I}(P^{\text{op}})}$. \square

Let \mathcal{C} be the category of complete prime algebraic lattices and Scott continuous maps and \mathcal{L} the luff subcategory of \mathcal{C} of sup-preserving maps between complete prime algebraic lattices. These are the Scott continuous maps which are linear in the sense that they preserve finite sups. The inclusion $I : \mathcal{L} \hookrightarrow \mathcal{C}$ has a left adjoint $!$ sending a prime algebraic lattice L to $\mathcal{P}_H(L)_\perp$, i.e. the poset of Scott closed subsets of L ordered by \subseteq , and a Scott continuous $f : L \rightarrow K$ to the linear continuous map $!f : \mathcal{P}_H(L)_\perp \rightarrow \mathcal{P}_H(K)_\perp$ sending a Scott closed subset A of L to $f[\overline{A}]$, the Scott closure of the image of A under f . The counit $\varepsilon_L : !L \rightarrow L$ is the linear map sending a Scott closed subset A of L to its supremum $\bigsqcup A$. From the bijection

$$\mathcal{L}(!L, K) \cong \mathcal{C}(L, I(K)) = \mathcal{C}(L, K)$$

natural in L and K it follows that \mathcal{C} is equivalent to the Kleisli category $\mathcal{L}_!$ where $!$ stands for the endofunctor $! \circ I$ on \mathcal{L} . Notice, moreover, that $\mathcal{D}(P^{\text{op}}) \cong \mathcal{D}(P)^{\text{op}}$ by taking complements. Thus \mathcal{L} is a model of classical linear logic as originally observed by M. Huth. Th. Ehrhard has recently (2009) shown that this model, the so-called *Scott model of Linear Logic*, can be obtained as the extensional collapse of the relational model of linear logic.

From the point of view of linear logic it is more convenient to describe \mathcal{L} as a category whose objects are posets and whose morphism from P to Q are downclosed $f \subseteq P^{\text{op}} \times Q$. The induced sup-preserving map $\mathcal{D}(P) \rightarrow \mathcal{D}(Q)$ also denoted by f sends $a \in \mathcal{D}(P)$ to $f(a) = \{q \in Q \mid \exists p \in a (p, q) \in f\} \in \mathcal{D}(Q)$. Vice versa a sup-preserving map $f : \mathcal{D}(P) \rightarrow \mathcal{D}(Q)$ is identified with its (*Scott trace*) $\{(p, q) \in P \times Q \mid q \in f(\downarrow p)\}$.

From this point of view the category \mathcal{L} is a $*$ -autonomous category with $P^\perp = P^{\text{op}}$ and $P \otimes Q = P \times Q$. Thus $P \multimap Q = P^{\text{op}} \times Q$ which is in accordance with $\mathcal{D}(P \multimap Q) \cong \mathcal{L}(\mathcal{D}(P), \mathcal{D}(Q))$. The unit for \otimes is given by $1 = \{*\}$. Thus $\perp = 1^\perp = 1$ and we have $\mathcal{D}(\perp) = \Sigma$. Notice that $P \wp Q = P \otimes Q$ and thus in this

respect the model is degenerate. The category \mathcal{L} has also finite products given by $\top = \emptyset$ and $P \& Q = P + Q$. The model is also degenerate in the sense that $0 = \top^\perp = \emptyset$ and $P \oplus Q = P \& Q = P + Q$.

Explicitating the adjunction $! \vdash I : \mathcal{L} \hookrightarrow \mathcal{C}$ gives rise to the comonad $! : \mathcal{L} \rightarrow \mathcal{L}$ sending a poset P to the poset $!P = (\mathcal{D}_0(P), \subseteq)$ and $f : P \multimap Q$ to the morphism

$$!f = \{(a, b) \in !P \times !Q \mid \forall q \in b \exists p \in a (p, q) \in f\}$$

from $!P$ to $!Q$. The comonad structure is given by

$$\varepsilon_P = \{(a, p) \in !P \times P \mid p \in a\} : !P \multimap P$$

called “deriliction” and

$$\delta_P = \{(a, A) \in !P \times !!P \mid \bigcup A \subseteq a\} : !P \multimap !!P$$

called “digging”. Moreover, the \otimes -comonoid structure on $!P$ is given by

$$w_P = !P \times 1 : !P \multimap 1$$

called “weakening” and

$$c_P = \{(a, (a_1, a_2)) \in !P \times (!P \times !P) \mid a_1 \cup a_2 \subseteq a\} : !P \multimap !P \otimes !P$$

called “contraction”.

Notice that $\mathcal{D}(!P)$ is isomorphic to $\mathcal{P}_H(\mathcal{D}(P))_\perp$, the lifting of the Hoare powerdomain of $\mathcal{D}(P)$. Accordingly, we have $\mathcal{D}(P \rightarrow \perp) = \mathcal{D}(!P)^\perp \cong \Sigma^{\mathcal{D}(P)}$. Thus $\mathcal{D}(?P)$ is isomorphic to $\Sigma^{\mathcal{D}(P^{\text{op}})}$. Moreover, we have $\mathcal{D}(!P \multimap Q) = \mathcal{D}(P \rightarrow Q) \cong \mathcal{D}(Q)^{\mathcal{D}(P)}$ where a Scott continuous map $f : \mathcal{D}(P) \rightarrow \mathcal{D}(Q)$ is identified with its Scott trace $\text{tr}_S(f) = \{(a, q) \in !P \times Q \mid q \in f(a)\}$. If $f_1 : \mathcal{D}(P_1) \rightarrow \mathcal{D}(P_2)$ and $f_2 : \mathcal{D}(P_2) \rightarrow \mathcal{D}(P_3)$ then $\text{tr}_S(f_2 \circ f_1) = \text{tr}_S(f_2) \circ !\text{tr}_S(f_1) \circ \delta_{P_1}$. Thus \mathcal{C} is equivalent to the Kleisli category $\mathcal{L}_!$.

The interpretation of call-by-name λ -calculus in $\mathcal{L}_!$ makes no positive (pun intended!) use of the $!$. This, however, is different when considering the call-by-value interpretation of λ -calculus in \mathcal{L} as described in L. Regnier’s Thesis. As usual base types are interpreted arbitrarily but function types $A \rightarrow B$ are interpreted as $!A \multimap !B$. Sequents $x_1 : A_1, \dots, x_n : A_n \vdash t : B$ are then interpreted as morphism $!A_1 \otimes \dots \otimes !A_n \multimap !B$, i.e. as Scott continuous maps $A_1 \times \dots \times A_n \rightarrow \mathcal{P}_H(B)_\perp$ since $\mathcal{D}(!P \multimap !Q) \cong [\mathcal{D}(P) \rightarrow \mathcal{P}_H(\mathcal{D}(Q))_\perp]$. Thus λ -abstraction can be interpreted as linear functional abstraction followed by promotion and function application as

$$!(A \multimap B) \otimes !A \xrightarrow{\varepsilon_{!A \multimap !B} \otimes !A} (!A \multimap !B) \otimes !A \xrightarrow{\text{ev}_\ell} !B$$

where the second arrow is linear application. From the point of view of \mathcal{C} this interpretation can be described as follows. Again base types are interpreted arbitrarily and $A \rightarrow B$ is interpreted as $[A \rightarrow \mathcal{P}_H(B)_\perp]$. Sequents $x_1 : A_1, \dots, x_n : A_n \vdash$

$t : B$ are interpreted as Scott continuous maps $A_1 \times \dots \times A_n \rightarrow \mathcal{P}_H(B)_\perp$. Functional abstraction is interpreted as currying followed by $\{\cdot\}$. Application is interpreted as $\text{ev} : (A \rightarrow !B) \times A \rightarrow !B$ in \mathcal{C} exploiting that in \mathcal{L} we have $!((A \rightarrow !B) \times A) \cong !(A \multimap !B) \otimes !A$. More concretely, the application of Scott closed $F \subseteq A \multimap !B$ to Scott closed $G \subseteq A$ is given by $\overline{\bigcup\{f(a) \mid f \in F, a \in G\}}$.

It is surprising how different the call-by-name and call-by-value interpretations of λ -calculus are in the Scott model of linear logic. The former gives rise to the call-by-name continuation semantics à la Lafont-Reus-Streicher and the latter to the partial correctness variant of the nondeterministic λ -calculus where outputs are arbitrary Scott closed subsets (including the empty one!). In call-by-name every term is interpreted as element of a \wp -monoid of the form Σ^X and in call-by-value every term is interpreted as element of a \otimes -comonoid of the form $\mathcal{P}_H(X)_\perp$. This may explain why Hoare powerdomains do not show up in the call-by-name interpretation though they do in the call-by-value interpretation.

195 A Connected Type Theoretic Universe

Let \mathbb{C} be the category with two objects V and E whose only nontrivial morphisms are $s, t : V \rightarrow E$. Obviously $\widehat{\mathbb{C}}$ is the topos of irreflexive graphs. Let \mathbf{U} be some Grothendieck universe. This gives rise to a universe U in $\widehat{\mathbb{C}}$ where $U(V) = \mathbf{U}$, $U(E)$ is the set of spans $s_X : X \rightarrow X_0$, $t_X : X \rightarrow X_1$ and $U(s)(X) = X_0$ and $U(t)(X) = X_1$, respectively.

In this universe isomorphic elements are not necessarily equal. But, nevertheless, the object U is obviously connected since for $X_0, X_1 \in \mathbf{U}$ we have $U(s)(\pi_0^{X_0, X_1}, \pi_1^{X_0, X_1}) = X_0$ and $U(t)(\pi_0^{X_0, X_1}, \pi_1^{X_0, X_1}) = X_1$. Thus, all maps from U to $\Delta(2)$ are constant.

This shows that in Martin-Löf type theory with a universe one cannot define any non-trivial, i.e. non-constant, map from a universe U to N_2 .

196 Ω is Connected in Sheaves on $\mathbf{Sp}(2^{\mathbb{N}}, 2^{\mathbb{N}})$ and in Johnstone's Topological Topos

Let $\mathbb{M} = \mathbf{Sp}(2^{\mathbb{N}}, 2^{\mathbb{N}})$ be the monoid of continuous endomaps on Cantor space $2^{\mathbb{N}}$ endowed with the Grothendieck topology \mathcal{J} generated by finite disjoint coverings of $2^{\mathbb{N}}$ by basic open sets.¹³⁵ The sheaf $\Delta(2)$ consist of all continuous maps from $2^{\mathbb{N}}$ to 2 on which \mathbb{M} acts by precomposition.¹³⁶ The subobject classifier Ω consists of \mathcal{J} -closed sieves on which \mathbb{M} acts as follows

$$\Omega(u)(S) = S \upharpoonright u = \{v \in \mathbb{M} \mid uv \in S\}$$

for $S \in \Omega$ and $u \in \mathbb{M}$.

Lemma 1 Every morphism $f : \Omega \rightarrow \Delta(2)$ is constant.

Proof: Since for every $u \in \mathbb{M}$ we have $\top \upharpoonright u = \top$ and $\perp \upharpoonright u = \perp$ the maps $f(\top)$ and $f(\perp)$ are constant.

For $\alpha \in 2^{\mathbb{N}}$ let $c_\alpha = \lambda\beta.\alpha \in \mathbb{M}$. Obviously, the sieve $S_\alpha = \{c_\alpha\} \in \Omega$. We have $S_\alpha \upharpoonright c_\alpha = \top$ and $S_\alpha \upharpoonright c_\beta = \perp$ whenever $\alpha \neq \beta$. Thus, for $\alpha, \beta \in 2^{\mathbb{N}}$ with $\alpha \neq \beta$ we have

$$f(\top) = f(S_\alpha \upharpoonright c_\alpha) = f(S_\alpha)c_\alpha \quad \text{and} \quad f(\perp) = f(S_\alpha \upharpoonright c_\beta) = f(S_\alpha)c_\beta$$

from which it follows that $f(\top) = f(\perp)$ since otherwise $f(S_\alpha)$ were not continuous.

Suppose $S \in \Omega$ and $\alpha \in 2^{\mathbb{N}}$. We have $f(S)c_\alpha = f(S \upharpoonright c_\alpha)$. Since $S \upharpoonright c_\alpha$ is either \top or \perp it follows that $f(S)c_\alpha = f(\top)c_\alpha = f(\perp)c_\alpha$. Since $f(\top) = f(\perp)$ is constant it follows that $f(S) = f(\top) = f(\perp)$.

¹³⁵basic open sets are those of the form $U_s = \{\alpha \in 2^{\mathbb{N}} \mid \bar{\alpha}(|s|) = s\}$ for some $s \in 2^*$

¹³⁶Notice that for every set S the map $\eta_S : S \rightarrow \Gamma\Delta(S)$ is an isomorphism and thus the sheaf topos is connected. However, by Lemma C.3.3.10 of the *Elephant* it is not locally connected since there are covers which are not connected.

Thus we have shown that f is constant. □

Let \mathbb{N}^+ be the one point compactification of the discrete space \mathbb{N} and \mathbb{M} the monoid $\mathbf{Sp}(\mathbb{N}^+, \mathbb{N}^+)$ of continuous endomaps on \mathbb{N}^+ . For an infinite subset I of \mathbb{N} let f_I be the unique monotone endomap on \mathbb{N}^+ whose image is $I \cup \{\infty\}$. Johnstone in his paper *On a Topological Topos* has characterized the canonical topology on $\mathbf{Sp}(\mathbb{N}^+, \mathbb{N}^+)$ as consisting of those sieves S such that

- (1) all constant maps are in S and
- (2) for every infinite $I \subseteq \mathbb{N}$ there is an infinite $J \subseteq I$ with $f_J \in S$.

The subobject classifier Ω consists of all sieves which are closed w.r.t. the canonical topology and on which \mathbb{M} acts as $S \upharpoonright u = \{v \in \mathbb{M} \mid uv \in S\}$. As before the sheaf $\Delta(2)$ is given by the set of all continuous maps from \mathbb{N}^+ to 2 on which \mathbb{M} acts by precomposition.

Lemma 2 Every morphism $f : \Omega \rightarrow \Delta(2)$ is constant.

Proof: Since for every $u \in \mathbb{M}$ we have $\top \upharpoonright u = \top$ and $\perp \upharpoonright u = \perp$ the maps $f(\top)$ and $f(\perp)$ are constant.

For $\alpha \in \mathbb{N}^+$ let $c_\alpha = \lambda\beta.\alpha \in \mathbb{M}$. Obviously, the sieve $S_\alpha = \{c_\alpha\} \in \Omega$. We have $S_\alpha \upharpoonright c_\alpha = \top$ and $S_\alpha \upharpoonright c_\beta = \perp$ whenever $\alpha \neq \beta$. Thus, for $\alpha, \beta \in \mathbb{N}^+$ with $\alpha \neq \beta$ we have

$$f(\top) = f(S_\alpha \upharpoonright c_\alpha) = f(S_\alpha)c_\alpha \quad \text{and} \quad f(\perp) = f(S_\alpha \upharpoonright c_\beta) = f(S_\alpha)c_\beta$$

from which it follows that $f(\top) = f(\perp)$ since otherwise $f(S_\alpha)$ were not continuous.

Suppose $S \in \Omega$ and $\alpha \in \mathbb{N}^+$. We have $f(S)c_\alpha = f(S \upharpoonright c_\alpha)$. Since $S \upharpoonright c_\alpha$ is either \top or \perp it follows that $f(S)c_\alpha = f(\top)c_\alpha = f(\perp)c_\alpha$. Since $f(\top) = f(\perp)$ is constant it follows that $f(S) = f(\top) = f(\perp)$.

Thus we have shown that f is constant. □

197 A Connected Universe in Johnstone's Topological Topos

Martin Escaró has asked whether Lemma 2 of the previous section 196 extends to universes in the topological topos. In $\widehat{\mathbb{M}}$ a universe U is given by the set of contravariant functors from $\mathbb{M}/*$ to a Grothendieck universe \mathcal{U} . For $u \in \mathbb{M}$ the action $U(u)$ is given by precomposition with Σ_u^{op} . The presheaf E over \mathbb{M} has underlying set $\{\langle A, a \rangle \mid A \in U(*), a \in A(\text{id}_{\mathbb{N}^+})\}$ and the action of \mathbb{M} acts on it is given by $E(u)(\langle A, a \rangle) = \langle U(u)(A), A(u : u \rightarrow \text{id}_{\mathbb{N}^+})(a) \rangle$. The universe in $\widehat{\mathbb{M}}$ is given by $\pi : E \rightarrow U$ sending $\langle A, a \rangle$ to A . The universe in the topological topos is obtained by sheafifying π , i.e. $\mathbf{a}(\pi)$ where \mathbf{a} is the left adjoint to the inclusion of the topological topos into $\widehat{\mathbb{M}}$.

Lemma In the topological topos the universe $\mathbf{a}(U)$ is connected, i.e. every $f : \mathbf{a}(U) \rightarrow \Delta(2)$ is constant.

Proof: Since $\Delta(2)$ is a sheaf it suffices to show that in $\widehat{\mathbb{M}}$ every map $f : U \rightarrow \Delta(2)$ is constant.

For every $A \in \mathcal{U}$ let A_c be the constant presheaf on $\mathbb{M}/*$ with value A . Obviously, for every $u \in \mathbb{M}$ we have $A_c \upharpoonright u = A_c$. Thus, for every $A \in \mathcal{U}$ the map $f(A_c)$ is constant.

Let $A \in \mathcal{U}$ and $\alpha \in \mathbb{N}^+$. Let A_α be the presheaf over $\mathbb{M}/*$ with $A_\alpha(c_\alpha) = A$, $A(u) = \emptyset$ for $u \in \mathbb{M} \setminus \{c_\alpha\}$ and $A(u : c_\alpha \rightarrow c_\alpha) = \text{id}_A$ for all $u \in \mathbb{M}$. Obviously, we have $A_\alpha \upharpoonright c_\alpha = A_c$ and $A_\alpha \upharpoonright c_\beta = \emptyset_c$ for $\beta \neq \alpha$.

Thus, for $\alpha, \beta \in \mathbb{N}^+$ with $\alpha \neq \beta$ we have

$$f(A_c) = f(A_\alpha \upharpoonright c_\alpha) = f(A_\alpha)c_\alpha \quad \text{and} \quad f(\emptyset_c) = f(A_\alpha \upharpoonright c_\beta) = f(A_\alpha)c_\beta$$

from which it follows that $f(A_c) = f(\emptyset_c)$ since otherwise $f(A_\alpha)$ were not continuous.

Suppose $F \in U$ and $\alpha \in \mathbb{N}^+$. Then $F \upharpoonright c_\alpha = A_c$ for a unique $A \in \mathcal{U}$. Thus, we have $f(F)c_\alpha = f(F \upharpoonright c_\alpha) = f(A_c) = f(\emptyset_c)$. Thus, we have shown that $f(F) = f(\emptyset_c)$ for all $F \in U$, i.e. that f is constant as desired. \square

198 Reflection of the Universe is Indiscrete

(14/02/13)

In Johnstone's *On a topological topos* it is observed that the $\neg\neg$ -topology on $\mathbb{M} = \mathbf{Sp}(\mathbb{N}^+, \mathbb{N}^+)$ consists of all sieves which contain all constant endomaps of \mathbb{N}^+ . Moreover, the $\neg\neg$ -sheaves of $\widehat{\mathbb{M}}$ are up to isomorphism all presheaves of the form $\nabla(S)$ whose underlying set is $S^{\mathbb{N}^+}$ on which \mathbb{M} acts by right composition. Thus, the subtopos of $\widehat{\mathbb{M}}$ of $\neg\neg$ -sheaves is given by the geometric inclusion $\Gamma \dashv \nabla : \mathbf{Set} \hookrightarrow \widehat{\mathbb{M}}$ where Γ is the global elements functor $\widehat{\mathbb{M}}(1, -)$. For an object X in $\widehat{\mathbb{M}}$ the reflection map $\eta_X : X \rightarrow \nabla \Gamma X$ sends an $x \in X$ to the function $\eta_X(x) : \Gamma Y(*) \rightarrow \Gamma(X) : c \mapsto x \cdot c$ (notice that $\Gamma Y(*)$ is the sieve of constant endomaps on \mathbb{N}^+). Thus, instantiating X by the universe U (constructed from Grothendieck universe \mathcal{U}) we observe that $\Gamma(U)$ is in bijective correspondence with \mathcal{U} and $\eta_U : U \rightarrow \nabla \Gamma U$ is epic. For this reason the reflection of U to $\mathbf{Sep}_{\neg\neg}(U)$ coincides with $\nabla \Gamma U$ (since it is obtained by epi-mono-factorization of $\eta_U : U \rightarrow \nabla \Gamma U$), i.e. is a codiscrete space!

199 Characterization of Non-Connected Universes

(A. Simpson 2012)

Let U be a type theoretic universe. If $(\Pi X : U) \neg X \vee \neg\neg X$ then by first projection we obtain a map $f : U \rightarrow 2$ with $f(0) = 0$ and $f(1) = 1$ and which sends

isomorphic objects of U to the same element of 2 . On the other hand when given a map $f : U \rightarrow 2$ with $f(A) = f(B)$ whenever $A \cong B$ and $A, B \in U$ with $f(A) = 0$ and $f(B) = 1$ then we may consider the map $h : U \rightarrow U$ with $h(X) = (\neg X \times A) + (X \times B)$. If $x \in X \in U$ then $h(X) \cong B$ and thus $(f \circ h)(X) = 1$. If $\neg X$ then $h(X) \cong A$ and thus $(f \circ h)(X) = 0$. Thus by contraposition we get that $(f \circ h)(X) = 0$ implies $\neg X$ and $(f \circ h)(X) = 1$ implies $\neg\neg X$, i.e. $(\prod X:U) \neg X \vee \neg\neg X$.

Thus $(\prod X:U) \neg X \vee \neg\neg X$ fails to hold iff all iso-preserving maps $U \rightarrow 2$ are constant, i.e. U is connected.

200 Small Maps are Stable under Localization

Let \mathcal{E} be a topos and $\mathcal{F} = \mathcal{E}_j$ be a subtopos. We write \mathbf{a} for the left adjoint (sheafification) of the inclusion $i : \mathcal{F} \hookrightarrow \mathcal{E}$. Suppose \mathcal{S} is a class of small maps in \mathcal{E} containing all monos. If \mathcal{S} is closed under sheafification then $\mathcal{S} \cap \mathcal{F}$ is a class of small maps for \mathcal{F} . Most of the desired properties of $\mathcal{S} \cap \mathcal{F}$ are immediate. We only show descent and collection axiom.

For showing descent suppose $a : A \rightarrow I$ is in \mathcal{F} and $e : J \rightarrow I$ is a regular epi in \mathcal{F} with $e^*a \in \mathcal{S}$. Let $e = m \circ c$ where $c : J \rightarrow K$ is a regular epi in \mathcal{E} and $m : K \rightarrow I$ is a j -dense mono. Consider the diagram

$$\begin{array}{ccccc} \cdot & \longrightarrow & \cdot & \xrightarrow{q} & A \\ e^*a \downarrow \lrcorner & & a' \downarrow \lrcorner & & \downarrow a \\ J & \xrightarrow{c} & K & \xrightarrow{m} & I \end{array}$$

in \mathcal{E} . By descent for \mathcal{S} the map a' is in \mathcal{S} . Since sheafification preserves finite limits and m is j -dense we have

$$\begin{array}{ccc} \cdot & \xrightarrow{\mathbf{a}(q)} & A \\ \mathbf{a}(a') \downarrow \lrcorner & & \downarrow a \\ \mathbf{a}(K) & \xrightarrow[\cong]{\mathbf{a}(m)} & I \end{array}$$

from which it follows that $a \in \mathcal{S}$ since $\mathbf{a}(a') \in \mathcal{S}$.

For showing collection suppose $a : A \rightarrow I$ is in $\mathcal{S} \cap \mathcal{F}$ and $e : C \rightarrow A$ is a regular epi in \mathcal{F} . Let $e = m \circ c$ where $c : C \rightarrow D$ is a regular epi in \mathcal{E} and $m : D \rightarrow A$ is a j -dense mono. Since collection holds for \mathcal{S} there is a quasi-pullback

$$\begin{array}{ccccc} B & \xrightarrow{q} & C & \xrightarrow{c} & D \\ b \downarrow & & & & \downarrow a \circ m \\ J & \xrightarrow{p} & & & I \end{array}$$

in \mathcal{E} with $b \in \mathcal{S}$ and p a regular epi. Since sheafification preserves smallness, regular epis and quasi-pullbacks and $\mathbf{a}(m)$ is an iso by applying \mathbf{a} to the previous diagram we obtain the quasi-pullback

$$\begin{array}{ccccc} \mathbf{a}(B) & \xrightarrow{\mathbf{a}(q)} & C & \xrightarrow{e} & A \\ \mathbf{a}(b) \downarrow & & & & \downarrow a \\ \mathbf{a}(J) & \xrightarrow{\mathbf{a}(p)} & & & I \end{array}$$

in \mathcal{F} where $\mathbf{a}(b) \in \mathcal{S}$ and $\mathbf{a}(p)$ is a regular epi in \mathcal{F} .

201 Lawvere's Fixpoint Theorem

Let \mathcal{C} be a cartesian closed category and $e : A \rightarrow B^A$ with $\Gamma(e) : \Gamma(A) \rightarrow \Gamma(B^A)$ surjective. Then every $f : B \rightarrow B$ has a fixpoint. Define $g : A \rightarrow B$ as $\lambda x:A.f(exx)$. Then there exists a global element a of A with $e(a) = g$. Then we have $ea a = g(a) = f(eaa)$, i.e. $ea a$ is a fixpoint of f .

If \mathcal{C} is a topos by internalizing the above argument one can show that for every epic $e : A \rightarrow B^A$ it holds that $\forall f:B^B.\exists y:B.y = f(y)$. Thus, if B admits an endomap not having a fixpoint then there cannot exist a surjection from A to B^A . In particular, there is no object A with a surjection from A to 2^A or $\Omega^A = \mathcal{P}(A)$ (as shown already by G. Cantor in the 19th century).

202 Uncountability of \mathbb{R} fails in $\text{Sh}(\mathbb{R})$

as shown by Rosolini and Spitters. Of course, like in a any topos there cannot exist a surjection from \mathbb{N} to $\mathcal{P}(\mathbb{N})$ but what fails is the statement

$$\forall x \in \mathbb{R}^{\mathbb{N}}.\exists y \in \mathbb{R}.\forall n \in \mathbb{N}.\exists m \in \mathbb{N}.|x_n - y| \geq \frac{1}{m}$$

for Dedekind reals \mathbb{R} .

Now if this would hold in $\text{Sh}(\mathbb{R})$ by Kripke-Joyal for every open U and $x \in \mathbb{R}^{\mathbb{N}}(U)$, i.e. any continuous function $x : U \times \mathbb{N} \rightarrow \mathbb{R}$, there exists a cover $(V_i)_{i \in I}$ of U and continuous $y_i : V_i \rightarrow \mathbb{R}$ such that $V_i \Vdash \forall n \in \mathbb{N}.\exists m \in \mathbb{N}.|x_n - y_i| \geq \frac{1}{m}$. But then for every $n \in \mathbb{N}$ there exists a cover $(W_{i,j})_{j \in J_i}$ of V_i and $m_{i,n,j} \in \mathbb{N} \setminus \{0\}$ such that

$$|x_n - y_i| \geq \frac{1}{m_{i,n,j}}$$

on $V_{i,j}$.

But for $U = \mathbb{R}$ and x a sequence of functions containing all $f_q(x) = q + x$ and $g_q(x) = q - x$ for $q \in \mathbb{Q}$ this is impossible for the following reason. For all nonempty open $V \subseteq \mathbb{R}$ and continuous maps $y : V \rightarrow \mathbb{R}$ there is an x_n such that x_n and y intersect, i.e. have the same value for some argument in V .

The latter claim can be seen as follows. Suppose $f_q(t) \neq y(t)$ and $g_q(t) \neq y(t)$ for all $t \in V$ and $q \in \mathbb{Q}$, i.e. for all $t \in V$ neither $y(t) - t$ nor $y(t) + t$ are elements of \mathbb{Q} . Since by the intermediate value theorem every non-constant continuous map from V to \mathbb{R} attains a rational value there exist $a, b \in \mathbb{R}$ with $y(t) - t = a$ and $y(t) + t = b$ for all $t \in V$. But then $2t = b - a$ for all $t \in V$ which is impossible since V being open and nonempty contains infinitely many elements.

203 Stone's Representation Theorem

A locale A is called *coherent* iff it is isomorphic to the *ideal completion* $\text{Idl}(D)$ of some distributive lattice D . A space X is called *coherent* or *spectral* iff it is sober and $\mathcal{O}(X)$ is a coherent locale. A spectral map between spectral spaces is a continuous map whose inverse image preserves compactness of open sets. The ensuing category **Spec** is equivalent to the dual of the category **DLat** of distributive lattices where for a spectral space X the corresponding distributive lattice is given by the poset $\mathcal{KO}(X)$ of compact open subsets of X ordered by \subseteq . For a distributive lattice D the corresponding spectral space $\text{spec}(D)$ consists of all prime ideals and its topology is given by the base consisting of all $D_a = \{I \in \text{spec}(D) \mid a \notin I\}$ with $a \in D$.

A spectral space X is Hausdorff iff $\mathcal{KO}(X)$ is a boolean algebra. Such spaces are called *Stone spaces* and can be characterized as those compact Hausdorff spaces whose clopen subsets form a basis for the topology. Obviously $\mathcal{KO}(X)$ consist precisely of the clopen subset of X . Since inverse images of continuous maps preserve clopen sets the duality between **Spec** and **DLat** restricts to one between **Bool** and **Stone** where the latter is the full subcategory of **Sp** on Stone spaces.

The duality between **Bool** and **Stone** is induced by the “schizophrenic” object 2 which as an object of **Bool** is the 2 element boolean algebra and as an object of **Sp** is the 2 element discrete space. For a Stone space X the boolean algebra of clopens is given by $\text{Stone}(X, 2)$ and for a boolean algebra B its spectrum is given by $\text{Bool}(B, 2)$, the space of points of the locale $\text{Idl}(B)$.

204 Kripke's Schema

claims that

$$\exists \alpha (A \leftrightarrow \exists n (\alpha(n) = 0))$$

i.e. that every proposition A is equivalent to $\exists n (\alpha(n) = 0)$ for some $\alpha \in \mathbb{N}^{\mathbb{N}}$. The idea is that α describes the “creative subject” (i.e. idealized mathematician) trying to prove A where $\alpha(n) = 0$ iff he has “proved” A at time n . In other words KS (Kripke's schema) says that every proposition is equivalent to a Σ_1^0 proposition (however α is not required to be primitive recursive). J. Myhill has shown that KS refutes both Church's Thesis and CONT_2 .

From KS (with parameters) it follows that there exists an $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $\{n\}(n) \uparrow$ iff $\exists m (\alpha(\langle n, m \rangle) = 0)$. By Church's Thesis such an α has to be recursive from which it follows¹³⁷ that $K = \{n \in \mathbb{N} \mid \{n\}(n) \downarrow\}$ is decidable. Contradiction!

From KS (with parameters) and CONT_2 there follows the existence of a continuous operator Φ on $\mathbb{N}^{\mathbb{N}}$ such that

$$(*) \quad \alpha = \lambda n.0 \leftrightarrow \exists k (\Phi(\alpha)(k) = 0)$$

for all $\alpha \in \mathbb{N}^{\mathbb{N}}$. For $\alpha = \lambda n.0$ there is a $k \in \mathbb{N}$ with $\Phi(\lambda n.0)(k) = 0$. By continuity of Φ there is an $m \in \mathbb{N}$ such that $\Phi(\alpha)(k) = 0$ for all α with $\alpha(i) = 0$ for $i < m$. Then for $\alpha = 0^m 1^\infty$ we have $\Phi(\alpha)(k) = 0$ but $\alpha \neq \lambda n.0$ contradicting (*).

¹³⁷see footnote 10 of Kreisel's *Church's thesis: A kind of reducibility axiom for constructive mathematics* for details

205 Cartesian Functors to Discrete Fibrations are Fibrations

Suppose $P : \mathbb{X} \rightarrow \mathbb{B}$ and $Q : \mathbb{Y} \rightarrow \mathbb{B}$ are fibrations and $F : Q \rightarrow P$ is a cartesian functor over \mathbb{B} . If P is discrete, i.e. all vertical arrows are identities, then F is a fibration itself.

Proof: Suppose $Y \in \mathbb{Y}$ and $f : X \rightarrow FY$ is a morphism in \mathbb{X} . Since Q is a fibration there exists a Q -cartesian arrow $\varphi : Z \rightarrow Y$ in \mathbb{Y} above $P(f)$. Since F is cartesian $F(\varphi) : FZ \rightarrow FY$ is P -cartesian. We have $P(F(\varphi)) = Q(\varphi) = P(f)$ and thus both $F(\varphi)$ and f are morphism to FY over $P(f)$. Since P is a discrete fibration it follows that $F(\varphi) = f$. It remains to show that φ is F -cartesian. For this purpose suppose $g : U \rightarrow X$ and $\psi : V \rightarrow Y$ with $F(\psi) = F(\varphi)g$. Then $Q(\psi) = Q(\varphi)P(g)$ and thus, since Q is a fibration, there exists a unique $\theta : V \rightarrow Z$ with $\varphi\theta = \psi$ and $Q(\theta) = P(g)$. Thus $F(\varphi)g = F(\psi) = F(\varphi)F(\theta)$ from which it follows that $F(\theta) = g$ since P is a discrete fibration. Suppose $\tilde{\theta} : V \rightarrow Z$ with $\varphi\tilde{\theta} = \psi$ and $F(\tilde{\theta}) = g$. Then $Q(\tilde{\theta}) = P(F(\tilde{\theta})) = P(g)$. Thus $\theta = \tilde{\theta}$ as desired. \square

In general F will not be a fibration. For example if \mathbb{B} is nontrivial, $Q = \text{Id}_{\mathbb{B}}$ and F is right adjoint to P , i.e. F picks a terminal object in each fibre, then F is not a fibration unless all fibres are equivalent to $\mathbb{1}$. In particular, if \mathbb{B} is the ordinal $\mathbb{2}$ and P is the fundamental fibration $P_{\mathbb{B}}$ of \mathbb{B} then $1 : \text{Id}_{\mathbb{B}} \rightarrow P_{\mathbb{B}}$ is not a fibration. Thus, it is not sufficient to require that P is faithful, i.e. that P is a posetal fibration, for F being a fibration, too.

206 “Generic Set” for Localic Geometric Morphisms

A geometric morphism $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ between elementary toposes is localic iff every $a : A \rightarrow FI$ in \mathcal{E} fits into a diagram

$$\begin{array}{ccc} A & \xleftarrow{e} & C \\ \downarrow a & & \downarrow m \\ FI & \xleftarrow{Fu} & FJ \end{array}$$

where e is epic and m is monic. Every mono $m : C \rightarrow FJ$ arises as pullback of $\top_{\mathcal{E}}$ along a unique $c : FJ \rightarrow \Omega_{\mathcal{E}}$. By transposition there exists a unique $v : J \rightarrow U\Omega_{\mathcal{E}}$ with

$$\begin{array}{ccccc} C & \xrightarrow{\quad} & G & \xrightarrow{\quad} & 1_{\mathcal{E}} \\ \downarrow m & \lrcorner & \downarrow m_G & \lrcorner & \downarrow \top_{\mathcal{E}} \\ FJ & \xrightarrow{Fv} & FU\Omega_{\mathcal{E}} & \xrightarrow{\varepsilon_{\Omega_{\mathcal{E}}}} & \Omega_{\mathcal{E}} \end{array}$$

where $\varepsilon_{\Omega_{\mathcal{E}}} Fv = c$. Thus, geometric morphism $F \dashv U$ is localic iff $m_G : G \rightarrow FU\Omega_{\mathcal{E}}$ is a generating family for the fibration $P_F = \partial_1 : \mathcal{E} \downarrow F \rightarrow \mathcal{S}$.¹³⁸

Notice that $A = U\Omega_{\mathcal{E}}$ is a complete Heyting algebra in \mathcal{S} with $\mathcal{E} \simeq \mathbf{Sh}_{\mathcal{S}}(A)$. In case $\mathcal{S} = \mathbf{Set}$ we have $\Delta(I) = \coprod_I 1_{\mathcal{E}}$ and $\Gamma(X) = \mathcal{E}(1, X)$ and thus $G = \coprod_{a \in A} y(a)$ and $m_G = \coprod_{a \in A} !_{y(a)} : \coprod_{a \in A} y(a) \rightarrow \coprod_{a \in A} 1_{\mathcal{E}}$ where $!_{y(a)} : y(a) \rightarrow 1_{\mathcal{E}}$ is the subterminal object corresponding to a . This G corresponds to the “generic set” (considered in forcing and Heyting valued models) since $\varepsilon_{\Omega_{\mathcal{E}}} : \coprod_{a \in A} 1_{\mathcal{E}} \rightarrow \Omega_{\mathcal{E}}$ is the source tupling of all global elements $a : 1_{\mathcal{E}} \rightarrow \Omega_{\mathcal{E}}$, i.e. $\varepsilon_{\Omega_{\mathcal{E}}}(a) = a$.

¹³⁸Recall that $F \dashv U$ is a *bounded* geometric morphism iff there is an object S in \mathcal{E} such that

$$\begin{array}{ccccc} & & G_S & \xrightarrow{\quad} & \exists_S \\ & \swarrow g_S & \downarrow & \lrcorner & \downarrow \\ FUP(S) & \xleftarrow{\pi} & FUP(S) \times S & \xrightarrow{\varepsilon_{\mathcal{P}(S)} \times S} & \mathcal{P}(S) \times S \end{array}$$

g_S is a generating family for $P_F = \partial_1 : \mathcal{E} \downarrow F \rightarrow \mathcal{S}$.

Suppose $F' \dashv U' : \mathcal{F} \rightarrow \mathcal{E}$ is another geometric morphism. Then we have

$$\begin{array}{ccccc}
 F'G & \longrightarrow & F'1_{\mathcal{E}} & \longrightarrow & 1_{\mathcal{F}} \\
 F'm_G \downarrow & \lrcorner & F'\top_{\mathcal{E}} \downarrow & \lrcorner & \downarrow \top_{\mathcal{F}} \\
 F'FU\Omega_{\mathcal{E}} & \xrightarrow{F'\varepsilon_{\Omega_{\mathcal{E}}}} & F'\Omega_{\mathcal{E}} & \xrightarrow{\tau} & \Omega_{\mathcal{F}}
 \end{array}$$

Notice that the transpose of the bottom arrow is

$$U'(\tau \circ F'\varepsilon_{\Omega_{\mathcal{E}}}) \circ \eta_{FU\Omega_{\mathcal{E}}} = U'\tau \circ U'F'\varepsilon_{\Omega_{\mathcal{E}}} \circ \eta_{FU\Omega_{\mathcal{E}}} = U'\tau \circ \eta_{\Omega_{\mathcal{E}}} \circ \varepsilon_{\Omega_{\mathcal{E}}} = \check{\tau} \circ \varepsilon_{\Omega_{\mathcal{E}}}$$

where $\check{\tau} = U'\tau \circ \eta_{\Omega_{\mathcal{E}}} : \Omega_{\mathcal{E}} \rightarrow U'\Omega_{\mathcal{F}}$ is the lower transpose of τ .¹³⁹ The further transpose of $\check{\tau} \circ \varepsilon_{\Omega_{\mathcal{E}}}$ is $U\check{\tau} : U\Omega_{\mathcal{E}} \rightarrow UU'\Omega_{\mathcal{F}}$ in \mathcal{S} which as shown by A. Joyal (unpublished¹⁴⁰) is a frame morphism in \mathcal{S} . Moreover, for every geometric morphism $f : \mathcal{F} \rightarrow \mathcal{S}$ every frame morphism $U\Omega_{\mathcal{E}} \rightarrow f_*\Omega_{\mathcal{F}}$ arises this way from a (unique up to isomorphism) geometric morphism $F' \dashv U' : \mathcal{F} \rightarrow \mathcal{E}$ with $f^* \cong F'F$ (and $f_* \cong UU'$).

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207 Cocartesian cartesian functors between Moens fibrations need not have cartesian right adjoints

Let \mathbb{B} be the category of countable sets and functions between them. Then $P_{\mathbb{B}} = \partial_1 : \mathbb{B}^2 \rightarrow \mathbb{B}$ is a Moens fibrations. Let $F : \mathbb{B}^2 \rightarrow \mathbb{B}^2$ be the functor sending

$$\begin{array}{ccc}
 B & \xrightarrow{f} & A \\
 b \downarrow & & \downarrow a \\
 J & \xrightarrow{u} & I
 \end{array}$$

¹³⁹If $p : X \rightarrow \Omega_{\mathcal{E}}$ classifies a mono $m : P \rightarrow X$ in \mathcal{E} then $\check{\tau} \circ p : X \rightarrow U'\Omega_{\mathcal{F}}$ is the transpose of the classifying map for $F'm$ in \mathcal{E} as is obvious from

$$\begin{array}{ccccc}
 F'P & \longrightarrow & F'1_{\mathcal{E}} & \longrightarrow & 1_{\mathcal{F}} \\
 F'm \downarrow & \lrcorner & F'\top_{\mathcal{E}} \downarrow & \lrcorner & \downarrow \top_{\mathcal{F}} \\
 F'X & \xrightarrow{F'p} & F'\Omega_{\mathcal{E}} & \xrightarrow{\tau} & \Omega_{\mathcal{F}}
 \end{array}$$

¹⁴⁰but see Fourman's *Continuous Truth II* (2013)

to (the outer square of)

$$\begin{array}{ccc}
 B \times N & \xrightarrow{f \times N} & A \times N \\
 b \times N \downarrow & & \downarrow a \times N \\
 J \times N & \xrightarrow{u \times N} & I \times N \\
 \pi \downarrow & \lrcorner & \downarrow \pi \\
 J & \xrightarrow{u} & I
 \end{array}$$

where N is an infinite object of \mathbb{B} . Since $(-)\times N : \mathbb{B} \rightarrow \mathbb{B}$ preserves pullbacks F is a cartesian functor from $P_{\mathbb{B}}$ to $P_{\mathbb{B}}$ which, obviously, is also cocartesian (since $(-)\times N$ like any functor preserves isomorphisms). However, the cartesian functor F does not have a cartesian right adjoint since $(-)\times N : \mathbb{B} \rightarrow \mathbb{B}$ does not have a right adjoint.

208 In a locally connected topos every object is isomorphic to the sum of its connected components

Let $f : \mathcal{E} \rightarrow \mathcal{S}$ be a locally connected geometric morphism, i.e. $f_! \dashv f^* \dashv f_*$ such that

$$\begin{array}{ccc}
 \begin{array}{ccc}
 B & \xrightarrow{g} & A \\
 b \downarrow & \lrcorner & \downarrow a \\
 f^* J & \xrightarrow{f^* u} & f^* I
 \end{array} & \text{implies} & \begin{array}{ccc}
 f_! B & \xrightarrow{f_! g} & f_! A \\
 \widehat{b} \downarrow & \lrcorner & \downarrow \widehat{a} \\
 J & \xrightarrow{u} & I
 \end{array}
 \end{array}$$

which condition expresses that $f_!$ is a fibred left adjoint to f^* .

For $a : A \rightarrow f^* I$ we have

$$\begin{array}{ccc}
 A & \overset{\text{=====}}{=} & A \\
 \eta_A \downarrow & & \downarrow a \\
 f^* f_! A & \xrightarrow{f^* \widehat{a}} & f^* I
 \end{array}$$

expressing that a is the sum of its connected components. The family of connected components of a is given by η_A which actually is a family of connected objects since $\widehat{\eta}_A$ is $\text{id}_{f_! A}$.

209 Co-Geometric Morphisms between Finite Limit Categories

As known from Moens’s Theorem for a finite limit category \mathbb{B} fibrations $P : \mathbb{X} \rightarrow \mathbb{B}$ of finite limit categories with stable and disjoint internal sums are equivalent to fibrations of the form $P_F = F^*P_{\mathbb{B}} = \partial_1 : \mathbb{C} \downarrow F \rightarrow \mathbb{B}$ where \mathbb{C} has and F preserves finite limits (notice that $FI = \Delta(I) = \coprod_I 1_I$ is determined by P). We write Δ_F for the cartesian functor from $P_{\mathbb{B}}$ to P_F which sends $u : J \rightarrow I$ to Fu . It is known that F has a right adjoint U iff $\Delta_F : P_{\mathbb{B}} \rightarrow P_F$ has a fibered right adjoint Γ_F (which at $I \in \mathbb{B}$ is given by $\eta_I^* \circ U_{/FI}$).

Thus, geometric morphisms to \mathbb{B} are Moens fibrations P over \mathbb{B} for which $\Delta : P_{\mathbb{B}} \rightarrow P$ has a fibered right adjoint Γ . One may now ask for the dual notion of a “co-geometric morphism” which should correspond to a Moens fibration P over \mathbb{B} where $\Delta : P_{\mathbb{B}} \rightarrow P$ has a fibered left adjoint Π , a “connected components” functor for P . Now if $F : \mathbb{B} \rightarrow \mathbb{C}$ is a finite limit preserving functor to a finite limit category \mathbb{C} then $\Delta_F : P_{\mathbb{B}} \rightarrow P_F$ has a fibered left adjoint Π_F iff F has a left adjoint L such that

$$\begin{array}{ccc}
 \begin{array}{ccc}
 B & \xrightarrow{f} & A \\
 b \downarrow & \lrcorner & \downarrow a \\
 FJ & \xrightarrow{Fu} & FI
 \end{array} & \text{implies} & \begin{array}{ccc}
 LB & \xrightarrow{Lf} & LA \\
 \hat{b} \downarrow & \lrcorner & \downarrow \hat{a} \\
 J & \xrightarrow{u} & I
 \end{array}
 \end{array}$$

where $\hat{}$ stands for taking the upper transpose. This property of the adjunction $L \dashv F$ is also known as “Funk’s Frobenius Reciprocity” and it is known from work of Ch. Townsend as “stably Frobenius” since, as one checks easily, it is equivalent to the requirement that for every $I \in \mathbb{B}$ the adjunction $L_I \dashv F_{/I}$ has the ordinary Frobenius reciprocity property.

Of course, if F has also a right adjoint U then the existence of Π_F means that the geometric morphism $F \dashv U$ is *locally connected*. But, of course, for Moens fibrations $P : \mathbb{X} \rightarrow \mathbb{B}$ one may study the independent¹⁴¹ properties that $\Delta_P : P_{\mathbb{B}} \rightarrow P$ has a fibered left adjoint Π_P or a fibered right adjoint Γ_P , respectively. Of course, the left adjoint Π_P is a “connected components” functor and Γ_P is a “global elements” functor for P .¹⁴²

One can show easily that co-geometric morphisms between finite limit categories are closed under composition.

¹⁴¹Let \mathcal{E} be the topos of reflexive graphs and $\Delta \dashv \Gamma \dashv \nabla : \mathbf{Set} \leftrightarrow \mathcal{E}$ then P_{∇} is locally connected but does not have small global sections. The same applies to $\Gamma \dashv \nabla : \mathbf{Set} \leftrightarrow \mathcal{E}$ where \mathcal{E} is a realizability topos.

Even more well known is the fact that not every geometric morphism is locally connected.

¹⁴²If \mathbb{C} is locally cartesian closed then P_F is a fibration of cartesian closed categories and thus existence of Γ_F is equivalent to P_F being locally small.

210 Weak vs. Strong Equivalences

A functor $F : \mathcal{A} \rightarrow \mathcal{B}$ is a *weak equivalence* iff F is full and faithful and *essentially surjective*, i.e. for every $Y \in \mathcal{B}$ there is an $X \in \mathcal{A}$ such that $Y \cong FX$. A functor $F : \mathcal{A} \rightarrow \mathcal{B}$ is a (*strong*) *equivalence* iff there exists a functor $G : \mathcal{B} \rightarrow \mathcal{A}$ with $GF \cong \text{Id}_{\mathcal{A}}$ and $FG \cong \text{Id}_{\mathcal{B}}$. Such a G is called a *quasi-inverse* of F . Of course, every strong equivalence is also a weak equivalence. But the reverse direction requires an axiom of choice for classes which can be seen as follows. Let \mathcal{A} be an *elementary* category, i.e. a posetal groupoid. Let \mathcal{B} be the discrete category arising from \mathcal{A} by identifying isomorphic objects and $Q : \mathcal{A} \rightarrow \mathcal{B}$ the corresponding quotient map. A quasi-inverse of Q is a section $S : \mathcal{B} \rightarrow \mathcal{A}$ of Q . Obviously, existence of S requires axiom of choice for classes.

Consider e.g. the Sierpiński topos $\mathcal{S} = \widehat{2}$ and the object $A = \Delta(2)$ in \mathcal{S} . Let E be the equivalence relation on A with $E_1 = \{(i, i) \mid i \in 2\}$ and $E_0 = 2 \times 2$. Let B be the quotient of A modulo E and $e : A \rightarrow B$ the corresponding quotient map which does not have a section $s : B \rightarrow A$ since otherwise $s_1 = \text{id}_2$ and thus $0 = s_0(*) = 1$ where $*$ is the unique element of B_0 , i.e.

$$\begin{array}{ccc}
 2 & \xlongequal{\quad} & 2 \\
 \parallel & & \downarrow \\
 2 & \xleftarrow{\quad s_0 \quad} & 1
 \end{array}$$

in diagrammatic terms, which clearly is impossible. Let \mathcal{A} be the elementary category in \mathcal{S} corresponding to the equivalence relation E , \mathcal{B} the discrete category in \mathcal{S} corresponding to B and $F : \mathcal{A} \rightarrow \mathcal{B}$ the internal functor corresponding to the epimorphism $e : A \rightarrow B$. Obviously F is a weak equivalence internal to \mathcal{S} and any quasi-inverse G of F in \mathcal{S} would give rise to a section s of e which does not exist in \mathcal{S} . Thus F is a weak equivalence in \mathcal{S} which, however, is not a strong equivalence.

In a mail to the CATEGORIES list in April 2013 J. Bénabou has observed that “Non evil is essentially evil” because for getting sufficiently many equivalences for replacing the “evil” concept of equality of objects by the “good” concept of equivalence requires the “very evil” principle of choice for classes.

211 Sobriety inside Equiological Spaces

(Bucalo & Rosolini)

Let (T, τ) be a topological T_0 where τ is the set of open sets of T . Inside **PEqu** the space (T, τ) gets represented as $(\mathcal{P}(\tau), \delta_T)$ where $V \delta_T W$ iff $\exists x \in T. V = \mathcal{N}(x) = W$.¹⁴³ Our aim is to determine the set $[\Sigma^{(T, \tau)}, \Sigma]$ of algebra morphisms (w.r.t. the continuation monad $\Sigma^{\Sigma^{(-)}}$).

After some explicitation one observes that the underlying algebraic lattice of $\Sigma^{(T, \tau)}$ is $\Sigma^{\mathcal{P}(\tau)}$. The equivalence relation on $\Sigma^{\mathcal{P}(\tau)}$ is the kernel of the map

$$q : \Sigma^{\mathcal{P}(\tau)} \rightarrow \tau : M \mapsto \bigcup_{\substack{F \in M \\ F \text{ finite}}} \bigcap F$$

since $q(M) = q(N)$ iff $\forall x \in T. \mathcal{N}_x \in M \leftrightarrow \mathcal{N}_x \in N$.

After some explicitation one observes that an algebra morphism from $\Sigma^{(T, \tau)}$ to Σ is a Scott continuous map $h : \Sigma^{\mathcal{P}(\tau)} \rightarrow \Sigma$ with $h(M) = h(N)$ whenever $q(M) = q(N)$ such that $h(M) = M(h \circ \eta_\tau)$ for all $M \in \Sigma^{\mathcal{P}(\tau)}$. Since h has to respect the equality on $\Sigma^{\mathcal{P}(\tau)}$ it is of the form $f \circ q$ for a unique $f : \tau \rightarrow \Sigma$. This allows one to further simplify the description of $[\Sigma^{(T, \tau)}, \Sigma]$. It consists of all maps $f : \tau \rightarrow \Sigma$ such that for all¹⁴⁴ $M \in \Sigma^{\mathcal{P}(\tau)}$ it holds that $f(q(M)) = M(f^{-1}(\top))$. Actually, it is a straightforward exercise to show that $f : \tau \rightarrow \Sigma$ is a frame morphism iff for all $M \in \Sigma^{\mathcal{P}(\tau)}$ it holds that $f(q(M)) = M(f^{-1}(\top))$.

Proof: Suppose $f : \tau \rightarrow \Sigma$ is a frame morphism and $M \in \Sigma^{\mathcal{P}(\tau)}$. Then we have $f(q(M)) = \top$ iff \exists finite $F \in M. f(\bigcap F) = \top$ iff \exists finite $F \in M. \forall U \in F. f(U) = \top$ iff $M(f^{-1}(\top))$.

For the reverse direction suppose $f : \tau \rightarrow \Sigma$ with $f(q(M)) = M(f^{-1}(\top))$ for all $M \in \Sigma^{\mathcal{P}(\tau)}$. We first show that f preserves finite meets. Suppose $F \subseteq_{\text{fin}} \tau$. Consider $M \in \Sigma^{\mathcal{P}(\tau)}$ with $M(W) = \top$ iff $F \subseteq W$. Obviously, we have $q(M) = \bigcap F$. Thus, by assumption on f we have $f(\bigcap F) = \top$ iff $f(q(M)) = \top$ iff $M(f^{-1}(\top)) = \top$ iff $F \subseteq f^{-1}(\top)$ iff $\bigcap f[F] = \top$. It remains to show that f preserves also arbitrary suprema. Suppose $S \subseteq \tau$. Consider $M \in \Sigma^{\mathcal{P}(\tau)}$ with $M(W) = \top$ iff $W \cap M \neq \emptyset$. Obviously, we have $q(M) = \bigcup S$. By assumption of f we have $f(\bigcup S) = \top$ iff $f(q(M)) = \top$ iff $M(f^{-1}(\top)) = \top$ iff $S \cap f^{-1}(\top) \neq \emptyset$ iff $\bigcup f[S] = \top$. \square

One knows that $[\Sigma^{(T, \tau)}, \Sigma]$ appears as equalizer of the maps $\eta_{\Sigma^2(T, \tau)}, \Sigma^2(\eta_{(T, \tau)}) : \Sigma^2(T, \tau) \rightarrow \Sigma^4(T, \tau)$. In my paper with Gruenhage in the Keimel Festschrift it has been shown that in $\omega\mathbf{PEqu}$ the equalizer of $\eta_{\Sigma^2(X)}$ and $\Sigma^2(\eta_X)$ does not always give an object whose Σ -topology is sober. Thus, the result cannot be generalised to all objects of **PEqu** (since the embedding of $\omega\mathbf{PEqu}$ into **PEqu** preserves all relevant structure).

¹⁴³where $\mathcal{P}(\tau)$ is considered as an algebraic lattice w.r.t. \subseteq and $\mathcal{N}_x = \{U \in \tau \mid x \in U\}$

¹⁴⁴since they automatically respect δ_T

212 Taylor’s “Abstract Stone Duality” (ASD)

Let \mathcal{C} be a category with finite products and an object Σ which is *basable* in the sense that Σ^X exists for all $X \in \mathcal{C}$. The self adjunction $\Sigma^{(-)}$ gives rise to the double dualization monad $\Sigma^2(-)$ on \mathcal{C} . Let \mathcal{A} be the category of algebras for this monad. There is a comparison functor $K : \mathcal{C}^{\text{op}} \rightarrow \mathcal{A}$ sending X in \mathcal{C} to the algebra $(\Sigma(X), \Sigma(\eta_X))$.

One can show that for every algebra (A, α) a map $P : Y \rightarrow \Sigma(A)$ equalizes the maps $\Sigma(\alpha), \eta_{\Sigma(A)} : \Sigma(A) \rightarrow \Sigma^3(A)$ iff its double transpose $H : A \rightarrow \Sigma(Y)$ is a homomorphism of algebras. Instantiating A by the free algebra $\Sigma(X)$ with structure map $\Sigma(\eta_X)$ one can show that η_X is an equalizer of $\Sigma^2(\eta_X)$ and $\eta_{\Sigma^2(X)}$ iff every algebra morphism from $\Sigma(X)$ to $\Sigma(Y)$ is of the form $\Sigma(f)$ for a unique $f : Y \rightarrow X$. Thus, the functor K is full and faithful iff for all X in \mathcal{C} the morphism η_X is an equalizer of $\Sigma^2(\eta_X)$ and $\eta_{\Sigma^2(X)}$. Moreover, under this assumption the comparison functor K is essentially surjective iff every algebra (A, α) is isomorphic to $(\Sigma(X), \Sigma(\eta_X))$ via the double transpose of the equalizer $X \rightarrow \Sigma(A)$ of $\Sigma(\alpha)$ and $\eta_{\Sigma(A)}$.

Alas, the only known (non-syntactic) model of ASD is the category of locally compact spaces (equivalent to the category of locally compact locales i.e. locales whose underlying lattice is continuous) with Sierpiński space Σ .

213 An Analogy Breaks! - Actually Not!

Generally locally presentable categories are considered as “glorified”, i.e. categorical, versions of algebraic lattices. As shown in the book by Makkai & Paré (following Gabriel & Ulmer) every Grothendieck topos is in particular also locally presentable.

Locally presentable categories can be characterized as reflective subcategories of presheaf categories where the right adjoint is accessible, i.e. preserves filtered colimits. Grothendieck toposes may be characterized as localizations¹⁴⁵ of presheaf categories.

The posetal version of presheaf categories are complete prime algebraic lattices, i.e. posets $D(P)$ of downward closed subsets of a poset P ordered by subset conclusion. The posetal analogon of locally presentable categories are algebraic lattices since they arise as images of complete prime algebraic lattices under Scott continuous closure operators. The posetal analogon of Grothendieck toposes are locales, i.e. complete Heyting algebras, since they arise as localizations of complete prime algebraic lattices.

However, not every locale is an algebraic lattice, e.g. $\mathcal{O}(\mathbb{R})$.

This “riddle” is based on a wrong reading of the statement that “every Grothendieck topos is locally presentable” as “every Grothendieck topos is locally **finitely** presentable”. What Makkai and Paré actually prove is that

every Grothendieck topos is locally κ -presentable for some regular cardinal κ

which depends on the size of covers of the site representing the Grothendieck topos. More precisely, if \mathcal{E} is $\mathbf{Sh}(\mathcal{C}, \mathcal{J})$ then the representing limit sketch \mathbb{T} contains for every cover S in \mathcal{J} a limiting cone whose size is that of S . Thus, for κ one may take the least regular cardinal strictly greater than the sizes of all covers in \mathcal{J} .

But for infinite regular cardinals κ the posetal version of locally κ -presentable category is not algebraic lattice but a version of it where “directed”, i.e. \aleph_0 -directed, is replaced by κ -directed. Choosing $\kappa > |L|$ a complete lattice L is κ -algebraic since every subset of L is κ -directed. Thus, locally presentable posets are nothing but arbitrary complete lattices and, accordingly, every complete Heyting algebra is locally κ -presentable for an appropriate regular cardinal κ .

¹⁴⁵i.e. full reflective subcategories where the reflector preserves finite limits

214 Molecular and Atomic Geometric Morphisms

A geometric morphism $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ is called *molecular* or *locally connected* iff F has a left adjoint $L : \mathcal{E} \rightarrow \mathcal{S}$ such that

$$\begin{array}{ccc}
 \begin{array}{ccc}
 B & \xrightarrow{f} & A \\
 \downarrow b & \lrcorner & \downarrow a \\
 FJ & \xrightarrow{Fu} & FI
 \end{array} & \text{implies} & \begin{array}{ccc}
 LB & \xrightarrow{Lf} & LA \\
 \downarrow \hat{b} & \lrcorner & \downarrow \hat{a} \\
 J & \xrightarrow{u} & I
 \end{array}
 \end{array}$$

Notice that a geometric morphism $F \dashv U$ is molecular iff F preserves dependent products (because this is equivalent to $\Delta : P_{\mathcal{S}} \rightarrow P_F$ preserving internal products which by the fibred adjoint functor theorem is equivalent to Δ having a fibred left adjoint Π).

A morphism $f : B \rightarrow A$ is called \mathcal{S} -*definable* iff it fits into a pullback

$$\begin{array}{ccc}
 B & \xrightarrow{f} & A \\
 \downarrow b & \lrcorner & \downarrow a \\
 FJ & \xrightarrow{Fu} & FI
 \end{array}$$

for some $u : J \rightarrow I$. Let $u = me$ with m monic and e epic then

$$\begin{array}{ccccc}
 B & \xrightarrow{e_f} & C & \xrightarrow{m_f} & A \\
 \downarrow b & \lrcorner & \downarrow c & \lrcorner & \downarrow a \\
 FJ & \xrightarrow{F_e} & FK & \xrightarrow{F_m} & FI
 \end{array}$$

with $f = m_f e_f$. Thus, if f is monic then e_f is an isomorphism.

Thus we have shown that $m : P \rightarrow A$ is an \mathcal{S} -definable mono iff

$$\begin{array}{ccc}
 P & \xrightarrow{m} & A \\
 \downarrow p & \lrcorner & \downarrow a \\
 FQ & \xrightarrow{F_n} & FI
 \end{array}$$

for some mono $n : Q \rightarrow I$.

Suppose $m : P \rightarrow A$ is an \mathcal{S} -definable mono as above. Thus we have

$$\begin{array}{ccc} LP & \xrightarrow{Lm} & LA \\ \hat{p} \downarrow & \lrcorner & \downarrow \hat{a} \\ Q & \xrightarrow{n} & I \end{array}$$

and thus

$$\begin{array}{ccc} P & \xrightarrow{m} & A \\ \eta_P \downarrow & \lrcorner & \downarrow \eta_A \\ FLP & \xrightarrow{FLm} & FLA \\ F\hat{p} \downarrow & \lrcorner & \downarrow F\hat{a} \\ FQ & \xrightarrow{Fn} & FI \end{array}$$

i.e. Lm is monic and m is isomorphic to the pullback of FLm along η_A .

On the other hand when starting from a mono $n : Q \rightarrow LA$ we may consider

$$\begin{array}{ccc} P & \xrightarrow{m} & A \\ p \downarrow & \lrcorner & \downarrow \eta_A \\ FQ & \xrightarrow{Fn} & FLA \end{array}$$

and thus, since $F \dashv U$ is locally connected, we have

$$\begin{array}{ccc} LP & \xrightarrow{Lm} & LA \\ \cong \downarrow & \lrcorner & \parallel \\ Q & \xrightarrow{n} & LA \end{array}$$

i.e. n is isomorphic to Lm .

Thus, for locally connected $F \dashv U$ we have established a 1-1-correspondence between \mathcal{S} -definable subobjects of A and ordinary subobjects of LA .

We show now that $F\top_{\mathcal{S}} : F1_{\mathcal{S}} \rightarrow F\Omega_{\mathcal{S}}$ is a classifier for \mathcal{S} -definable monos in \mathcal{E} . Suppose $m : P \rightarrow A$ is an \mathcal{S} -definable mono. Then we have

$$\begin{array}{ccc} LP & \xrightarrow{Lm} & LA \\ \downarrow & \lrcorner & \downarrow \chi \\ 1_{\mathcal{S}} & \xrightarrow{\top_{\mathcal{S}}} & \Omega_{\mathcal{S}} \end{array}$$

and thus

$$\begin{array}{ccc} P & \xrightarrow{m} & A \\ \eta_P \downarrow & \lrcorner & \downarrow \eta_A \\ FLP & \xrightarrow{FLm} & FLA \\ \downarrow & \lrcorner & \downarrow F\chi \\ F1_{\mathcal{S}} & \xrightarrow{F\top_{\mathcal{S}}} & F\Omega_{\mathcal{S}} \end{array}$$

On the other hand if

$$\begin{array}{ccc} P & \xrightarrow{m} & A \\ \downarrow & \lrcorner & \downarrow \pi \\ F1_{\mathcal{S}} & \xrightarrow{F\top_{\mathcal{S}}} & F\Omega_{\mathcal{S}} \end{array}$$

then

$$\begin{array}{ccc} LP & \xrightarrow{Lm} & LA \\ \downarrow & \lrcorner & \downarrow \hat{\pi} \\ 1_{\mathcal{S}} & \xrightarrow{\top_{\mathcal{S}}} & \Omega_{\mathcal{S}} \end{array}$$

and thus $\chi = \hat{\pi}$, i.e. $F\chi \circ \eta_A = \pi$.

Accordingly, for locally connected $F \dashv U$ all monos in \mathcal{E} are \mathcal{S} -definable iff $F\top_{\mathcal{S}}$ is a subobject classifier. Locally connected geometric morphism whose

inverse image part preserves subobject classifiers are called *atomic*.¹⁴⁶ They can be characterized as those geometric morphism whose inverse image part is logical.

By Cor. A.2.2.10 of PTJ's *Elephant* a logical functor $F : \mathcal{S} \rightarrow \mathcal{E}$ has a left adjoint iff it has a right adjoint. Accordingly, logical functors $F : \mathcal{S} \rightarrow \mathcal{E}$ having a left or a right adjoint are precisely the inverse image parts of atomic geometric morphisms between toposes. From considerations above it follows that for such F for all $I \in \mathcal{S}$ the functors $F_{/I} : \mathcal{S}/I \rightarrow \mathcal{E}/FI$ restrict to equivalences between $\text{Sub}_{\mathcal{S}}(I)$ and $\text{Sub}_{\mathcal{E}}(FI)$ thus providing an alternative proof of Lemma A.2.4.8 in PTJ's *Elephant*.

As shown in Lemma A.2.3.8 of the *Elephant* up to equivalence functors of the form $I^* : \mathcal{E} \rightarrow \mathcal{E}/I$ can be characterized as logical functors $F : \mathcal{S} \rightarrow \mathcal{E}$ having a faithful left adjoint L (where $I \cong L1$). Alternatively, one may characterize such functors as inverse image parts of localic and atomic geometric morphisms to \mathcal{S} often also called *local homeomorphisms*.

On the other hand the inverse image parts of connected atomic geometric morphisms are precisely those full and faithful logical functors $F : \mathcal{S} \rightarrow \mathcal{E}$ having both adjoints. These are necessarily hyperconnected since the right adjoint of F preserves subobject classifiers. Obviously, connected local homeomorphisms are precisely equivalences of toposes.

215 LF canonically isomorphic to $(-)\times L1$

Let $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ be a molecular geometric morphism, i.e. F has a left adjoint L fibered over \mathcal{S} . Since

$$\begin{array}{ccc} FI & \xrightarrow{!_{FI}} & 1 \\ \parallel & \lrcorner & \downarrow \cong \\ FI & \longrightarrow & F1 \end{array}$$

is a pullback (because $F1$ is terminal) it follows from $F \dashv U$ being a molecular geometric morphism that

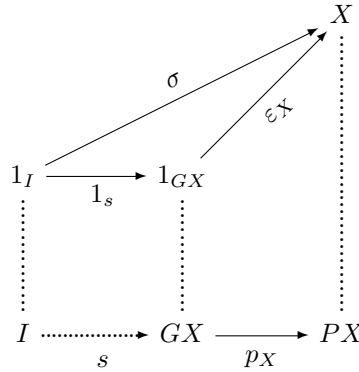
$$\begin{array}{ccc} LFI & \xrightarrow{L!_{FI}} & L1 \\ \varepsilon_I \downarrow & \lrcorner & \downarrow \\ I & \longrightarrow & 1 \end{array}$$

is a pullback, too, and thus $\langle \varepsilon_I, L!_{FI} \rangle : LFI \rightarrow I \times L1$ is an isomorphism.

¹⁴⁶An object C of \mathcal{E} is called *connected* if LC is terminal. Thus, if \mathcal{S} is 2-valued, e.g. if \mathcal{S} is **Set**, a connected object of \mathcal{E} has only trivial subobjects, i.e. is an *atom*.

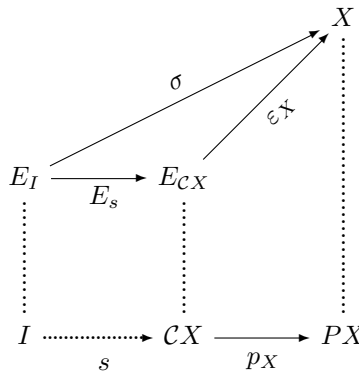
216 Categorical Semantics of Linear Type Theory

A fibration $P : \mathbb{X} \rightarrow \mathbb{B}$ is a fibration of categories with terminal objects if P has a rari (right adjoint right inverse) 1 , i.e. $P \dashv 1$ and $P \circ 1 = \text{id}_{\mathbb{B}}$. Such a fibration has *Lawvere comprehension* iff, moreover, the functor 1 has a right adjoint G .¹⁴⁷ More explicitly, that means that for every $X \in \mathbb{X}$ there is a map $\varepsilon_X : 1_{GX} \rightarrow X$ such that for every $\sigma : 1_I \rightarrow X$ there is a unique $s : I \rightarrow GX$ with $\varepsilon_X \circ 1_s = \sigma$ as depicted in



where $p_X = P\varepsilon_X$.

Now if P is a fibred monoidal category over \mathbb{B} it is natural to postulate comprehension as the requirement that the functor $E : \mathbb{B} \rightarrow \mathbb{X}$ picking the unit for \otimes in each fibre¹⁴⁸ has a right adjoint which we call C (for “comprehension”). More explicitly, that means that for every $X \in \mathbb{X}$ there is a map $\varepsilon_X : E_{CX} \rightarrow X$ such that for every $\sigma : E_I \rightarrow X$ there is a unique $s : I \rightarrow CX$ with $\varepsilon_X \circ E_s = \sigma$ as depicted in



¹⁴⁷This is not Lawvere’s original formulation but a generalisation which can be found in Th. Ehrhard’s Thèse. Actually, the functor G should be thought of as $\text{hom}(1, -)$ when hom is understood in the sense of Bénabou’s notion of local smallness for fibrations.

¹⁴⁸notice that E is right inverse but typically not right adjoint to P

where $p_X = P\varepsilon_X$.¹⁴⁹

Under this correspondence sections of p_X , i.e. maps $s : PX \rightarrow CX$ with $p_X \circ s = \text{id}_{PX}$, correspond to vertical maps $\sigma : E_{PX} \rightarrow X$ (where $\sigma = \varepsilon_X \circ E_s$).

On the other hand for X and Y over $I \in \mathbb{B}$ morphisms $f : p_X \rightarrow p_Y$ in \mathbb{B}/I correspond to morphisms $\varphi : E_{CX} \rightarrow Y$ over p_X . If, moreover, P is a bifibration then we have

$$\begin{array}{ccc} E_{CX} & \xrightarrow{E_f} & E_{CY} \\ \psi_X \downarrow & \searrow \varphi & \downarrow \varepsilon_Y \\ !X & \xrightarrow{\phi} & Y \end{array}$$

where $\psi_X : E_{CX} \rightarrow !X$ is cocartesian and $\phi : !X \rightarrow Y$ is vertical. For $X = Y$ and $f = \text{id}_{CX}$ the corresponding vertical map is $\text{read}_X : !X \rightarrow X$. From this point of view $!X$ is $CX \cdot E$, i.e. the $\text{hom}(E, X)$ -fold copower of the unit E .

It appears as most natural to assume that cocartesian arrows in \mathbb{X} of the form ψ_X satisfy an appropriate version of the Chevalley condition. First observe that for cartesian maps $\varphi : X \rightarrow Y$ the square

$$\begin{array}{ccc} CX & \xrightarrow{C\varphi} & CY \\ p_X \downarrow & \lrcorner & \downarrow p_Y \\ PX & \xrightarrow{P\varphi} & PY \end{array}$$

is a pullback.¹⁵⁰ Since $E_{C\varphi}$ is cartesian and ψ_X and ψ_Y are cocartesian the Chevalley condition requires that the unique morphism $!\varphi : !X \rightarrow !Y$ above $P\varphi$ making the diagram

$$\begin{array}{ccc} E_{CX} & \xrightarrow{E_{C\varphi}} & E_{CY} \\ \psi_X \downarrow & & \downarrow \psi_Y \\ !X & \xrightarrow{!\varphi} & !Y \end{array}$$

commute is actually cartesian.

¹⁴⁹Notice that C should be thought of as $\text{hom}(E, -)$ where E is the unit for \otimes .

¹⁵⁰Suppose $u : I \rightarrow PX$ and $t : I \rightarrow CY$ with $P\varphi \circ u = p_Y \circ t$. We have to show that there is a unique map $s : I \rightarrow CX$ with $p_X \circ s = u$ and $C\varphi \circ s = t$. The latter condition is equivalent to $\varepsilon_Y \circ E_{C\varphi \circ s} = \varepsilon_Y \circ E_t$. Since $\varepsilon_Y \circ E_{C\varphi \circ s} = \varepsilon_Y \circ E_{C\varphi} \circ E_s = \varphi \circ \varepsilon_X \circ E_s$ it is equivalent to $\varphi \circ \varepsilon_X \circ E_s = \varepsilon_Y \circ E_t$. Since $p_X \circ s = u$ is equivalent to $P(\varepsilon_X \circ E_s) = u$ it suffices to show that there is a unique $\sigma : E_I \rightarrow X$ with $P\sigma = u$ and $\varphi \circ \sigma = \varepsilon_Y \circ E_t$ which, however, is immediate from the assumption that φ is cartesian.

217 Two Remarks on Models of Set Theory

Consistency Weaker than Existence of a Natural Model

If ZF or ZFC are consistent one may consistently add the formalized statements of their consistency. Since ZF allows one to prove Gödel's Completeness Theorem these statements are provably equivalent to the claim that there is a model, i.e. a set with a binary relation validating the required axioms. A different story, however, is to require that a natural model exists, i.e. a set M such that $(M, \{(x, y) \in M^2 \mid x \in y\})$ is a model.

As shown by J. Shepherdson if there exists a natural model then there exists also a *minimal*¹⁵¹ natural model M which, moreover, is countable. This model M can itself not contain a natural model as an element since this would contradict its minimality.

But since consistency is a Π_1^0 sentence it holds in M iff it holds in the ambient model. Thus starting from a model of set theory in which formalized consistency holds the minimal natural model M_0 still validates formalized consistency though it doesn't host a natural model.

Thus formalized consistency is weaker than the claim that there exists a natural model. This observation is due to P. Cohen.

Countable Transitive Models

Let T be a consistent extension of ZF. Let T^* be the theory in the language of set theory extended by a constant c consisting of T , all relativizations ϕ^c of $\phi \in T$ to c and the sentence claiming that c is countable and transitive. As observed by J. Shoenfield T^* is conservative over T w.r.t. set theoretic formulas since by the Reflection Theorem for every finite $F \subseteq T$ the theory T proves that F has a countable transitive model.

Thus, in particular, the theories T and T^* are equiconsistent. In T^* one may construct generic extensions $c[G]$ of the countable transitive model c . Notice, however, that T^* does not prove the formalized statement that c is a model of T . However, this is not needed for the purposes of forcing.

¹⁵¹i.e. least w.r.t. \subseteq

218 Forcing from a Topos Point of View

It is well known that a Grothendieck topos can be characterized as a locally small elementary topos \mathcal{E} with small sums and a small generating family. Grothendieck toposes relative to a base topos \mathcal{S} can be characterized as geometric morphisms $f : \mathcal{E} \rightarrow \mathcal{S}$ (i.e. $f_* : \mathcal{E} \rightarrow \mathcal{S}$ together with a left adjoint f^* preserving finite limits) which are *bounded* in the sense that there is a $B \in \mathcal{E}$ such that every $X \in \mathcal{E}$ appears as subquotient¹⁵² of some $B \times f^*I$ for some $I \in \mathcal{S}$.

Localic toposes are those Grothendieck toposes where subobjects of 1 form a generating family. Localic toposes relative to base toposes \mathcal{S} correspond to localic geometric morphisms $f : \mathcal{E} \rightarrow \mathcal{S}$, meaning that $1_{\mathcal{E}}$ is a bound, i.e. every $X \in \mathcal{E}$ appears as subquotient of some f^*I .

One easily shows that bounded geometric and also localic geometric morphisms are closed under composition. This fact amounts to an *iteration* theorem for Grothendieck and localic toposes.

A boolean localic topos \mathcal{E} is equivalent to $\mathbf{Sh}(B)$ where B is the complete boolean algebra of subobjects of $1_{\mathcal{E}}$. If \mathcal{S} is a boolean base topos then localic boolean toposes over \mathcal{S} correspond to localic geometric morphisms $\mathcal{E} \rightarrow \mathcal{S}$ with \mathcal{E} boolean. In the light of the remarks above this gives rise to an iteration theorem for boolean localic toposes.

Another useful view often adopted is that a localic geometric morphism to \mathcal{S} corresponds to a complete Heyting algebra internal to \mathcal{S} . If A is such a cHa in \mathcal{S} then the corresponding localic geometric morphism is $\Delta \dashv \Gamma : \mathbf{Sh}_{\mathcal{S}}(A) \rightarrow \mathcal{S}$ fibred over \mathcal{S} .

As is well known there is an inclusion of \mathbf{Loc} into the 2-category \mathbf{BTop} of Grothendieck toposes and geometric morphism between them (which are necessarily bounded). This inclusion sends a locale A to $\mathbf{Sh}(A)$ and a frame morphism $h : A \rightarrow B$ to the geometric morphism $f : \mathbf{Sh}(B) \rightarrow \mathbf{Sh}(A)$ whose direct image f_* is given by precomposition with h (which can be shown to have a left adjoint f^* given by left Kan extension followed by sheafification). The left adjoint to the inclusion of \mathbf{Loc} into the 2-category \mathbf{BTop} sends a Grothendieck topos \mathcal{E} to the cHa $\mathbf{Sub}_{\mathcal{E}}(1_{\mathcal{E}})$.¹⁵³

I finish this excursion with a couple of more down to earth remarks. Boolean localic toposes, i.e. boolean valued models, can be characterized as those Grothendieck toposes where all epimorphisms split. This latter property is known as EAC (External Axiom of Choice). The weaker IAC (Internal Axiom of Choice) just says that for all objects X the functor $(-)^X$ preserves epimorphisms. EAC is equivalent to IAC + SS, where SS (support splits) claims that every epimorphism to a subterminal object has a section. This implies that true existential statements (without free variables) are witnessed by global elements (which, quite surprisingly, holds for boolean valued models of set theory). Toposes of actions of a non-posetal group(oid) are examples of toposes validating IAC but

¹⁵²quotient of a subobject

¹⁵³As described in Johnstone's *Elephant* localic geometric morphism form the right part of a factorization system in \mathbf{BTop} whose left part are the so called *hyperconnected* geometric morphisms.

not EAC.

In Freyd and Scedrov's book *Categories, Allegories* in 1.972 they characterize Grothendieck toposes validating EAC as those boolean Grothendieck toposes for which 1 is projective (i.e. every epi to 1 splits, i.e. every well supported object has a global element) and in 1.978 they characterize Grothendieck toposes validating IAC as those Grothendieck toposes \mathcal{E} which are boolean etendues, i.e. \mathcal{E} is boolean and \mathcal{E}/X is localic for some well supported X in \mathcal{E} .

But in any case it is a nice fact that **boolean valued models** can be characterized as **locally small elementary toposes with small sums where every epimorphism splits**.

As follows from Johnstone's paper *Quotients of Decidable Objects in a Topos* (1983) **boolean Grothendieck toposes** can be characterized as **boolean toposes localic over the Schanuel topos $C(G_0)$** of continuous actions of the topological group G_0 on **Set**.¹⁵⁴ Thus, boolean Grothendieck toposes may be considered as boolean valued models over toposes $C(G)$ of continuous actions of some topological group G on **Set** for which reason they may be considered as (mild generalisations of the) **symmetric boolean valued models** as considered in set theory e.g. for the purpose of constructing models for ZF refuting AC as described in Jech's 1973 book on *The Axiom of Choice*.

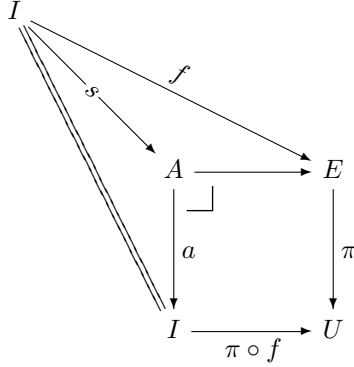
219 Quotients of Decidable Objects in a Topos (Johnstone 1983)

In Johnstone's 1983 paper *Quotients of Decidable Objects in a Topos* he shows that a Grothendieck topos is localic over the Schanuel topos if and only if every object appears as quotient of an object with decidable equality. Moreover, he also shows in this paper that for every Grothendieck topos \mathcal{E} there is a connected atomic geometric morphism $f : \mathcal{F} \rightarrow \mathcal{E}$ where \mathcal{F} is localic over the Schanuel topos. Recall that f being connected and atomic means that $f^* : \mathcal{E} \rightarrow \mathcal{F}$ is full and faithful and logical and has both adjoints. Necessarily, such f are also hyperconnected, i.e. f_* preserves subobject classifiers.

¹⁵⁴Here G_0 is the group of all permutations of \mathbb{N} endowed with the subspace topology induced by $G_0 \subseteq \mathbb{N}^{\mathbb{N}}$ where $\mathbb{N}^{\mathbb{N}}$ carries the Baire topology.

220 Universes in Presheaf Toposes (Hofmann and Streicher around 1997)

Let \mathcal{C} be a category with pullbacks. A *universe* in \mathcal{C} is a map $\pi : E \rightarrow U$. We write \mathcal{F}_π for the class of maps in \mathcal{C} which can be obtained as pullbacks of π along some arrow in \mathcal{C} . As can be seen from the diagram

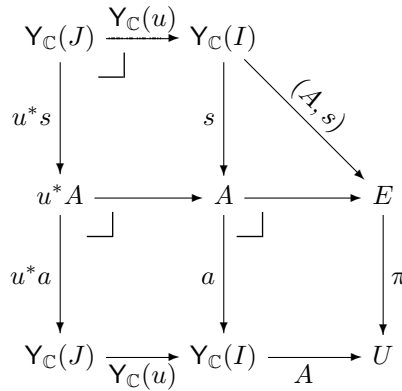


generalized elements of E correspond to sections of maps in \mathcal{F}_π .

Let \mathbb{C} be a small category living in a Grothendieck universe \mathcal{U} . We want to come up with a universe $\pi : E \rightarrow U$ in the presheaf topos in $\widehat{\mathbb{C}} = \mathbf{Set}^{\mathbb{C}^{\text{op}}}$ such that maps in \mathcal{F}_π with codomain $\mathbf{Y}_{\mathbb{C}}(I)$ are the morphisms whose fibres are small in the sense of \mathcal{U} , i.e. are isomorphic to $\int A$ for some $A \in \mathcal{U}^{\mathbb{C}/I}{}^{\text{op}}$. Thus, by Yoneda $U(I) = \mathcal{U}^{\mathbb{C}/I}{}^{\text{op}}$ and $E(I) = \{(A, s) \mid A \in U(I) \text{ and } s \in A(\text{id}_I)\}$. Also by Yoneda for $u : J \rightarrow I$ we have

$$U(u)(A) = A \circ \Sigma_u^{\text{op}} \quad E(u)(A, s) = (U(u)(A), A(u : u \xrightarrow{u} \text{id}_I)(s))$$

as can be seen from the diagram



where s and u^*s are sections of a and u^*a , respectively.

221 An observation by E. Faber (2018)

stated in more general form in Appendix 3 of his 2019 Cambridge PhD Thesis.

Let \mathbb{C} be a small category and $\tau : F \rightarrow Y_{\mathbb{C}}$ a cartesian natural transformation in $\widehat{\mathbb{C}}$. Since τ is cartesian for all $u : J \rightarrow I$ in \mathbb{C} we have

$$\begin{array}{ccc} F(J) & \xrightarrow{F(u)} & F(I) \\ \tau_J \downarrow & \lrcorner & \downarrow \tau_I \\ Y_{\mathbb{C}}(J) & \xrightarrow{Y_{\mathbb{C}}(u)} & Y_{\mathbb{C}}(I) \end{array}$$

and thus also

$$\begin{array}{ccc} F(J)(J) & \xrightarrow{F(u)_J} & F(I)(J) \\ (\tau_J)_J \downarrow & \lrcorner & \downarrow (\tau_I)_J \\ Y_{\mathbb{C}}(J)(J) & \xrightarrow{Y_{\mathbb{C}}(u)_J} & Y_{\mathbb{C}}(I)(J) \end{array}$$

for which reason the map $i_{I,J} : \mathbb{C}(J, I) \times (\tau_J)_J^{-1}(\text{id}_J) \rightarrow F(I)(J) : \langle u, b \rangle \mapsto F(u)_J(b)$ is a bijection.

Let $A \in \widehat{\mathbb{C}}$ with $A(I) = (\tau_I)_I^{-1}(\text{id}_I)$ and $A(u)(a)$ the unique $b \in A(J) = (\tau_J)_J^{-1}(\text{id}_J)$ with $i_{I,J}(u, b) = a$. One readily checks¹⁵⁵ that i induces a natural isomorphism $\iota : Y_{\mathbb{C}} \times A \rightarrow F$ putting $(\iota_I)_J = i_{I,J}$ and that $\tau = \pi \circ \iota^{-1}$ where $\pi : Y_{\mathbb{C}} \times A \rightarrow Y_{\mathbb{C}}$ is the componentwise projection.

I have serious doubts that this works! But see Faber's Thesis!

But if \mathbb{C} has a terminal object 1 then the claim is true since we have

$$\begin{array}{ccccc} FJ & \xrightarrow{Fu} & FI & \xrightarrow{F!_I} & F1 \\ \tau_J \downarrow & \lrcorner & \downarrow \tau_I & \lrcorner & \downarrow \tau_1 \\ Y_{\mathbb{C}}(J) & \xrightarrow{Y_{\mathbb{C}}(u)} & Y_{\mathbb{C}}(I) & \xrightarrow{!_I} & Y_{\mathbb{C}}(1) \end{array}$$

for all $u : J \rightarrow I$ in \mathbb{C} and thus F is naturally isomorphic to $Y_{\mathbb{C}} \times A$ with $A = F1$ (since $Y_{\mathbb{C}}(1) \cong 1_{\widehat{\mathbb{C}}}$).

Without the assumption that \mathbb{C} has a terminal object one has to take for A instead of $F1$ the colimit of F .

¹⁵⁵Suppose $u : J \rightarrow I, v : K \rightarrow J$ and $c \in A(K), b \in A(J), a \in A(I)$ with $F(v)_K(c) = b$ and $F(u)_J(b) = a$. Then we have $F(uv)_K(c) = F(u)_K(F(v)_K(c)) = F(u)_K(b) = a$.

222 Coherent categories have initial objects

A *coherent* category is a regular category \mathcal{C} where all $\text{Sub}_{\mathcal{C}}(A)$ have finite suprema which are preserved by $f^{-1} : \text{Sub}_{\mathcal{C}}(A) \rightarrow \text{Sub}_{\mathcal{C}}(B)$ for all $f : B \rightarrow A$ in \mathcal{C} . We write $0_A \twoheadrightarrow A$ for the least subobject of A and denote 0_1 by 0 . We will show that 0 is a strict initial object in \mathcal{C} .

Suppose there is a morphism $A \rightarrow 0$ then

$$\begin{array}{ccccc} 0_A & \longrightarrow & 0 & \xlongequal{\quad} & 0 \\ \downarrow & \lrcorner & \parallel & \lrcorner & \downarrow \\ A & \longrightarrow & 0 & \longrightarrow & 1 \end{array}$$

from which it follows that $0_A \twoheadrightarrow A$ is isomorphic to id_A and thus is an isomorphism, i.e. id_A is the least subobject of A . If there is a morphism $A \rightarrow 0$ there is also a morphism $A \times A \rightarrow 0$ and thus it follows from the previous consideration that $\delta_A = \langle \text{id}_A, \text{id}_A \rangle : A \rightarrow A \times A$ is an isomorphism, i.e. A is subterminal. Thus $A \rightarrow 0$ is monic from which it follows that $A \rightarrow 0$ is also an isomorphism (since $0 \twoheadrightarrow 1$ is the least subobject of 1). Thus, we have in the diagram

$$\begin{array}{ccc} 0_A & \xrightarrow{\cong} & 0 \\ \downarrow & \lrcorner & \downarrow \\ A & \longrightarrow & 1 \end{array}$$

from which it follows that there is a morphism $0 \rightarrow A$. If $f, g : 0 \rightarrow A$ then their equalizer $e : E \twoheadrightarrow 0$ is an isomorphism and thus $f = g$. Thus we have shown that 0 is initial (and isomorphic to all 0_A).

223 A locally small topos is complete iff it is cocomplete

Let \mathcal{E} be a locally small elementary topos. Following the argument in Freyd & Scedrov's *Categories, Allegories* (1.968) we show that \mathcal{E} has small products iff \mathcal{E} has small sums.

Suppose \mathcal{E} has small sums. Let $(A_i)_{i \in I}$ be a small family of objects in \mathcal{E} and S its sum. Then $\prod_I S$ is given by $S^{\Delta(I)}$ where $\Delta(I) = \coprod_I 1$. For every $i \in I$ consider the pullback

$$\begin{array}{ccc} P_i & \longrightarrow & \prod_I S \\ \downarrow & \lrcorner & \downarrow \pi_i \\ A_i & \xrightarrow{\iota_i} & S \end{array}$$

and let $P = \bigcap_{i \in I} P_i$. Then the family $(P \rightarrow P_i \rightarrow A_i)_{i \in I}$ is a product cone.

Suppose \mathcal{E} has small products. Let $(A_i)_{i \in I}$ be a small family of objects in \mathcal{E} . Consider the product $\prod_{i \in I} (A_i + 1)$. For every $i \in I$ we have a map $u_i : A_i \rightarrow \prod_{i \in I} (A_i + 1)$ such that $u_i u_i^\circ = \text{id}_{A_i}$ and $u_i u_j^\circ = 0$ for $i \neq j$. Let S be the union of the images of the u_i and $\iota_i : A_i \rightarrow S$ be the pullback of u_i along the inclusion of S into $\prod_{i \in I} (A_i + 1)$. Then the family $(\iota_i : A_i \rightarrow S)_{i \in I}$ exhibits S as sum of the A_i .

224 Splitting Ehrhard's Dictoses (13/10/2015)

Let \mathcal{C} together with $p_{\text{Prop}} : \text{Prf} \rightarrow \text{Prop}$ be a dictos. Let \mathcal{U}_0 be a Grothendieck universe in \mathbf{Set} such that \mathcal{C} lives in \mathcal{U}_0 and \mathcal{U} be a Grothendieck universe in \mathbf{Set} with $\mathcal{U}_0 \in \mathcal{U}$.

Then $\widehat{\mathcal{C}} = \mathbf{Set}^{\mathcal{C}^{\text{op}}}$ is a locally cartesian closed category and $Y : \mathcal{C} \rightarrow \widehat{\mathcal{C}}$ preserves the locally cartesian closed structure. Let $\mathbf{set}_{\mathcal{U}}(\mathcal{C})$ be the presheaf over \mathcal{C} with $\mathbf{set}_{\mathcal{U}}(\mathcal{C})(I) = \{A \in \mathcal{U}^{(\mathcal{C}/I)^{\text{op}}} \mid A \text{ representable}\}$. Then $p_{\mathbf{set}_{\mathcal{U}}(\mathcal{C})} : E_{\mathbf{set}_{\mathcal{U}}(\mathcal{C})} \rightarrow \mathbf{set}_{\mathcal{U}}(\mathcal{C})$ exhibits \mathcal{C} as a full internal subcategory of $\widehat{\mathcal{C}}$.

Notice that the representable presheaf $\widehat{\text{Prop}} = Y_{\mathcal{C}}(\text{Prop}) : \mathcal{C}^{\text{op}} \rightarrow \mathcal{U}_0$ is an element of $\mathcal{U}_0^{\mathcal{C}^{\text{op}}}$ and thus $\widehat{\text{Prop}}$ is a global element of $\mathbf{set}_{\mathcal{U}}(\mathcal{C})$. For similar reasons $Y_{\mathcal{C}}(p_{\text{Prop}})$ is a morphism in the full subcategory represented by $\mathbf{set}_{\mathcal{U}}(\mathcal{C})$ making it a dictos internal to $\widehat{\mathcal{C}}$ which, moreover, is a full subcategory of $\widehat{\mathcal{C}}$.

This allows us to use Voevodsky's method of universes for splitting the original dictos within $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$.

225 Splitting Ehrhard's Dictoses (7/1/2016)

A most natural notion of model for the Calculus of Constructions (CC) is T. Ehrhard's notion of a *dictos*, a locally cartesian closed category \mathcal{C} together with a map $p_{\text{Prop}} : \text{Prf} \rightarrow \text{Prop}$ such that the class \mathcal{P} of maps in \mathcal{C} which can be obtained as pullback of p_{Prop} are closed under dependent products in \mathcal{C} , i.e. $\Pi_u a \in \mathcal{P}$ whenever $a : A \rightarrow J$ in \mathcal{P} and $u : J \rightarrow I$ in \mathcal{C} . However, in order to interpret CC in a dictos we have to "split" it for the sake of interpreting the syntax of CC.

As is well known the Yoneda functor $Y_{\mathcal{C}} : \mathcal{C} \rightarrow \mathbf{Set}^{\mathcal{C}^{\text{op}}}$ preserves finite limits and dependent products. Moreover, if \mathcal{U} is a Grothendieck universe such that \mathcal{C} lives in \mathcal{U} , i.e. \mathcal{C} is internal to the category \mathcal{U} , then the Yoneda functor $Y : \mathcal{C} \rightarrow \mathbf{Set}^{\mathcal{C}^{\text{op}}}$ factors through the inclusion $\mathcal{U}^{\mathcal{C}^{\text{op}}} \hookrightarrow \mathbf{Set}^{\mathcal{C}^{\text{op}}}$. We also write $Y_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{U}^{\mathcal{C}^{\text{op}}}$ for the corresponding corestriction of $Y_{\mathcal{C}} : \mathcal{C} \rightarrow \mathbf{Set}^{\mathcal{C}^{\text{op}}}$ and notice that it also preserves finite limits and dependent products.

If we choose \mathcal{U} big enough for \mathcal{C} being internal to \mathcal{U} we may consider \mathcal{C} as a small full subcategory of $\widehat{\mathcal{C}} = \mathbf{Set}^{\mathcal{C}^{\text{op}}}$ as induced by a certain representable morphism $p_U : E_U \rightarrow U$ in $\widehat{\mathcal{C}}$ which can be described as follows. For $I \in \mathcal{C}$ let $U(I) = \{A \in \mathcal{U}^{(\mathcal{C}/I)^{\text{op}}} \mid A \text{ representable}\}$ and for $u : J \rightarrow I$ let $U(u) = \mathcal{U}^{\Sigma_u}$. We define p_U via its corresponding presheaf E_U over $\text{Elts}(U)$ as follows: $E_U(I, A) = A(\text{id}_I)$ and $E_U(u : u^*A \rightarrow A) = A(u : u \rightarrow \text{id}_I)$. Notice that p_U is universal among the class \mathcal{S} of representable morphisms in $\widehat{\mathcal{C}}$. This class \mathcal{S} is stable under pullbacks along arbitrary morphisms in $\widehat{\mathcal{C}}$ and is it stable under dependent products in $\widehat{\mathcal{C}}$ (since $Y_{\mathcal{C}}$ preserves Π).

We always can choose \mathcal{U} so big that there is a Grothendieck universe $\mathcal{U}_0 \in \mathcal{U}$ with \mathcal{C} internal to \mathcal{U}_0 . Let U_P be the subpresheaf of U where $U_P(I)$ consists of all presheaves $A : (\mathcal{C}/I)^{\text{op}} \rightarrow \mathcal{U}_0$ representable by a map in \mathcal{P} with codomain I .

Then the map p_P in

$$\begin{array}{ccc} E_P & \hookrightarrow & E_U \\ p_P \downarrow & \lrcorner & \downarrow p_U \\ U_P & \hookrightarrow & U \end{array}$$

is universal for the class \mathcal{S}_P of \mathcal{P} -representable morphisms, i.e. morphisms $f : Y \rightarrow X$ which for all $x : Y(I) \rightarrow X$ fit into a pullback diagram

$$\begin{array}{ccc} Y(A) & \longrightarrow & Y \\ Y(a) \downarrow & \lrcorner & \downarrow f \\ Y(I) & \xrightarrow{x} & X \end{array}$$

for some $a \in \mathcal{P}$.

Notice also that the terminal projection of U_P is in \mathcal{S} . **NO! and that is a serious problem!**

Lemma 225.1 *If $f : Y \rightarrow X$ is in \mathcal{S} and $g : Z \rightarrow Y$ is in \mathcal{S}_P then $\Pi_f g$ is in \mathcal{S}_P .*

Proof: By assumption on f and g for all $x : Y_I(I) \rightarrow X$ we have

$$\begin{array}{ccc} Y(A) & \longrightarrow & Z \\ Y(a) \downarrow & \lrcorner & \downarrow g \\ Y(J) & \longrightarrow & Y \\ Y(u) \downarrow & \lrcorner & \downarrow f \\ Y(I) & \xrightarrow{x} & X \end{array}$$

for some $u : J \rightarrow I$ in \mathcal{C} and $a : A \rightarrow J$ in \mathcal{P} .

Since Yoneda preserves Π we have $x^* \Pi_f g \cong \Pi_{Y(u)} Y(a) \cong Y(\Pi_u a)$ from which the claim follows since $\Pi_u a$ is in \mathcal{P} . \square

Now we can apply Voevodsky's "method of universes" for splitting the original dictos within $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$. Dependent products for representable morphisms are dealt with as in *loc.cit.*

For impredicative universal quantification we proceed as follows. Consider the generic context

$$\Gamma_G \equiv A : U, p : U_P^{E_U(A)}$$

and the families

$$a_G \equiv \Gamma_G \vdash A \quad \text{and} \quad p_G \equiv \Gamma_G, a : A \vdash E_P(P(a))$$

in \mathcal{S} and \mathcal{S}_P , respectively. By Lemma 225.1 conclude that $\Pi_{a_G} p_G$ is in \mathcal{S}_P . Thus, since p_P is generic for \mathcal{S}_P there is a morphism $\forall : \Gamma_G \rightarrow U_P$ such that

$$\begin{array}{ccc} & \longrightarrow & E_P \\ \Pi_{a_G} p_G \downarrow & \lrcorner & \downarrow p_P \\ \Gamma_G & \xrightarrow{\forall} & U_P \end{array}$$

and allows us to interpret impredicative quantification.

226 Can one split dictoses? (February 2016)

A most natural notion of model for the Calculus of Constructions (CC) is T. Ehrhard's notion of a *dictos*, i.e. a locally cartesian closed category \mathbb{C} together with a map $p_{\text{Prop}} : \text{Prf} \rightarrow \text{Prop}$ such that the class \mathcal{P} of maps in \mathbb{C} which can be obtained as pullback of p_{Prop} are closed under dependent products in \mathbb{C} , i.e. $\Pi_u a \in \mathcal{P}$ whenever $a : A \rightarrow J$ in \mathcal{P} and $u : J \rightarrow I$ in \mathbb{C} . However, in order to interpret CC in a dictos we have to “split” it for the sake of interpreting the syntax of CC.

As well known the Yoneda functor $Y_{\mathbb{C}} : \mathbb{C} \rightarrow \mathbf{Set}^{\text{C}^{\text{op}}}$ preserves finite limits and dependent products. Moreover, if \mathcal{U} is a Grothendieck universe such that \mathbb{C} lives in \mathcal{U} , i.e. \mathbb{C} is internal to the category \mathcal{U} , then the Yoneda functor $Y_{\mathbb{C}} : \mathbb{C} \rightarrow \mathbf{Set}^{\text{C}^{\text{op}}}$ factors through the inclusion $\mathcal{U}^{\text{C}^{\text{op}}} \hookrightarrow \mathbf{Set}^{\text{C}^{\text{op}}}$. We also write $Y_{\mathbb{C}} : \mathbb{C} \rightarrow \mathcal{U}^{\text{C}^{\text{op}}}$ for the corresponding corestriction of $Y_{\mathbb{C}} : \mathbb{C} \rightarrow \mathbf{Set}^{\text{C}^{\text{op}}}$ and notice that it also preserves finite limits and dependent products. Most of the time, however, we will simply write Y for $Y_{\mathbb{C}}$.

If we choose \mathcal{U} big enough for \mathbb{C} being internal to \mathcal{U} we may consider \mathbb{C} as a small full subcategory of $\widehat{\mathbb{C}} = \mathbf{Set}^{\text{C}^{\text{op}}}$ as induced by a certain representable morphism $p_U : E_U \rightarrow U$ in $\widehat{\mathbb{C}}$ which can be described as follows. For $I \in \mathbb{C}$ let $U(I) = \{A \in \mathcal{U}^{(\mathbb{C}/I)^{\text{op}}} \mid A \text{ representable}\}$ and for $u : J \rightarrow I$ in \mathbb{C} let $U(u) = \mathcal{U}^{\Sigma_u^{\text{op}}}$. We define p_U via its corresponding presheaf E_U over $\text{Elts}(U)$ as follows: $E_U(I, A) = A(\text{id}_I)$ and $E_U(u : u^*A \rightarrow A) = A(u : u \rightarrow \text{id}_I)$. Notice that p_U is universal among the class \mathcal{S} of representable morphisms in $\widehat{\mathbb{C}}$. This class \mathcal{S} is stable under pullbacks along arbitrary morphisms in $\widehat{\mathbb{C}}$ and is stable under dependent products in $\widehat{\mathbb{C}}$ (since $Y_{\mathbb{C}}$ preserves Π).

We always can choose \mathcal{U} so big that there is a Grothendieck universe $\mathcal{U}_0 \in \mathcal{U}$ with \mathbb{C} internal to \mathcal{U}_0 . Let U_P be the subpresheaf of U where $U_P(I)$ consists of all presheaves $A : (\mathbb{C}/I)^{\text{op}} \rightarrow \mathcal{U}_0$ representable by a map in \mathcal{P} with codomain I . Then the map p_P in

$$\begin{array}{ccc} E_P & \hookrightarrow & E_U \\ \downarrow p_P & \lrcorner & \downarrow p_U \\ U_P & \hookrightarrow & U \end{array}$$

is universal for the class \mathcal{S}_P of \mathcal{P} -representable morphisms, i.e. morphisms $f : Y \rightarrow X$ which for all $x : Y(I) \rightarrow X$ fit into a pullback diagram

$$\begin{array}{ccc} Y(A) & \longrightarrow & Y \\ \downarrow Y(a) & \lrcorner & \downarrow f \\ Y(I) & \xrightarrow{x} & X \end{array}$$

for some $a \in \mathcal{P}$.

Lemma 226.1 *If $f : Y \rightarrow X$ is in \mathcal{S} and $g : Z \rightarrow Y$ is in \mathcal{S}_P then $\Pi_f g$ is in \mathcal{S}_P .*

Proof: By assumption on f and g for all $x : Y_I(I) \rightarrow X$ we have

$$\begin{array}{ccc} Y(A) & \longrightarrow & Z \\ \downarrow Y(a) & \lrcorner & \downarrow g \\ Y(J) & \longrightarrow & Y \\ \downarrow Y(u) & \lrcorner & \downarrow f \\ Y(I) & \xrightarrow{x} & X \end{array}$$

for some $u : J \rightarrow I$ in \mathbb{C} and $a : A \rightarrow J$ in \mathcal{P} .

Since Yoneda preserves Π we have $x^* \Pi_f g \cong \Pi_{Y(u)} Y(a) \cong Y(\Pi_u a)$ from which the claim follows since $\Pi_u a$ is in \mathcal{P} . \square

Now, if U_P were in U we could apply Voevodsky's "method of universes" for splitting the original dictos within $\mathbf{Set}^{\mathbb{C}^{\text{op}}}$. Dependent products for representable morphisms are dealt with as in *loc.cit.* For impredicative universal quantification we proceed as follows. Consider the generic context

$$\Gamma_G \equiv A : U, p : U_P^{E_U(A)}$$

and the families

$$a_G \equiv \Gamma_G \vdash E_U(A) \quad \text{and} \quad p_G \equiv \Gamma_G, a : E_U(A) \vdash E_P(P(a))$$

in \mathcal{S} and \mathcal{S}_P , respectively. By Lemma 226.1 conclude that $\Pi_{a_G} p_G$ is in \mathcal{S}_P . Thus, since p_P is generic for \mathcal{S}_P there is a morphism $\forall : \Gamma_G \rightarrow U_P$ with

$$\begin{array}{ccc} & \longrightarrow & E_P \\ \Pi_{a_G} p_G \downarrow & \lrcorner & \downarrow p_P \\ \Gamma_G & \xrightarrow{\forall} & U_P \end{array}$$

which allows us to interpret impredicative quantification.

Unfortunately, the presheaf U_P is not representable as opposed to $\mathsf{Y}(\mathsf{Prop})$. But one can define a morphism $|S| : \mathsf{Y}(\mathsf{Prop}) \rightarrow U_P$ in $\widehat{\mathbb{C}}$ sending $a : I \rightarrow \mathsf{Prop}$ to the presheaf $|S|_I(a)$ over \mathbb{C}/I with $|S|_I(a)(u) = \{f : I \rightarrow \mathsf{Prf} \mid p_{\mathsf{Prop}} \circ f = a \circ u\}$ and $|S|_I(a)(v : uv \rightarrow u)(f) = f \circ v$. Obviously, every $A \in U_P(I)$ is isomorphic to $|S|_I(a)$ for some $a : I \rightarrow \mathsf{Prop}$ in the category $\mathcal{U}_0^{(\mathbb{C}/I)^{\text{op}}}$. Let us write S_P for the split fibration sending $I \in \mathbb{C}$ to the full subcategory of $\mathcal{U}_0^{(\mathbb{C}/I)^{\text{op}}}$ on representable presheaves. Obviously, we have $U_P = |S_P|$. Let sS_P be the split fibration where $sS_P(I)$ is the category whose objects are morphisms from I to Prop and where $sS_P(I)(a, b) = \widehat{\mathbb{C}}(|S|_I(a), |S|_I(b))$. Obviously, we have $\mathsf{Y}(\mathsf{Prop}) = |sS_P|$. We write S for the split cartesian functor from sS_P to S_P whose object part is given by $|S|_P$ and which is the identity on morphisms. Then in the category $\mathbf{Sp}(\mathbb{C})$ of split fibrations over \mathbb{C} we have

$$\begin{array}{ccc} |sS_P| & \xrightarrow{|S|} & |S_P| \\ \downarrow & & \downarrow \\ sS_P & \xrightarrow[S]{\simeq} & S_P \end{array}$$

where the vertical arrows are the canonical¹⁵⁶ maps $|sS_P| \rightarrow sS_P$ and $|S_P| \rightarrow S_P$, respectively, and S is a weak equivalence, i.e. all S_I are weak equivalences in the ordinary sense.

The reason why we can't work with $\mathsf{Y}(\mathsf{Prop})$ instead of U_P is that the latter is closed under the respective type forming operations up to equality and not just up to isomorphism as the former is.

If we start from a situation where \mathcal{E} is some finite limit category, \mathcal{S} is a pullback stable class of maps closed under composition and Π , contains all regular monos and a generic family $p_U : E_U \rightarrow U$ then we can use Voevodsky's "method of universes" for obtaining a split model (using global choice). If, moreover, we have a pullback stable subclass \mathcal{S}_P of \mathcal{S} such that

- (P1) $\Pi_f a \in \mathcal{S}_P$ whenever $a \in \mathcal{S}_P$ and $f \in \mathcal{S}$
- (P2) there is a subobject $m_P : U_P \rightarrow U$ with $m_P^* p_U$ generic for \mathcal{S}_P and the terminal projection $U_P \rightarrow 1_{\mathcal{E}}$ in \mathcal{S}

then the above splitting of \mathcal{S} restricts to one of \mathcal{S}_P . The attempt described in this note was motivated by establishing such a situation with $\mathcal{E} = \widehat{\mathbb{C}}$. It "only" failed in the respect that we couldn't get $U_P \rightarrow 1_{\mathcal{E}}$ in \mathcal{S} .

Quite generally, there arises the question whether for a finite limit category \mathbb{C} together with a map $p_U : E \rightarrow U$ in \mathbb{C} there does exist a splitting $S_{\mathbb{C}}$ of the fundamental fibration $P_{\mathbb{C}} = \partial_1 : \mathbb{C}^2 \rightarrow \mathbb{C}$ such that the cartesian equivalence $F :$

¹⁵⁶including its presheaf of objects into a split fibration

$P_{\mathbb{C}} \xrightarrow{\simeq} S_{\mathbb{C}}$ restricts to an equivalence between the full subfibrations generated by p_U and $F_U(p_U)$, respectively.

Well, we may achieve something quite close to this even for general Grothendieck fibrations $P : \mathbb{X} \rightarrow \mathbb{B}$ using the left adjoint splitting $L(P)$ of P . We consider the variant of $L(P)$ used by Lumsdaine and Warren making use of a normalized cleavage Cart_P of P . For $X \in \mathbb{X}$ let $P|X$ be the full subfibration of P on those objects from which there exists a cartesian arrow to X and $L(P)|X$ the full split subfibration of $L(P)$ on objects of the form (u, X) where the codomain of u is $P(X)$. Notice that for (u, X) there is a unique v with $(u, X) = v^*(\text{id}_{P(X)}, X)$, namely u . One may find a non-split equivalence $E_X : P|X \rightarrow L(P)|X$ such that the diagram

$$\begin{array}{ccc}
 P & \xrightarrow{F_P} & L(P) \\
 \uparrow & \simeq & \uparrow \\
 P|X & \xrightarrow[E_X]{\simeq} & L(P)|X
 \end{array}$$

commutes up to isomorphism in $\mathbf{Fib}(\mathbb{B})$.

227 A question by V. Gregoriades

Let 2^ω be Cantor space which we identify with $\mathcal{P}(\omega)$. For $A \in \mathcal{P}(\omega)$ its *Turing jump* A' is defined as $\{n \in \omega \mid \{n\}^A(n) \downarrow\}$. If $x \in 2^\omega$ is the characteristic function of A we write x' for the characteristic function of A' .

There is a recursive function $\sigma : \omega \rightarrow \omega$ such that

$$n \in A \quad \text{iff} \quad \sigma(n) \in A'$$

for all $A \in \mathcal{P}(\omega)$. Let $\tilde{\sigma} : 2^\omega \rightarrow 2^\omega$ with $\tilde{\sigma}(x)(n) = x(\sigma(n))$. Then for all $x \in 2^\omega$ we have

$$\tilde{\sigma}(x')(n) = 1 \quad \text{iff} \quad x'(\sigma(n)) = 1 \quad \text{iff} \quad x(n) = 1$$

i.e. $\tilde{\sigma}(x') = x$, i.e. $(\cdot)'$: $2^\omega \rightarrow 2^\omega$ is a monomorphism split by $\tilde{\sigma}$. Thus, a map $\varphi : 2^\omega \rightarrow 2^\omega$ factors through $(\cdot)'$: $2^\omega \rightarrow 2^\omega$ iff $(\cdot)' \circ \tilde{\sigma} \circ \varphi = \varphi$. Vassili's question now was whether for every continuous monic $\varphi : 2^\omega \rightarrow 2^\omega$ with $(\cdot)' \circ \tilde{\sigma} \circ \varphi = \varphi$ the map $\varphi \circ (\cdot)'$ is continuous, too.

Instead of starting from continuous monic $\varphi : 2^\omega \rightarrow 2^\omega$ with $(\cdot)' \circ \tilde{\sigma} \circ \varphi = \varphi$ one could equally well start from $\tilde{\sigma} \circ \varphi$, i.e. a continuous monic map $\theta : 2^\omega \rightarrow 2^\omega$ such that $(\cdot)' \circ \theta$ is continuous. Vassili's question then amounts to asking whether then $\theta \circ (\cdot)'$ is always continuous, too.

228 Toposes Fibered over a Topos

(12. March 2016, Correction 17.-24. April 2017)

Let \mathcal{S} be a topos and $P : \mathcal{X} \rightarrow \mathcal{S}$ a fibration. Then P is a *fibration of toposes*, i.e. all fibers are toposes and all reindexing functors u^* are logical, if and only if \mathcal{X} is a topos and P is a logical functor.

Since for a topos \mathcal{E} the family fibration $Fam(\mathcal{E}) : Fam(\mathcal{E}) \rightarrow \mathbf{Set}$ is a fibration of toposes the category $Fam(\mathcal{E})$ is a topos and the functor $Fam(\mathcal{E})$ is logical.

This is WRONG! One can show that (see sections 75 and 46) if $P : \mathcal{X} \rightarrow \mathcal{S}$ is a fibration of toposes with \mathcal{X} a topos and P logical (or just preserving the cartesian closed structure) then P has internal sums. By Jibladze's Theorem if P has internal sums then P is equivalent to $P_\Delta = \partial_1 : \mathcal{X}_1 \downarrow \Delta \rightarrow \mathcal{S}$ where $\Delta : \mathcal{S} \rightarrow \mathcal{E} : I \mapsto \coprod_I 1_I$. Thus, a fibered topos $P : \mathcal{X} \rightarrow \mathcal{S}$ has internal sums iff \mathcal{X} is a topos and P is a logical functor.

Let \mathcal{E} be the free topos (with nno) or any other elementary topos not having small sums (as e.g. a nontrivial realizability topos) then $Fam(\mathcal{E})$ is a fibration of toposes but even if $Fam(\mathcal{E})$ were a topos it does not preserve the cartesian closed structure since \mathcal{E} does not have small sums.

P. Hofstra in his Thesis proved (Th.6.2.3) that $Fam(\mathcal{C})$ is a topos iff \mathcal{C} is an atomic category (in the sense of Johnstone's 1977 book on Topos Theory, exercise 12 on p. 257). But in atomic categories all morphisms are epic. Thus, for a topos \mathcal{E} the category $Fam(\mathcal{E})$ is a topos iff all maps of \mathcal{E} are epic iff \mathcal{E} is trivial. **But, as pointed out by Menni Hofstra's result is wrong!** since for Grothendieck toposes \mathcal{E} the category $Fam(\mathcal{E})$ is equivalent to $\mathcal{E} \downarrow \Delta$ which is a topos because $\Delta : \mathbf{Set} \rightarrow \mathcal{E} : I \mapsto \coprod_I 1_{\mathcal{E}}$ preserves finite limits (Artin glueing).

From discussion with PTJ on 24th April 2017 we obtained the following

Theorem 228.1 *For an elementary topos \mathcal{E} the category $Fam(\mathcal{E})$ is cartesian closed iff \mathcal{E} has small sums.*

Proof: If \mathcal{E} has small sums then $Fam(\mathcal{E})$ is equivalent to $\mathcal{E} \downarrow \Delta$ which is a topos and thus cartesian closed.

Suppose $Fam(\mathcal{E})$ is cartesian closed. Since it has a subobject classifier anyway it is also a topos. The fibration $Fam(\mathcal{E})$ is equivalent to $U^* : Fam(\mathcal{E}) \rightarrow Fam(\mathcal{E})/U$ for $U = (1, (0))$ since every morphism to 0 is an iso in \mathcal{E} . Thus $Fam(\mathcal{E})$ is logical since U^* is logical. Thus, by the considerations above $Fam(\mathcal{E})$ has internal sums from which it follows that \mathcal{E} is cocomplete, i.e. has small sums. \square

Thus $Fam(\mathcal{E})$ is not cartesian closed if \mathcal{E} is the free topos (with nno) or a nontrivial realizability topos though $Fam(\mathcal{E})$ certainly is a fibered topos.

On 22nd April 2017 Peter Johnstone came up on the CATEGORIES list with a proof of the following theorem which was a precursor of Th. 228.1.

Theorem 228.2 *If \mathcal{E} is an elementary topos such that $Fam(\mathcal{E})$ cartesian closed then \mathcal{E} has copowers of 1.*

Proof: Suppose $\text{Fam}(\mathcal{E})$ is cartesian closed. First we show that $\text{Fam}(\mathcal{E})(1)$ is an exponential ideal in $\text{Fam}(\mathcal{E})$. One easily sees that $(1, (A))^{(1, (B))}$ is isomorphic to $(1, (A^B))$ in $\text{Fam}(\mathcal{E})$. Moreover $(1, A)^{(I, (B_i))}$ is isomorphic to $\prod_{i \in I} (1, A)^{(1, (B_i))}$ since $(I, (B_i))$ is isomorphic to $\prod_{i \in I} (1, (B_i))$ in $\text{Fam}(\mathcal{E})$. Since $\text{Fam}(\mathcal{E})$ has a left adjoint (picking initial objects in each fiber) the functor $\text{Fam}(\mathcal{E})$ preserves products. Thus $\prod_{i \in I} (1, A)^{(1, (B_i))}$ is in the fiber over a terminal object for which reason we can choose $(1, A)^{(I, (B_i))}$ as an object in the fiber over 1. Thus, in particular, the exponential $(1, A)^{(I, (1))}$ is isomorphic to $(1, C)$ where C is an I -fold power of A in \mathcal{E} . So \mathcal{E} has arbitrary set-indexed powers. But \mathcal{E}^{op} is monadic over \mathcal{E} , so it also has set-indexed powers, i.e. \mathcal{E} has set-indexed copowers. \square

229 $\text{Sp}(\mathbb{B})$ as a reflective subcategory of Cat/\mathbb{B}

The obvious forgetful functor from $\text{Sp}(\mathbb{B})$ to Cat/\mathbb{B} has a left adjoint S assigning to every $F : \mathbb{A} \rightarrow \mathbb{B}$ a split fibration $S(F)$ which is obtained via the Grothendieck construction from the presheaf of categories sending $I \in \mathbb{B}$ to the category $I \downarrow F$ and $u : J \rightarrow I$ to the functor $u^* : I \downarrow F \rightarrow J \downarrow F$ operating by precomposition with u . The underlying fibration of the split $S(F)$ is $\partial_0 : \text{Id}_{\mathbb{B}} \downarrow F \rightarrow \mathbb{B}$ and whose split cartesian arrows are those of the form

$$\begin{array}{ccc} J & \xrightarrow{v} & I \\ uv \downarrow & & \downarrow u \\ FA & \equiv & FA \end{array}$$

The unit of the adjunction at F is given by $E_F : F \rightarrow S(F)$ sending X to id_{FX} and $g : Y \rightarrow X$ to $(F(g), g)$. Its universal property can be seen as follows. Let $P : \mathbb{X} \rightarrow \mathbb{B}$ be a split fibration and $G : \mathbb{A} \rightarrow \mathbb{X}$ with $F = P \circ G$. Then there exist a unique split cartesian functor $\tilde{G} : S(F) \rightarrow P$ with $G = \tilde{G} \circ E_F$. An object $u : I \rightarrow FA$ is sent by \tilde{G} to u^*GA and a morphism

$$\begin{array}{ccc} J & \xrightarrow{v} & I \\ w \downarrow & & \downarrow u \\ FB & \xrightarrow{Ff} & FA \end{array}$$

is sent to the unique cartesian arrow $\varphi : w^*GB \rightarrow u^*GA$ over v with $\text{Cart}(u, GA) \circ \varphi = Gf \circ \text{Cart}(w, GB)$, i.e.

$$\begin{array}{ccc} w^*GB & \xrightarrow{\varphi} & u^*GA \\ \text{Cart}(w, GB) \downarrow & & \downarrow \text{Cart}(u, GA) \\ GB & \xrightarrow{Gf} & GA \end{array}$$

where \mathbf{Cart} is the split cleavage of P .

I think this construction can be found in J. Gray's old paper "The Categorical Comprehension Scheme".

230 Fibrations with internal sums à la Gray

With every category \mathbb{B} one may associate the span $\mathbb{B} \xleftarrow{\partial_1} \mathbb{B}^2 \xrightarrow{\partial_0} \mathbb{B}$ which is a monoid $\mathbb{M}(\mathbb{B})$ in $\mathbf{Span}(\mathbb{B}, \mathbb{B})$. A fibration over \mathbb{B} with internal sums is a functor $P : \mathbb{X} \rightarrow \mathbb{B}$ together with an action of the monoid $\mathbb{M}(\mathbb{B})$ on P where $(f, g) \cdot X$ is $\coprod_f g^* X$. Notice that the isomorphism $s \cdot (t \cdot X) \cong (s \cdot t) \cdot X$ correspond to the Chevalley condition of internal sums.

I think this can be found in J. Gray's book *Formal Category Theory*.

231 Fibrations in Bicategories à la Baković (Feb. 16)

Let \mathfrak{B} be a bicategory. Then a 1-cell $P : X \rightarrow B$ in \mathfrak{B} is called a fibration iff for all 1-cells $F : C \rightarrow B$ there exists a lax square

$$\begin{array}{ccc} F^*X & \xrightarrow{P^*F} & X \\ E_F \downarrow & \xRightarrow{\varepsilon} & \downarrow P \\ C & \xrightarrow{F} & B \end{array}$$

such that for all lax squares

$$\begin{array}{ccc} Y & \xrightarrow{G} & X \\ H \downarrow & \xRightarrow{\varphi} & \downarrow P \\ C & \xrightarrow{F} & B \end{array}$$

there exists unique $K : Y \rightarrow F^*X$ and $\psi : P^*F \circ K \Rightarrow G$ such that $E_F \circ K = H$ and $P\psi \circ \varepsilon K = \varphi$ as depicted in the following diagram

$$\begin{array}{ccccc} & & Y & & \\ & & \swarrow & & \\ & & K & & G \\ & & \searrow & & \searrow \\ & & & F^*X & \xrightarrow{P^*F} & X \\ & & & \downarrow E_F & \xRightarrow{\varepsilon} & \downarrow P \\ & & & C & \xrightarrow{F} & B \\ & & & \swarrow & & \\ & & & H & & \end{array}$$

This is ok when \mathfrak{B} is a 2-category but when \mathfrak{B} is a genuine bicategory one should weaken the requirement by just claiming $E_F \circ K \cong H$ instead of $E_F \circ K = H$.

Explicitate what it means to be a fibration in the bicategory \mathfrak{Dist} of small categories and distributors.

The intuition is that Σ_P has right adjoint P^* but for “lax” slices \mathfrak{B}/X where morphism from $G' : Y' \rightarrow X$ to $G : Y \rightarrow X$ are pairs (K, ψ) with $K : Y' \rightarrow Y$ and $\psi : GK \Rightarrow G'$. That’s adapted from Johnstone’s 1993 paper *Fibrations and Partial Products in a 2-Category*.

232 Grothendieck Toposes from Moral Tripases

“Moral Tripases” over a base topos \mathcal{S} are those posets fibred over \mathcal{S} which give rise to a topos by “adding subquotients”. Up to equivalence they can be characterized as finite limit preserving functors $F : \mathcal{S} \rightarrow \mathcal{E}$ where \mathcal{E} is a topos such that $1_{\mathcal{E}}$ is a *bound* for F , i.e. every object of \mathcal{E} appears as subquotient of some FI . Thus, moral tripases may be understood as “weakly localic geometric morphisms” from \mathcal{E} to \mathcal{S} .

Tripases over \mathcal{S} as originally introduced by Hyland, Johnstone and Pitts in the late 1970s in [HJP] are moral tripases F such that the fibred poset $F^*\text{Sub}_{\mathcal{E}}$ has a generic family $\tau : T \rightarrow F\Sigma$.

We give examples of moral tripases over **Set** whose associated toposes are non-localic Grothendieck toposes. Let \mathcal{E} be the topos of reflexive graphs or the topos $\mathbf{Set}^{\Delta^{\text{op}}}$ of simplicial sets and $F = \nabla$, the right adjoint of $\Gamma = \mathcal{E}(1, -) : \mathcal{E} \rightarrow \mathbf{Set}$. One can show that every objects of \mathcal{E} appears as subquotient of FI for some set I . But in both cases $F^*\text{Sub}_{\mathcal{E}}$ is just a moral tripase. Nevertheless, this is somewhat surprising since moral tripases over **Set** are “weakly localic geometric morphisms to **Set**”. It is an open question whether non-localic Grothendieck toposes may arise from genuine tripases over **Set**.

233 Are Set-based tripases determined by their associated toposes? (June 2016)

Tripases over a base topos \mathcal{S} correspond to finite limit preserving functors F from \mathcal{S} to some topos \mathcal{E} such that (1) every A in \mathcal{E} appears as subquotient of FI for some I in \mathcal{S} and (2) $F^*\text{Sub}_{\mathcal{E}}$ admits a generic family $\tau : T \rightarrow F\Sigma$, i.e. every mono $m : P \rightarrow FI$ fits into a pullback diagram

$$\begin{array}{ccc} P & \longrightarrow & T \\ m \downarrow & \lrcorner & \downarrow \tau \\ FI & \xrightarrow{Fp} & F\Sigma \end{array}$$

for some $p : I \rightarrow \Sigma$ in \mathcal{S} typically not unique with this property. Such functors F are called *constant object functors* and the corresponding tripase is $F^*\text{Sub}_{\mathcal{E}}$. For a tripase \mathcal{P} over \mathcal{S} the corresponding constant objects functor is denoted as $\Delta_{\mathcal{P}}$ or simply Δ when \mathcal{P} is clear from the context.

For the case where the base topos \mathcal{S} is **Set** it has been shown already in Th. 4.1 of [HJP] that the tripase $F^*\text{Sub}_{\mathcal{E}}$ is localic, i.e. locally small as a fibration, iff F preserves small sums (of 1s). But it is not clear whether any constant objects functor from **Set** to a localic topos \mathcal{E} does preserve coproducts of 1 and thus is uniquely determined by \mathcal{E} . What if F is a constant objects functor from **Set** to a Grothendieck topos \mathcal{E} , does it necessarily preserve coproducts of 1? If this were the case then \mathcal{E} were automatically localic. But as can be seen from

the counterexamples of section 232 there are at least moral triposes over **Set** giving rise to non-localic Grothendieck toposes.

Quite generally one may ask whether all functors from **Set** to \mathcal{E} corresponding to triposes (all necessarily giving rise to the same topos \mathcal{E}) are necessarily isomorphic? For moral triposes the answer is negative at least when \mathcal{E} is **Set**. For natural numbers $n > 0$ the functor $F_n : \mathbf{Set} \rightarrow \mathbf{Set} : S \mapsto S^n$ gives rise to a moral tripos since F_n preserves finite limits and $F_n(S)$ covers S for all sets S . But for $n > m > 0$ the moral triposes corresponding to F_n and F_m are not equivalent (since $2^n \neq 2^m$) although they both give rise to the same localic topos **Set**. Thus, there are infinitely many non-equivalent moral triposes over **Set** all giving rise to the same localic topos **Set**.

Of course, whenever \mathcal{E} is a (nontrivial) localic tripos over **Set** then besides the unique geometric morphism $\Delta \dashv \Gamma : \mathcal{E} \rightarrow \mathbf{Set}$ for every natural number $n > 0$ there is a moral tripos $F_n : \mathbf{Set} \rightarrow \mathcal{E}$ sending $I \in \mathbf{Set}$ to $F_n(I) = \Delta(I)^n \cong \Delta(I^n)$ (since F_n clearly preserves finite limits and every $A \in \mathcal{E}$ appears as quotient of some subobject $m : P \twoheadrightarrow \Delta(I)$ and thus also of $\langle m \rangle_{i=1, \dots, n} : P \twoheadrightarrow \Delta(I)^n = F_n(I)$). Obviously, for $n > m > 0$ the functors F_n and F_m are not isomorphic though they both induce the same topos \mathcal{E} .

234 Summary of van Oosten and Zou (2016)

Let \mathcal{D} be the downset monad on the 2-category **BCO** of *basic combinatorial objects* (BCOs). It is a KZ-monad and thus being a \mathcal{D} -algebra is a property of BCOs.

With every BCO Σ there is associated a preorder $[-, \Sigma]$ fibered over **Set**. Hofstra has shown that a BCO Σ is equivalent to a filtered opca (A, A') iff $[-, \mathcal{D}\Sigma]$ is a tripos. Tripuses of the form $[-, \mathcal{D}(A, A')]$ are called *relative realizability tripuses* and their associated toposes are called *relative realizability toposes*.

In [vOZ16] (Cor.1.17) it has been shown that tripuses of the form $[-, \Sigma]$ for some BCO Σ arise as subtripuses of *relative realizability tripuses*. In Th.1.21 they have given a characterization of those BCOs Σ for which $[-, \Sigma]$ is a tripos as *pre-implicative opcas*.

[It follows from a result of Hofstra that toposes arising from such tripuses are Grothendieck iff they are localic. **But see section 233!**]

It would be nice to have a characterization of those subtripuses of relative realizability tripuses which are of the form $[-, \Sigma]$ for some BCO Σ . It would be nice to see at least an example of a subtripos of a relative realizability tripos which is not of this form.

The main result is that classical realizability tripuses are precisely the boolean subtripuses of relative realizability tripuses.

[They are all induced by a filtered opca and thus by a BCO from which it follows that classical realizability toposes which are Grothendieck are already localic. **NO! see section 233!** Thus one cannot use any of the Grothendieck toposes refuting countable choice for showing that not all classical realizability tripuses validate countable choice.]

The tripuses obtained by booleanization from (generalized) relative realizability tripuses are not localic. Alas, we don't know whether this holds also for the induced toposes. Nor do we know that they are not Grothendieck since by section 163 tripuses over **Set** may induce Grothendieck toposes.

Another important result of their paper is that a prominent ordinary relative realizability tripos, namely the one inducing the Kleene-Vesley topos \mathcal{KV} , contains a lot of boolean subtripuses which are not localic. More precisely, they show that the subtripos induced by $\neg_U \neg_U$ is not localic iff U is a subset of Baire space containing no computable elements such that all elements of U are isolated in the subspace topology. Typical such examples are finite (non-empty) sets of non-recursive functions as e.g. $U = \{\tau\}$ where τ decides the halting problem. It is an open problem whether the boolean subtopos of \mathcal{KV} induced by this U validates countable choice.

235 Preservation of ICC under Booleanization

Let \mathcal{E} be a topos with a mno N . Then \mathcal{E} validates *Internal Countable Choice* (ICC) iff e^N is epic whenever e is epic.

In all Grothendieck toposes \mathcal{E} a natural numbers object N is given by $\coprod_{\mathbb{N}} 1$ for which reason $(-)^N : \mathcal{E} \rightarrow \mathcal{E}$ is isomorphic to $(-)^{\mathbb{N}} = \prod_{\mathbb{N}} (-) \circ \mathbb{N}^* : \mathcal{E} \rightarrow \mathcal{E}$. As

observed by A. Simpson (Jan. 2018) all presheaf toposes validate ICC since in them morphisms are epic iff all their components are onto and this property is preserved by $(-)^{\mathbb{N}}$ which is isomorphic to $(-)^N$. Since there are Grothendieck toposes not validating ICC sheafification does not preserve ICC in general. Since there are boolean Grothendieck toposes not validating ICC booleanization of Grothendieck toposes does not preserve ICC.

Analogously, generalized relative realizability toposes all validate ICC but presumably their booleanizations, i.e. classical realizability toposes, do not all validate ICC.

236 Tribes via Representable Morphisms

Let \mathbb{B} be a category with pullbacks. A class of *display maps* or *tribe* (recent terminology of Joyal) in \mathbb{B} is a class \mathcal{D} of morphisms in \mathbb{B} stable under arbitrary pullbacks in \mathbb{B} . For any tribe \mathcal{D} one can find a representable morphism $p_{\mathcal{D}} : E_{\mathcal{D}} \rightarrow U_{\mathcal{D}}$ in $\widehat{\mathbb{B}}$ such that $u : J \rightarrow I$ is in \mathcal{D} iff there is a pullback

$$\begin{array}{ccc} Y(J) & \longrightarrow & E_{\mathcal{D}} \\ \downarrow Y(u) & \lrcorner & \downarrow p_{\mathcal{D}} \\ Y(J) & \longrightarrow & U_{\mathcal{D}} \end{array}$$

in $\widehat{\mathbb{B}}$. Such a $p_{\mathcal{D}}$ can be obtained as

$$\begin{array}{ccc} E_{\mathcal{D}} & \hookrightarrow & E_{\mathbf{set}(\mathbb{B})} \\ \downarrow p_{\mathcal{D}} & \lrcorner & \downarrow p_{\mathbf{set}(\mathbb{B})} \\ U_{\mathcal{D}} & \hookrightarrow & \mathbf{set}(\mathbb{B}) \end{array}$$

where $U_{\mathcal{D}}$ is the subobject of $\mathbf{set}(\mathbb{B})$ (c.f. section 92) consisting of all presheaves over some \mathbb{B}/I which are representable by a map in \mathcal{D} with codomain I .

As observed independently by Fiore and Awodey a model of dependent type theory over \mathbb{B} is given by a representable morphism $p : E \rightarrow U$ in $\widehat{\mathbb{B}}$ together with a choice of a pullback

$$\begin{array}{ccc} Y(I.A) & \xrightarrow{q_A} & E \\ \downarrow p_A & \lrcorner & \downarrow p \\ Y(I) & \xrightarrow{A} & U \end{array}$$

for every $A \in U(I)$. For every $u : J \rightarrow I$ in \mathbb{B} let

$$\begin{array}{ccc} Y(J.Au) & \xrightarrow{q(u, A)} & I.A \\ \downarrow p_{Au} & \lrcorner & \downarrow p_A \\ Y(J) & \xrightarrow{Y(u)} & I \end{array}$$

with $q_A \circ q(u, A) = q_{Au}$ (where we write Au for $A \circ Y(u)$).

A predicate on U is given by a subpresheaf P of U giving rise to

$$\begin{array}{ccc} E_P & \hookrightarrow & E \\ p_P \downarrow & \lrcorner & \downarrow p \\ P & \hookrightarrow & U \end{array}$$

where for $A \in P(I)$ the chosen pullback is

$$\begin{array}{ccc} Y(I, A) & \xrightarrow{q_A} & E_P \\ p_A \downarrow & \lrcorner & \downarrow p_P \\ Y(I) & \xrightarrow{A} & P \end{array}$$

giving rise to a submodel in a very strong sense.

The characteristic predicate of $P \subseteq U$ is the map $p : U \rightarrow \Omega$ with $p_I(A) = \{u : J \rightarrow I \mid Au \in P\}$. We say that $P \subseteq U$ is *definable* (in the sense of Bénabou) iff for all $A \in P(I)$ the subobject $p_I(A)$ of $Y(I)$ is representable, i.e. there is a subobject m_A of I in \mathbb{B} such that

- (1) $Am_A \in P$ and
- (2) $u : J \rightarrow I$ factors through m_A whenever $Au \in P$.

Obviously $P \subseteq U$ is definable iff the inclusion $P \hookrightarrow U$ is a representable morphism. Let $\omega \subseteq \Omega$ be the subpresheaf consisting of representable sieves. Then $\top : 1 \rightarrow \omega$ classifies representable monomorphisms. Thus $P \subseteq U$ is definable iff its characteristic map p factors through ω .

A model of dependent type theory is called *small* iff U is representable, i.e. $p : E \rightarrow U$ is isomorphic to a map in \mathbb{B} . Thus, a small model of dependent type theory amounts to a map $p : E \rightarrow U$ in \mathbb{B} together with a choice of pullbacks for all morphisms in \mathbb{B} with codomain U for which reason a small model is traditionally called a *universe* (in \mathbb{B}). Submodels P of small models U need not be small in general but they are iff they are definable.

237 An Observation by I. Orton (Summer 2017)

In extensional type theory we consider universes U_0 and U_1 which are nested in the sense that $U_0 \in U_1$ and $U_0 \subseteq U_1$. We further assume that there is a proof irrelevant universe Prop with $\text{Prop} \in U_1$ and $\text{Prop} \subseteq U_0$. Let N_1 be the unit type containing precisely one element $*$. We assume $N_1 \in U_0$. In presheaf toposes $\widehat{\mathbb{C}}$ we get such universes U_0 and U_1 from Grothendieck universes \mathcal{U}_0 and \mathcal{U}_1 with $\mathcal{U}_0 \in \mathcal{U}_1$ and $\mathbb{C} \in \mathcal{U}_0$.

Suppose $P \in (\Gamma|U_1)(\Gamma \rightarrow U_0) \rightarrow \text{Prop}$ which is pullback stable in the sense that there is a P' in $(\Gamma, \Delta|U_1)(u : \Delta \rightarrow \Gamma)(A : \Gamma \rightarrow U_0) P(A) \rightarrow P(A \circ u)$. We write $\text{PFam} : U_1^{\text{op}} \rightarrow U_1$ for the functor with $\text{PFam}(\Gamma) = \{A : \Gamma \rightarrow U_0 \mid P(A)\}$ and $\text{PFam}(u)(A) = A \circ u$ (one needs P' for defining the morphism part of PFam since subset types are actually Σ -types). We further assume that there is a $U \in U_1$, $E : \text{PFam}(U)$ and $\text{code} \in (\Gamma|U_1)\text{PFam}(\Gamma) \rightarrow \Gamma \rightarrow U$ with $E \circ \text{code}(A) = A$ for $A \in \text{PFam}(\Gamma)$. Let $\text{PFib} \in (\Gamma|U_1)(A : \Gamma \rightarrow U_0) \rightarrow \text{Prop}$ with $\text{PFib}(A) = (x : \Gamma)P(A(x))$ (where as usual we identify elements of a type X with functions from N_1 to X).

Now the surprising observation of Ian Orton is that

$$\text{PFib}(A) \rightarrow P(A)$$

for all $A : \Gamma \rightarrow U_0$.

Proof: Suppose $\Gamma \in U_1$ and $A : \Gamma \rightarrow U_0$ with $\text{PFib}(A)$. Let $\text{AU} : \Gamma \rightarrow U$ with $\text{AU}(x) = \text{code}(A(x))$. We have $E(\text{AU}(x)) = E(\text{code}(A(x))) = A(x)$ and thus by function extensionality that $E \circ \text{AU} = A$ from which it follows that $P(A)$ since $P(E \circ \text{AU})$ because $P(E)$ and P is pullback stable (by P'). \square

A triple (P, U, E, code) as above may be called an *internal universe*.

In a recent paper from January 2018 by Licata, Orton, Pitts and Spitters with title *Internal Universes in Models of Homotopy Type Theory* the authors show that there is no internal universe of CCHM^{157} fibrations based on an axiomatically postulated interval object \mathbb{I} with two distinct global elements 0 and 1 since otherwise the family $P = \lambda i : \mathbb{I}. \text{Id}_{\mathbb{I}}(0, i)$ were CCHM fibrant from which it follows that $\text{Id}_{\mathbb{I}}(0, 1)$.¹⁵⁸

¹⁵⁷stands for Cohen, Coquand, Huber and Mörtberg, the authors of the paper from 2015 where they introduce a Cubical Type Theory together with a presheaf model for it. This model was presented in a more axiomatic way in 2016 by Orton and Pitts.

¹⁵⁸Here Id stands for extensional identity types and P is a CCHM fibration by Orton's observation since all its items are CCHM fibrant because they all contain at most one element.

238 Why the universe of small Kan objects is Kan itself in sSet (Nov. 2017)

If in the topos $\mathbf{sSet} = \mathbf{Set}^{\Delta^{\text{op}}}$ of simplicial sets one wants to construct a universe of Kan objects then following Yoneda one sees that it has to look as follows. Let \mathcal{U} be a Grothendieck universe. In order to come up with a weak classifier for Kan fibrations whose fibers have a cardinality living in \mathcal{U} Yoneda tells us to construct it as the map $\pi : E \rightarrow U$ where $U([n])$ consists of all $A \in \mathcal{U}^{(\Delta/[n])^{\text{op}}}$ such that $\int A \rightarrow \Delta[n]$ obtained by Grothendieck construction is a Kan fibration, $E([n], A) = A(\text{id}_{[n]})$ and π is first projection of $\int E$ to U . If $p : Y \rightarrow X$ is a Kan fibration whose fibers have cardinality in \mathcal{U} then p fits into a pullback

$$\begin{array}{ccc}
 Y & \hookrightarrow & E \\
 \downarrow p & \lrcorner & \downarrow \pi \\
 X & \xrightarrow{f} & U
 \end{array}
 \quad (1)$$

where $f_I(x)(u : J \rightarrow I) = p_J^{-1}(x)$ and $f_I(x)(v : uv \rightarrow u)$ is the restriction of $Y(v)$ to a map from $f_I(x)(u)$ to $f_I(x)(uv)$. We call f the “canonical” weak classifier for p . Notice that we implicitly identify maps to X with presheaves over $\mathbf{Elts}(X)$, the category of elements of X , as justified by the well known equivalence $\widehat{\mathcal{C}}/X \simeq \widehat{\mathbf{Elts}(X)}$ for arbitrary small categories \mathcal{C} .

Joyal has given an argument why for every anodyne mono $i : X \rightarrow X'$ every \mathcal{U} -small Kan fibration $p : Y \rightarrow X$ fits into a pullback diagram

$$\begin{array}{ccc}
 Y & \xrightarrow{i'} & Y' \\
 \downarrow p & \lrcorner & \downarrow p' \\
 X & \xrightarrow{i} & X'
 \end{array}
 \quad (2)$$

where p' is a Kan fibration. Since in presheaf toposes all monos factor as inclusions followed by isos we may assume that i' in (2) is actually an inclusion.

For showing that U is Kan it suffices to show that given (2) with i' an inclusion the canonical weak classifiers f and f' for p and p' , respectively, validate the equality

$$f = f' \circ i, \text{ i.e. } Y' \circ \mathbf{Elts}(i) = Y$$

which can be checked as follows

$$\begin{aligned}
 (f' \circ i)(x)(u : J \rightarrow I) &= Y'(J, u^*(i(x))) = Y'(J, i(u^*x)) = Y(J, u^*x) = f(x)(u) \\
 (f' \circ i)(x)(v : uv \rightarrow u) &= f'(i(x))(v : uv \rightarrow u) = Y'(\mathbf{Elts}(i)((uv)^*x \rightarrow u^*x)) = \\
 &= (Y' \circ \mathbf{Elts}(i))((uv)^*x \rightarrow u^*x) = Y((uv)^*x \rightarrow u^*x) = \\
 &= f(x)(v : uv \rightarrow u)
 \end{aligned}$$

239 “Propositional Truncation” in $\mathbf{Set}^{\Delta_1^{\text{op}}}$

Consider the topos $\mathcal{E} = \mathbf{Set}^{\Delta_1^{\text{op}}}$ of reflexive graphs. We have a sequence of adjoints $\Pi \dashv \Delta \dashv \Gamma \dashv \nabla : \mathbf{Set} \rightarrow \mathcal{E}$ where Δ and ∇ are full and faithful. Like in every topos the inclusion of subterminals into \mathcal{E} has a left adjoint supp sending X in \mathcal{E} to the image of $X \rightarrow 1$. Of course, in the topos \mathcal{E} of reflexive graphs 1 has only two subobjects, namely 0 and 1 . Notice that $N = \Delta(\mathbb{N})$ is the natural numbers object of \mathcal{E} .

But for every $X \in \mathcal{E}$ we may consider $\eta_X : X \rightarrow \nabla\Gamma X$ which is the identity on nodes and where $\nabla\Gamma X$ is the full graph on the set of nodes of X . We may consider $\nabla\Gamma X$ as the “propositional truncation” of X in the sense of HoTT since all nodes are considered as equal without really identifying them. Thus “propositions” are those X for which η_X is an iso, i.e. objects isomorphic to some ∇I . Notice that there are class many propositions up to iso whereas there are just two subterminals up to iso.

Let X be the subobject of $\nabla(\mathbb{N} \times \mathbb{N})$ where there is an edge between (n, m) and (n', m') iff $n \neq n'$ or $m = m'$ and $p : X \rightarrow \nabla(\mathbb{N})$ be the map with $p(n, m) = n$. Every map $f : \mathbb{N} \rightarrow \mathbb{N}$ gives rise to a section s_f of p with $s_f(n) = (n, f(n))$ and every section s of p equals s_f for a unique $f \in \mathbb{N}^{\mathbb{N}}$. In particular, we have the section $s = s_{\text{id}_{\mathbb{N}}}$ of p . This s may be understood as a semantic analogue of N. Kraus’s hack allowing to “unpack” elements of $\nabla(\mathbb{N})$. But notice that p is not isomorphic to $\pi : \nabla(\mathbb{N}) \times N \rightarrow \nabla(\mathbb{N})$ although for every $n : 1 \rightarrow \nabla(\mathbb{N})$ we have $n^*p \cong n^*\pi \cong \Delta(\mathbb{N})$. Since there is an edge between (n, m) and (n', m') in $\nabla(\mathbb{N}) \times N$ iff $m = m'$ it is rather the case that π is a proper subobject of p in $\mathcal{E}/\nabla(\mathbb{N})$.

240 Nerve as Left Kan Extension (P.-A. Melliès)

Let $i : \Delta \hookrightarrow \mathbf{Cat}$. Then the nerve functor $\text{Nv} : \mathbf{Cat} \rightarrow \widehat{\Delta}$ is given by $\text{Nv}(\mathcal{C}) = \mathbf{Cat}(i(-), \mathcal{C})$. It has a left adjoint τ_1 given by left Kan extension of i along Y_{Δ} . But as observed by Paul-André Melliès the nerve functor itself appears as left Kan extension of Y_{Δ} along i because $\partial_0 : i \downarrow \mathcal{C} \rightarrow \Delta$ is isomorphic to $\int \text{Nv}(\mathcal{C}) : \text{Elts}(\text{Nv}(\mathcal{C})) \rightarrow \Delta$ and $\text{Nv}(\mathcal{C})$ is canonically isomorphic to the colimit of $Y_{\Delta} \circ \int \text{Nv}(\mathcal{C})$.

241 Morita Equivalence in terms of Distributors

Categories \mathcal{A} and \mathcal{B} are called *Morita equivalent* iff their categories of presheaves $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{B}}$ are equivalent. Thus \mathcal{A} and \mathcal{B} are Morita equivalent iff they are equivalent in the bicategory \mathbf{Dist} of distributors since distributors between categories correspond to cocontinuous functors between the associated categories of presheaves. Categories where idempotents split are Morita equivalent iff they are equivalent as ordinary categories since distributors from \mathcal{A} to \mathcal{B} having a right adjoint correspond to ordinary functors from \mathcal{A} to the “Cauchy completion” of \mathcal{B} obtained by splitting idempotents.

242 Test Categories

Grothendieck in his *Pursuing Stacks* asked the question for which small categories \mathcal{C} one may impose a model structure on $\widehat{\mathcal{C}}$ which is equivalent to the canonical model structure on the topos $\mathbf{sSet} = \widehat{\Delta}$ of simplicial sets.

In this context one only considers model structures where monos are the cofibrations, i.e. Cisinski model structures. The canonical model structure on the topos $\widehat{\Delta}$ of simplicial is a Cisinski model structure where a map w is a weak equivalence iff its geometric realization $|w|$ is a homotopy equivalence in \mathbf{Sp} .

Let $\mathbf{Nv} : \mathbf{Cat} \rightarrow \widehat{\Delta}$ be the *nerve* functor sending a small category \mathcal{A} to the simplicial set $\mathbf{Cat}(i, \mathcal{A})$ where $i : \Delta \rightarrow \mathbf{Cat}$ is the obvious inclusion. The left adjoint \mathbf{C} to \mathbf{Nv} is called *categorical realization*. Thus $\mathbf{C} \dashv \mathbf{Nv}$ is the adjunction induced by the inclusion $i : \Delta \rightarrow \mathbf{Cat}$. A functor w in \mathbf{Cat} is called a weak equivalence iff $\mathbf{Nv}(w)$ is a weak equivalence in \mathbf{sSet} , i.e. iff $|\mathbf{Nv}(w)|$ is a homotopy equivalence of spaces.

For a small category \mathcal{C} let $i_{\mathcal{C}} : \widehat{\mathcal{C}} \rightarrow \mathbf{Cat}$ be the functor sending X to $\mathbf{Elts}(X) = \mathbf{Y}_{\mathcal{C}} \downarrow X$, the category of elements of X . Its right adjoint $i_{\mathcal{C}}^* : \mathbf{Cat} \rightarrow \widehat{\mathcal{C}}$ is given by sending \mathcal{A} to $\mathbf{Cat}(\mathcal{C}/c, \mathcal{A})$. Thus $i_{\mathcal{C}} \dashv i_{\mathcal{C}}^*$ is the adjunction induced by the functor $\mathcal{C}/(-) : \mathcal{C} \rightarrow \mathbf{Cat}$. The class $W_{\infty}(\mathcal{C})$ of weak equivalences in $\widehat{\mathcal{C}}$ consist of all maps which are sent by $i_{\mathcal{C}}$ to a weak equivalence in \mathbf{Cat} .

A small category \mathcal{C} is called a *weak test category* iff $i_{\mathcal{C}}^*$ sends weak equivalences in \mathbf{Cat} to weak equivalences in $W_{\infty}(\mathcal{C})$. This induces a Cisinski model structure on $\widehat{\mathcal{C}}$ called *test model structure* such that $i_{\mathcal{C}} \dashv i_{\mathcal{C}}^*$ is a Quillen equivalence between $W_{\infty}(\mathcal{C})$ and \mathbf{Cat} with the respective model structures.

For $\mathcal{C} = \Delta$ the adjunction $i_{\Delta} \dashv i_{\Delta}^*$ is a Quillen equivalence between the canonical model structures on \mathbf{sSet} and \mathbf{Cat} respectively. In particular, a functor w between small categories is a weak equivalence, i.e. $\mathbf{Nv}(w)$ is a weak equivalence in \mathbf{sSet} , iff $i_{\Delta}^*(w)$ is a weak equivalence in \mathbf{sSet} . Notice that $i_{\Delta} \dashv i_{\Delta}^*$ is a Quillen equivalence though $\mathbf{C} \dashv \mathbf{Nv}$ in general is not.

A small category is called *aspherical* iff its terminal projection is a weak equivalence in \mathbf{Cat} and $F \in \widehat{\mathcal{C}}$ is called *aspherical* iff $i_{\mathcal{C}}(F)$ aspherical. Grothendieck has shown that \mathcal{C} is a weak test category iff $i_{\mathcal{C}}^*(\mathcal{A})$ is aspherical for all $\mathcal{A} \in \mathbf{Cat}$ with a terminal object.

Alas, the notion of weak test category is not stable under slicing. One calls \mathcal{C} a *local test category* iff all its slices \mathcal{C}/c are weak test categories. Finally, one calls \mathcal{C} a *test category* iff \mathcal{C} and all its slices are weak test categories.

Grothendieck has characterized local test categories as the categories \mathcal{C} such that $\Sigma_{\mathcal{C}}^* \Omega_{\widehat{\mathcal{C}}}$ is an aspherical presheaf over \mathcal{C}/c for all $c \in \mathcal{C}$. Notice that the sub-object classifier $\Omega_{\widehat{\mathcal{C}}}$ is isomorphic to $i_{\mathcal{C}}^*[2]$ where 2 is the poset $0 \rightarrow 1$ considered as an object of \mathbf{Cat} .

The category \mathbf{FL} of finite lattices and monotone maps between them is known to be a test category. However, it is an open problem iff the test model structure on $\mathbf{cSet} = \widehat{\mathbf{FL}}$ coincides with the minimal Cisinski model structure generated by open box inclusions.

243 Descent and Stacks

Let $P \in \mathbf{Fib}(\mathbb{B})$. Then a *descent map w.r.t. P* is a morphism $f : X \rightarrow Y$ in $\mathbf{dFib}(\mathbb{B})$, the full sub-2-category of $\mathbf{Fib}(\mathbb{B})$ on discrete fibrations, such that

$$\mathbf{Fib}(\mathbb{B})(f, P) : \mathbf{Fib}(\mathbb{B})(Y, P) \rightarrow \mathbf{Fib}(\mathbb{B})(X, P)$$

is an equivalence of (ordinary) categories. We write $f \perp P$ for f being a descent map w.r.t. P .

If \mathfrak{J} is a Grothendieck topology on \mathbb{B} then a \mathfrak{J} -stack is a fibration $P \in \mathbf{Fib}(\mathbb{B})$ such that $i_S : S \hookrightarrow y(I)$ is a descent map w.r.t. P for all \mathfrak{J} -covers S of I , i.e.

$$\mathbf{Fib}(\mathbb{B})(i_S, P) : \mathbf{Fib}(\mathbb{B})(y(I), P) \rightarrow \mathbf{Fib}(\mathbb{B})(S, P)$$

for all \mathfrak{J} covers S of I .¹⁵⁹ Obviously, a discrete fibration is a \mathfrak{J} -stack iff it is a \mathfrak{J} -sheaf.

For $P \in \mathbf{Fib}(\mathbb{B})$ we may consider just the monos m in $\widehat{\mathbb{B}} \simeq \mathbf{dFib}(\mathbb{B})$ with $m \perp P$ and, actually, for the purposes of stacks it is enough to consider those sieves $S \subseteq y(I)$ for which $i_S \perp P$.

244 N. Rasekh's Higher Elementary Toposes

are defined (in his Thesis) as an infinite dimensional variant of the following 1-categorical notion. A category \mathcal{E} is a *topos with sufficiently many universes* iff \mathcal{E} is a category with finite limits, disjoint and stable finite sums and a subobject classifier such that every morphism of \mathcal{E} appears as pullback of some $\pi : E \rightarrow U$ which is a *small subuniverse of \mathcal{E}* in the sense that the full subfibration of $P_{\mathcal{E}}$ consisting of pullbacks of π is locally small, i.e. $f \rightarrow_I g$ is a pullback of π whenever f and g are pullbacks of π .

Notice that for f_1, \dots, f_n in \mathcal{E} there exists a small subuniverse $\pi : E \rightarrow U$ of \mathcal{E} such that all f_i appear as pullbacks of π (just take a subuniverse of \mathcal{E} of which $f_1 + \dots + f_n$ appears as pullback).

¹⁵⁹Vistoli's survey paper *Notes on Grothendieck topologies, fibered categories and descent theory* (2008) proves this definition of \mathfrak{J} -stack to be equivalent to the traditional definition in terms of "descent data" though both Grothendieck and Giraud used the one we have given. As I understand "descent data" are just a "concrete" way of describing cartesian functors from S to P in an "elementary" way avoiding the abstract but convenient way it is done using the language of fibrations and cartesian functors between them.

E.g. in B.1.5 of Johnstone's *Elephant* he only gives the formulation in terms of "descent data". However, this is already more general than the formulation one finds at many places in the literature where one just considers the special case of the inclusion of $S_u = \{uv \mid \text{dom}(u) = \text{cod}(v)\}$ into $y(I)$ for $u : J \rightarrow I$ in \mathbb{B} and calls u a descent map w.r.t. P iff the inclusion of S_u into $y(I)$ is a descent map w.r.t. P in the original sense of Grothendieck and Giraud (which is also recalled in Bunge and Paré's CTGD article). This restricted notion is not general enough for defining \mathfrak{J} -stack in general but suffices in case where \mathfrak{J} is generated by \mathfrak{J} -covers of this restricted form as e.g. in case of the regular cover topology which is generated by the S_e where e is a regular epi in \mathbb{B} .

245 Quasicategories inside Bisimplicial Sets

The underlying idea of the 2017 Riehl-Shulman paper is to consider a model of type theory within *bisimplicial sets*, i.e. $\mathbf{bsSet} = \mathbf{sSet}^{\Delta^{\text{op}}} \cong \mathbf{Set}^{\Delta^{\text{op}} \times \Delta^{\text{op}}}$.

There is an obvious inclusion $F^* : \mathbf{sSet} \rightarrow \mathbf{bsSet}$ where $F : \Delta \times \Delta \rightarrow \Delta$ is second projection, i.e. F^* sends a simplicial set X to the constant functor in $\mathbf{sSet}^{\Delta^{\text{op}}}$ with value X . This F^* is the inverse image part of a geometric morphism whose direct image part is given by $U^* : \mathbf{bsSet} \rightarrow \mathbf{sSet}$ where U is right adjoint to F , i.e. $U : \Delta \rightarrow \Delta \times \Delta$ sends $[n]$ to $\langle [0], [n] \rangle$. Thus U^* sends an $A \in \mathbf{Set}^{\Delta^{\text{op}} \times \Delta^{\text{op}}}$ to $A([0])$. Thus, the geometric morphism $F^* \dashv U^*$ exhibits \mathbf{bsSet} as simplicial sets over base topos \mathbf{sSet} .

Of course, there is also a geometric morphism $G^* \dashv V^*$ from \mathbf{bsSet} to \mathbf{sSet} where $G : \Delta \times \Delta \rightarrow \Delta$ is first projection and V sends $[n]$ to $\langle [n], [0] \rangle$. Notice that V^* is $\Gamma^{\Delta^{\text{op}}} : \mathbf{sSet}^{\Delta^{\text{op}}} \rightarrow \mathbf{Set}^{\Delta^{\text{op}}} = \mathbf{sSet}$ where Γ is $\mathbf{sSet}(\Delta[0], -)$. Thus V^* sends a simplicial set in \mathbf{sSet} to a simplicial set by applying Γ in each component. The functor G^* on the other hand is $\Delta^{\Delta^{\text{op}}} : \mathbf{Set}^{\Delta^{\text{op}}} \rightarrow \mathbf{sSet}^{\Delta^{\text{op}}}$ where the base $\Delta : \mathbf{Set} \rightarrow \mathbf{sSet}$ left adjoint to Γ sends I to $\coprod_I 1$, the I -fold copower of 1.

Now on $\mathbf{sSet}^{\Delta^{\text{op}}}$ we may consider the *injective* model structure whose cofibrations are the monos and whose anodyne cofibrations are those monos m all whose components $m_{[n]}$ are anodyne cofibrations in \mathbf{sSet} . Weak equivalences are all those maps w in $\mathbf{sSet}^{\Delta^{\text{op}}}$ all whose components $w_{[n]}$ are weak equivalences in \mathbf{sSet} . A fibration in \mathbf{bsSet} is a morphism f with $i \perp f$ for all anodyne cofibrations in \mathbf{bsSet} . The ensuing Cisinski model structure is right proper for which reason fibrations are closed under dependent products.

Let i_1^2 be the inclusion of $\Lambda_1[2]$ into $\Delta[2]$. A *Segal type* is a fibrant object X such that $X^{G^* i_1^2}$ is a weak equivalence in \mathbf{bsSet} w.r.t. the injective model structure. The *Segal types* are thought of as ∞ -precategories.

A *complete Segal* or *Rezk type* is a Segal type X such that $X^{G^* i}$ is a weak equivalence in \mathbf{bsSet} where $i : \Delta[0] \rightarrow E$ picks the first object of E , the nerve of the generic iso, i.e. the category with two objects and one iso between them. This completeness condition may be understood as a *univalence condition* requiring isomorphic objects to be equal. The complete Segal types are thought of as ∞ -categories, i.e. ∞ -precategories which are *univalent* in the sense that isomorphic types are equal.

A *discrete type* is a complete Segal type X such that all X_n are Kan objects, i.e. $X^{G^* !\Delta[1]}$ is a weak equivalence. A better name than “discrete” would be *univalent ∞ -groupoid*.

Remark In a sense it would be more natural to refer to Segal types as ∞ -categories and to Rezk types as univalent ∞ -categories. But people consider univalent ∞ -categories more “natural” since the usual nerve functor N sends an ordinary category \mathcal{C} to a simplicial set $N(\mathcal{C})$ which already happens to be a quasicategory!

One may understand **Rezk types as quasicategories constructed inside the \mathbf{sSet} model of \mathbf{HoTT}** . Joyal and Tierney have shown that $G^* \dashv V^*$ is a Quillen equivalence between quasicategories and Rezk categories.

One may construct inside **bsSet** a universe U where $U([n], [m])$ consist of all small presheaves A over $\Delta[n, m] = \Delta \times \Delta \downarrow ([n], [m])$ such that $\int A$ is a fibration in the model structure on **bsSet**. One may further cut down U to a universe U_R of small Rezk types consisting of those $A \in U([n], [m])$ such that $\int A$ is a Grothendieck cofibration (necessarily of Rezk types).

A Univalent Universe in Bisimplicial Sets à la [Cis14]

In his 2014 paper [Cis14] Cisinski has shown that for a local¹⁶⁰ and proper¹⁶¹ model structure on a presheaf topos $\widehat{\mathcal{C}}$ there is a generic small fibration $E_f \rightarrow U_f$ obtained by pulling back the universe à la Yoneda $E \rightarrow U$ along the inclusion $U_f \hookrightarrow U$ where $U_f(I)$ consists of all small presheaves A over \mathcal{C}/I such that $\int A \rightarrow Y_{\mathcal{C}}(I)$ is a fibration w.r.t. the model structure under consideration. Moreover, as shown in [Cis14] if \mathcal{C} is an Eilenberg-Zilber category the universe $E_f \rightarrow U_f$ is univalent and U_f is a fibrant object.

Since $\Delta \times \Delta$ is an EZ-category and the injective model structure on **bsSet** = **sSet** ^{Δ^{op}} \simeq **Set** ^{$\Delta \times \Delta$} is both local and proper we obtain a univalent universe $E_f \rightarrow U_f$ in **bsSet** in the way described above.

246 $\mathcal{U}(\Delta/[n])^{\text{op}}$ vs. $\mathcal{U}[n]$

Let $i_n : [n]^{\text{op}} \rightarrow \Delta/[n]$ be the functor sending k to the inclusion of $[n-k]$ into $[n]$. For a Grothendieck universe \mathcal{U} the change of base functor $i_n^* : \mathcal{U}(\Delta/[n])^{\text{op}} \rightarrow \mathcal{U}[n]$ has a left adjoint L_n sending $A : [n] \rightarrow \mathcal{U}$ to the functor $L_n(A)$ which sends $\alpha : [k] \rightarrow [n]$ to $A(k_\alpha)$ where k_α is the greatest k in $[n]$ such that $n-k$ is an upper bound for α and $\gamma : \beta \rightarrow \alpha$ to $A(k_\alpha \leq k_\beta)$.

247 $N(\mathcal{U})$ as subobject of U in **sSet**

Let $F_n : \Delta/[n] \rightarrow [n]^{\text{op}} : \alpha \mapsto \alpha(0)$. Obviously, we have

$$\begin{array}{ccc} \Delta/[m] & \xrightarrow{\Sigma_u} & \Delta/[n] \\ F_m \downarrow & & \downarrow F_n \\ [m]^{\text{op}} & \xrightarrow{u^{\text{op}}} & [n]^{\text{op}} \end{array}$$

for $u : [m] \rightarrow [n]$. The functor F_n has a right adjoint U_n sending k to $U_n(k) : [n-k] \rightarrow [n] : i \mapsto k+i$. Obviously U_n is full and faithful and $F_n \circ U_n = \text{Id}$.

¹⁶⁰i.e. a map $p : Y \rightarrow X$ is a fibration iff x^*p is a fibration for all maps x to X with representable domain

¹⁶¹A model structure is *left proper* iff weak equivalences are preserved by pushout along cofibrations and it is called *right proper* iff weak equivalences are preserved by pullback along fibrations and it is *proper* iff it is both left and right proper.

Thus, for Grothendieck universes \mathcal{U} the (full and faithful) functor $F_n^* : \mathcal{U}^{[n]} \rightarrow \mathcal{U}^{(\Delta/[n])^{\text{op}}}$ has right adjoint U_n^* .

Applying $\mathcal{U}^{(-)^{\text{op}}}$ to the diagram above we obtain

$$\begin{array}{ccc}
 U([m]) & \xleftarrow{U(u)} & U([n]) \\
 F_m^* \uparrow & & \uparrow F_n^* \\
 \mathcal{U}^{[m]} & \xleftarrow{\mathcal{U}^u} & \mathcal{U}^{[n]}
 \end{array}$$

for all $u : [m] \rightarrow [n]$ in Δ . Thus, the nerve $N(\mathcal{U})$ of \mathcal{U} appears as subobject of U via the natural mono whose component at $[n]$ is given by F_n^* . But $\mathcal{U}([1])$ is much more complicated than $\mathcal{U}^{[1]}$.

A similar reasoning applies when replacing Δ by its subcategory of monos in Δ_2 . Then $U([1])$ consists of spans of sets and into which $\mathcal{U}^{[1]}$ embeds by sending a map $f : A \rightarrow B$ to the span (id_A, f) .

248 A natural map from $A, B : U \vdash A \rightarrow B$ to $U^{\Delta[1]} \rightarrow U \times U$ in $\mathbf{sSet}/U \times U$

First observe that the source of (the interpretation of) $A, B : U \vdash A \rightarrow B$ is given by $E_2 \in \mathbf{sSet}$ with $E_2([n]) = \mathcal{U}^{(\Delta/[n])^{\text{op}} \times [1]} \cong U([n])^{[1]}$ and $E_2(u : [m] \rightarrow [n]) = \mathcal{U}^{\Sigma_u^{\text{op}} \times [1]} \cong U(u)^{[1]}$. Thus $E_2([n])$ consists of natural transformations in $\mathcal{U}^{(\Delta/[n])^{\text{op}}}$ and $u^* \tau = \Sigma_u^* \tau$ for $u : [m] \rightarrow [n]$.

Second observe that $U^{\Delta[1]}([n]) = \mathcal{U}^{\text{Elts}(\Delta[n] \times \Delta[1])^{\text{op}}} \cong \mathbf{sSet}(\Delta[n] \times \Delta[1], U)$ and $U^{\Delta[1]}(u) = \mathcal{U}^{\text{Elts}(\Delta[u] \times \Delta[1])^{\text{op}}}$ for $u : [m] \rightarrow [n]$.

Now we define $i : E_2 \rightarrow U^{\Delta[1]}$. For $B : (\Delta/[n])^{\text{op}} \times [1] \rightarrow \mathcal{U}$ let $i(B) = B \circ G_n$ where $G_n : \text{Elts}(\Delta[u] \times \Delta[1])^{\text{op}} \rightarrow \Delta/[n]^{\text{op}} \times [1]$ is defined as follows: for $u : [m] \rightarrow [n]$ and $p : [m] \rightarrow [n]$ we put $G_n(u, p) = (u, F_1^{\text{op}}(p))$ and for $v : [k] \rightarrow [m]$ we put $G_n(v : (uv, pv) \rightarrow (u, p)) = (v : uv \rightarrow u, F_1^{\text{op}}(v : pv \rightarrow p))$.

One would hope that i is monic but that is unlikely since the functor G_n is not surjective on morphism (since $(\text{id}_{[0]} : u \rightarrow u, 0 \rightarrow 1)$ will not be in the image of G_n for any $u : [0] \rightarrow [n]$).

Moreover, when restricting to the univalent subuniverse U_f of U consisting of small Kan fibrations we know that $U_f^{\Delta[1]}$ is weakly equivalent to $(\Sigma A, B : U_f) \mathbf{Weq}(A, B)$ and not to $(\Sigma A, B : U_f) A \rightarrow B$. This is another evidence for i not being monic!

249 Simplicial Sets inside Cubical Sets

(Aug.18-Mar.19)

Let Δ be the small category of finite ordinals greater 0 and monotone maps between them and \square the small category of finite powers of 2 and monotone maps between them. Presheaves over Δ are called *simplicial sets* and presheaves over \square are called *cubical sets*. The categories of simplicial and cubical sets are denoted by **sSet** and **cSet**, respectively.

Kapulkin and Voevodsky have observed in 2017 that one may obtain **sSet** as a subtopos of **cSet** in the following way. The nerve functor $N : \mathbf{Cat} \rightarrow \mathbf{sSet}$ is known to be full and faithful and so is its restriction $u : \square \rightarrow \mathbf{sSet}$ to the full subcategory \square of **Cat**. This functor u induces an adjunction $u_! \dashv u^* : \mathbf{sSet} \rightarrow \mathbf{cSet}$ where $u^*(X) = \mathbf{sSet}(u(-), X)$ and $u_!$ is the left Kan extension of u along $Y_\square : \square \rightarrow \mathbf{cSet}$. It follows from general topos theoretic results that $u_! \dashv u^*$ exhibits **sSet** as a subtopos of **cSet** induced by the Grothendieck topology \mathcal{J} consisting of those sieves in \square which are sent by u to jointly epic families in **sSet**.

Most type theoretic constructions in the cubical model of type theory do not lead out of sheaves, i.e. stay within the subtopos **sSet** of **cSet**. It remains to be seen whether the corresponding universe is a \mathcal{J} -sheaf itself.

Even more recently Ch. Sattler has given an alternative description of the situation based on the fact that splitting idempotents in \square gives rise to the category **FL** of finite lattices and monotone maps between them. Thus **cSet** is equivalent to $\widehat{\mathbf{FL}} = \mathbf{Set}^{\mathbf{FL}^{\text{op}}}$. The inclusion functor $i : \Delta \rightarrow \mathbf{FL}$ induces an essential geometric morphism $i_! \dashv i^* \dashv i_*$ which, moreover, is injective, i.e. i_* and thus also $i_!$ are full and faithful. The inverse image part i^* restricts presheaves over **FL** to presheaves over Δ (by precomposition with i^{op}). The direct image part i_* is given by $i_*(X) = \mathbf{sSet}(N(-), X)$ (since N restricted to **FL** is given by $i^* \circ Y_{\mathbf{FL}}$). The cocontinuous functor $i_!$ is the left Kan extension of $Y_{\mathbf{FL}} \circ i$ along Y_Δ . It sends $X \in \mathbf{sSet}$ to the colimit of $\Delta \downarrow X \xrightarrow{\partial_0} \Delta \xrightarrow{i} \mathbf{FL} \xrightarrow{Y_{\mathbf{FL}}} \mathbf{cSet}$.

Obviously, a sieve $S \subseteq Y_{\mathbf{FL}}(L)$ is a cover iff $i^*S = i^*Y_{\mathbf{FL}}(L)$, i.e. S contains all chains in L , i.e. all monotone maps $c : [n] \rightarrow L$. Thus a sieve $S \subseteq Y_\square(2^n)$ covers iff for every maximal chain $C \subseteq 2^n$ there is an idempotent $r \in S$ whose image is C . Obviously, such an S contains all monotone maps to 2^n whose image is contained in C . Thus the collection of all monotone maps to 2^n whose image is contained in a (maximal) chain in 2^n is the least covering sieve for 2^n .¹⁶²

Within **cSet** and **sSet** one may consider those monos which are “open box inclusions”, i.e. subobjects of $(\{\varepsilon\} \times X) \cup (I \times Y) \hookrightarrow I \times X$ where I is the interval, $\varepsilon \in \{0, 1\}$ and Y is a subobject of X . Notice that i^* reflects open box

¹⁶²A concrete take on the topology on \square :

As above let \square be the category whose objects are natural numbers and whose morphisms from m to n are monotone maps from 2^m to 2^n . Composition in \square is just composition of such functions. For every n let S_n be the sieve of all monotone functions p from some 2^m to 2^n such that the image of p is linearly ordered in 2^n . We say that a sieve C on 2^n covers iff $C \supseteq S_n$. Since every $p : 2^m \rightarrow 2^n$ sends chains to chains we have $pq \in S_n$ whenever $q \in S_n$ and thus the collection of all S_n provides a coverage of \square .

inclusions in \mathbf{cSet} to \mathbf{sSet} . Both in \mathbf{cSet} and \mathbf{sSet} we define fibrations as those maps which are weakly right orthogonal to all open box inclusions for which reason a map f in \mathbf{sSet} is a fibration iff i_*f is a fibration in \mathbf{cSet} . Writing \mathcal{F} for the class of fibrations in \mathbf{cSet} the class of Kan fibrations in \mathbf{sSet} is given by $\mathcal{F} \cap \mathbf{sSet}$ (considering \mathbf{sSet} as full subcategory of \mathbf{cSet} via i_*).

If A and B are fibrant in \mathbf{sSet} a map $f : A \rightarrow B$ is a weak equivalence in \mathbf{sSet} iff it is a weak equivalence in \mathbf{cSet} which can be seen as follows. One can show that $i_*(\mathbf{hfib}(f)) \simeq \mathbf{hfib}(i_*f)$ and thus $\forall m \in \mathbf{Mono}(\mathbf{cSet})(m \perp \mathbf{hfib}(i_*f))$ iff $\forall m \in \mathbf{Mono}(\mathbf{cSet})(m \perp i_*(\mathbf{hfib}(f)))$ iff $\forall m \in \mathbf{Mono}(\mathbf{cSet})(i^*m \perp \mathbf{hfib}(f))$ iff¹⁶³ $\forall m \in \mathbf{Mono}(\mathbf{sSet})(i^*m \perp \mathbf{hfib}(f))$, i.e.¹⁶⁴ i_*f is a weak equivalence in \mathbf{cSet} iff f is a weak equivalence in \mathbf{sSet} . But as observed by Sattler there is actually a simpler argument: in both \mathbf{sSet} and \mathbf{cSet} weak equivalences between fibrant objects are just homotopy equivalences and these are preserved by i^* and i_* since these functors preserve I and finite products.

One knows that $i^* \dashv i_*$ is a Quillen pair but not whether it is a Quillen equivalence. For that purpose one would have to show that for fibrant $B \in \mathbf{sSet}$ and $A \in \mathbf{cSet}$ a map $f : i^*A \rightarrow B$ is a weak equivalence in \mathbf{sSet} iff $\check{f} : A \rightarrow i_*B$ is a weak equivalence in \mathbf{cSet} . But, alas, this question is open. For some cubical sites different from \mathbf{FL} a counterexample is given by $A = \Delta[2]$ and $B = \Delta[0]$ but for \mathbf{FL} this does not work since it has connections.

Universes in $\widehat{\mathbf{FL}}$

Given a Grothendieck universe \mathcal{U} this induces a universe à la Yoneda $\pi : E \rightarrow U$ in $\widehat{\mathbf{FL}}$ which is generic for a class \mathcal{S} of \mathcal{U} -small maps in $\widehat{\mathbf{FL}}$. As described in [GS17] there is a universe $\pi_c : E_c \rightarrow U_c$ weakly generic for \mathcal{U} -small fibrations in \mathbf{cSet} such that U_c is fibrant. Moreover, there is a morphism $u_c : U_c \rightarrow U$ which is full and faithful as a (split) cartesian functor (when considering U_c and U as (split) fibrations and not just as presheaves) with π_c isomorphic to $u_c^*\pi$.¹⁶⁵

Since by Sattler's Theorem (see subsection 249.1) i^* preserves small fibrations $i^*\pi_c$ is weakly generic for small fibrations in \mathbf{sSet} . As proved by Voevodsky there is a univalent universe $\pi_s : E_s \rightarrow U_s$ weakly generic for small fibrations in \mathbf{sSet} . Thus, we have

$$\begin{array}{ccccc}
 E_s & \longrightarrow & i^*E_c & \longrightarrow & E_s \\
 \pi_s \downarrow & \lrcorner & \downarrow i^*\pi_c & \lrcorner & \downarrow \pi_s \\
 U_s & \xrightarrow{e} & i^*U_c & \xrightarrow{p} & U_s
 \end{array}$$

¹⁶³since the monos in \mathbf{sSet} are precisely the sheafifications of monos in \mathbf{cSet}

¹⁶⁴as shown by Voevodsky for fibrant objects A and B a map $w : A \rightarrow B$ is a weak equivalence iff $\mathbf{hfib}(w)$ is a trivial cofibrations, i.e. $m \perp \mathbf{hfib}(w)$ for all monos m

¹⁶⁵As described in [GS17] $U_c(L)$ does not simply consist of \mathcal{U} -small fibrations over $\mathbf{Y}_{\mathbf{FL}}(L)$ but rather such fibrations together with a functorial choice of fillers which are forgotten by u_c .

and $pe \sim \text{id}_{U_s}$ (i.e. pe and id_{U_s} are homotopy equivalent) since the universe π_s is univalent. Since u_* preserves fibrations, pullbacks and \sim for maps between fibrant objects we have

$$\begin{array}{ccccc}
i_*E_s & \longrightarrow & i_*i^*E_c & \longrightarrow & i_*E_s \\
\downarrow i_*\pi_s & \lrcorner & \downarrow i_*i^*\pi_c & \lrcorner & \downarrow i_*\pi_s \\
i_*U_s & \xrightarrow{i_*e} & i_*i^*U_c & \xrightarrow{i_*p} & i_*U_s
\end{array}$$

with $i_*p \circ i_*e = i_*(p \circ e) \sim i_*(\text{id}_{U_s}) = \text{id}_{i_*U_s}$. Thus $i_*i^*\pi_c$ is generic for small fibrations which are families of sheaves and $i_*\pi_s$ is a univalent such universe.¹⁶⁶ Thus, pulling back $i_*i^*\pi_c$ along the homotopy equalizer of $i_*e \circ i_*p$ and $\text{id}_{i_*i^*U_c}$ gives rise to a univalent universe in U_c weakly generic for small fibrations which are families of sheaves.

249.1 Proof of Sattler's Theorem

claiming that $i_! \dashv i^* : \mathbf{cSet} \rightarrow \mathbf{sSet}$ is a Quillen pair, i.e. $i_!$ preserves monos and i^* preserves fibrations when \mathbf{cSet} is endowed with the minimal Cisinski model structure.

First recall that $i_!$ sends $X \in \mathbf{sSet}$ to the presheaf $i_!X$ over \mathbf{FL} where for $A \in \mathbf{FL}$ we have $i_!X(A) = \Pi_0(A \downarrow ip_X)$ where $p_X : \text{Els}(X) \rightarrow \Delta$ is the discrete fibration obtained from X via the Grothendieck construction.

Let $f : X \rightarrow Y$ be a map in \mathbf{sSet} . Then we write p_f for the map from p_X to p_Y over Δ as given by $\text{Els}(f) : \text{Els}(X) \rightarrow \text{Els}(Y)$ which is a discrete fibration. Considering the diagram

$$\begin{array}{ccccccc}
A \downarrow ip_X & \xrightarrow{A \downarrow ip_f} & A \downarrow ip_Y & \longrightarrow & A \downarrow i & \longrightarrow & A \downarrow \mathbf{FL} \\
\downarrow \partial_1 & \lrcorner & \downarrow \partial_1 & \lrcorner & \downarrow \partial_1 & \lrcorner & \downarrow \partial_1 \\
\text{Els}(X) & \xrightarrow{\text{Els}(f)} & \text{Els}(Y) & \xrightarrow{p_Y} & \Delta & \xrightarrow{i} & \mathbf{FL}
\end{array}$$

we observe that the vertical arrows are discrete cofibrations and $\text{Els}(f)$ and thus also $A \downarrow ip_f$ is a discrete fibration.

¹⁶⁶we have used here that $a : A \rightarrow I$ is a family of sheaves iff

$$\begin{array}{ccc}
A & \xrightarrow{\eta_A} & i_*i^*A \\
\downarrow a & \lrcorner & \downarrow i_*i^*a \\
I & \xrightarrow{\eta_I} & i_*i^*I
\end{array}$$

Suppose f is monic. Then $\text{Elts}(f)$ is 1-1 on objects and thus a sieve in $\text{Elts}(Y)$ and so is $A \downarrow ip_f$.

Since Δ is elegant Reedy¹⁶⁷ $\text{Elts}(f)$ is a cofibration when restricted to degeneracy maps in Δ and so is $A \downarrow ip_f$. Thus, when considering $A \downarrow ip_f$ as a presheaf over $A \downarrow ip_Y$ it inverts degeneracy maps. Thus, zig-zag's in $A \downarrow ip_Y$ lift along $A \downarrow ip_f$ to zig-zag's in $A \downarrow ip_Y$ for which reason $\Pi_0(A \downarrow ip_f)$ is monic.

Thus, we have shown that $i_!f(A) = \Pi_0(A \downarrow ip_f)$ is monic whenever f is. Accordingly, we conclude that $i_!$ preserves monos, i.e. cofibrations.

For showing that i^* preserves fibrations it suffices to show that $i_!$ sends horn inclusions to weak equivalences.

First notice that for every natural number n the map $[n] \rightarrow [0]$ is a homotopy equivalence in \mathbf{cSet} and thus also a weak equivalence. By 2-out-of-3 we conclude that $\mathbf{Y}_{\mathbf{FL}} \circ i = i_! \circ \mathbf{Y}_\Delta$ takes values in weak equivalences in \mathbf{cSet} .

We now generalize the claim from horn inclusions to generalized horn inclusions $\Lambda_K^n \hookrightarrow \Delta^n$ where $K \subseteq [n]$ and its complement are nonempty where Λ_K^n is defined as the union of all subobjects $\Delta^{[n] \setminus \{j\}}$ where j ranges over $[n] \setminus K$.¹⁶⁸

We prove now by induction on n and the size of K that $\Lambda_K^n \hookrightarrow \Delta^n$ is sent by $i_!$ to a weak equivalence in \mathbf{cSet} . If the complement of K is a singleton $\{j\}$, then $\Lambda_K^n \rightarrow \Delta^n$ is the inclusion $\Delta^{[n] \setminus \{j\}} \hookrightarrow \Delta^n$ which lies in the image of \mathbf{Y}_Δ . Otherwise, we pick j in the complement of K and look at the diagram $\Lambda_{K \setminus \{j\}}^n \hookrightarrow \Lambda_K^n \hookrightarrow \Delta^n$. The composite is sent by $i_!$ to a weak equivalence by induction hypothesis. The first map is a pushout of $\Delta_K^{[n] \setminus \{j\}} \beta \Delta^{[n] \setminus \{j\}}$ which by induction hypothesis is also sent by $i_!$ to a weak equivalence. Thus, since $i_!$ is cocontinuous and trivial cofibrations are closed under pushouts it follows that $i_!$ sends $\Lambda_{K \setminus \{j\}}^n \hookrightarrow \Lambda_K^n$ to a weak equivalence. By induction hypothesis the inclusion $\Lambda_{K \setminus \{j\}}^n \hookrightarrow \Delta^n$ is sent by $i_!$ to a weak equivalence. Thus, by 2-out-of-3 for weak equivalences it follows that $i_!$ sends $\Lambda_K^n \hookrightarrow \Delta^n$ to a weak equivalence.

Test Category Model Structure on \mathbf{cSet} and its Relation to the ‘‘Type-Theoretic’’ One

The inclusion functor $i : \Delta \hookrightarrow \mathbf{FL}$ is aspherical, i.e. $\text{Nv}(\text{Elts}(i))$ is a weak equivalence in \mathbf{sSet} . Thus by Th.1.2.9 of Maltiniotis’s book *La Théorie de l’Homotopie de Grothendieck* the functor $i^* : \mathbf{cSet} \rightarrow \mathbf{sSet}$ preserves and reflects weak equivalences of the respective test model structures. Since i^* also preserves monos the adjunction $i^* \dashv i_*$ is a Quillen equivalence between \mathbf{cSet} and \mathbf{sSet} .

Let $\varepsilon_X : i_!i^*X \rightarrow X$ be the counit of $i_! \dashv i^*$ and $\eta_X : X \rightarrow i_*i^*X$ be the unit of $i^* \dashv i_*$. These are weak equivalences w.r.t. the test model structure on \mathbf{cSet} since both are sent to isos and thus weak equivalences by applying i^* .

The ‘‘type-theoretic’’ model structure on \mathbf{cSet} is the minimal Cisinski model structure whose anodyne cofibrations are generated by open box inclusions. We

¹⁶⁷i.e. every degenerate object arises in a unique way from a nondegenerate one

¹⁶⁸If $K = [n] \setminus \{j\}$ then $\Lambda_K^n \hookrightarrow \Delta^n$ is the j -th face inclusion $\Delta^{[n] \setminus \{j\}} \hookrightarrow \Delta^n$. If $K = \{j\}$ then $\Lambda_K^n \hookrightarrow \Delta^n$ is the j -th horn inclusion $\Lambda_j^n \hookrightarrow \Delta^n$.

know that both $i_! \dashv i^*$ and $i^* \dashv i_*$ are Quillen pairs when \mathbf{cSet} is endowed with the “type-theoretic” model structure.

Thus, if the “type-theoretic” model structure on \mathbf{cSet} coincides with the test model structure then all $\eta_X : X \rightarrow i_* i^* X$ and $\varepsilon_X : i_! i^* X \rightarrow X$ are weak equivalences w.r.t. the “type-theoretic” model structure on \mathbf{cSet} .

But if all ε_X are weak equivalences w.r.t. the “type-theoretic” model structure then it coincides with the test model structure which can be seen as follows. Suppose $m : Y \rightarrow X$ is an anodyne cofibration w.r.t. the test model structure then $i^* m$ is an anodyne cofibration in \mathbf{sSet} from which it follows that $i_! i^* m$ is an anodyne cofibration w.r.t. the “type-theoretic” model structure on \mathbf{cSet} . But since

$$\begin{array}{ccc} i_! i^* Y & \xrightarrow{\varepsilon_Y} & Y \\ i_! i^* m \downarrow & & \downarrow m \\ i_! i^* X & \xrightarrow{\varepsilon_X} & X \end{array}$$

commutes it follows by the 2-out-of-3 property for weak equivalences that m is a weak equivalence and thus an anodyne cofibration w.r.t. the “type-theoretic” model structure on \mathbf{cSet} .

Thus, summarizing the above considerations we conclude that the “type-theoretic” and the test model structure on \mathbf{cSet} coincide if and only if all $\varepsilon_X : i_! i^* X \rightarrow X$ are weak equivalences in the “type-theoretic” model structure on \mathbf{cSet} .

Some observations

1. For the square $S = [1] \times [1]$ the subobject ε_S consists of all generalised elements $L \rightarrow S$ whose image in S is a chain, i.e. ε_S is the “boundary” of the square S . Nevertheless ε_S is a weak equivalence in the “type-theoretic” model structure on \mathbf{cSet} which can be seen as follows. The map $p = (0, 0) : [0] \rightarrow S$ is a point of S which is also in $i_! i^* S$ since p factors through ε_S . But both points are weak equivalences since id_S and $p \circ !_S$ are homotopic by $(z, x, y) \mapsto (z \wedge x, z \wedge y) : [1] \times S \rightarrow S$. Thus, the map ε_S is also a weak equivalence by their 2-out-of-3 property. Possibly one can extend this argument to show that ε_X is a weak equivalence w.r.t. the “type theoretic” model structure for all representable objects X (actually, it suffices to show this for X which are finite powers of $[1]$). Finally, one might extend this from representable objects to arbitrary objects of \mathbf{cSet} .
2. Notice that both i^* and i_* preserve the interval \mathbb{I} . Thus the interval is a sheaf and since sheaves are closed under limits all representable objects of \mathbf{cSet} are sheaves.
3. The functor $i_!$ preserves the interval \mathbb{I} but doesn’t preserve binary products since $\mathbb{I} \times \mathbb{I}$ is not in the image of $i_!$.

250 Modal type theories arising from cohesive toposes \mathcal{E} over \mathcal{S}

Let \mathcal{E} and \mathcal{S} be toposes and $L \dashv F \dashv U \dashv R : \mathcal{S} \rightarrow \mathcal{E}$ such that $F \dashv U$ is locally connected, local and hyperconnected. Then we have

$$\begin{array}{ccc} \mathcal{E} \downarrow F & & \mathcal{S} \downarrow U \\ P_F \downarrow & & \downarrow P_U \\ \mathcal{S} & \xrightarrow{F} & \mathcal{E} \end{array}$$

where (the codomain functors) P_F and P_U are geometric fibrations and $F^* P_U \simeq P_{\mathcal{S}}$ since $UF \cong \text{Id}_{\mathcal{S}}$. Moreover, by change of base along F the fibered geometric morphism $\Delta \dashv \Gamma : P_U \rightarrow P_{\mathcal{E}}$ over \mathcal{E} is sent to the fibered geometric morphism $\Gamma \dashv \nabla : P_{\mathcal{S}} \rightarrow P_F$ over \mathcal{S} .

251 Universes as Stacks?

Following Bénabou a category over a base category \mathbb{C} is a fibration $P : \mathbb{X} \rightarrow \mathbb{C}$. If, moreover, the base category \mathbb{C} is endowed with a Grothendieck topology \mathcal{J} one may require P to be a \mathcal{J} -stack, i.e. for every $S \in \mathcal{J}(I)$ the functor $\mathbf{Fib}(\mathbb{C})(i_S, P) : \mathbf{Fib}(\mathbb{C})(I, P) \rightarrow \mathbf{Fib}(\mathbb{C})(S, P)$ is an equivalence where $i_S : S \hookrightarrow \mathbb{C}/I$.

Split fibrations over \mathbb{C} correspond to categories internal to $\widehat{\mathbb{C}}$. Coquand and Manna have suggested that in case \mathbb{C} is endowed with a topology \mathcal{J} the split fibration over \mathbb{C} should also be a \mathcal{J} -stack. This, however, does not mean that the corresponding category C internal to $\widehat{\mathbb{C}}$ is also internal to $\mathbf{Sh}(\mathbb{C}, \mathcal{J})$ simply because $\text{Ob}(C) \in \widehat{\mathbb{C}}$ need not be a \mathcal{J} -sheaf. However, we may consider instead $\mathbf{a}(C)$, the sheafication of C , which is a category internal to $\mathbf{Sh}(\mathbb{C}, \mathcal{J})$ since $\mathbf{a} : \widehat{\mathbb{C}} \rightarrow \mathbf{Sh}(\mathbb{C}, \mathcal{J})$ preserves finite limits.

Universes in toposes \mathcal{E} are categories U internal to \mathcal{E} whose externalisation \underline{U} is a full subfibration of $P_{\mathcal{E}}$, the fundamental fibration of \mathcal{E} . Thus, universes in $\mathcal{E} = \mathbf{Sh}(\mathbb{C}, \mathcal{J})$ should be considered as categories U internal to $\mathbf{Sh}(\mathbb{C}, \mathcal{J})$ whose externalisation is a full subfibration of $P_{\mathbf{Sh}(\mathbb{C}, \mathcal{J})}$ which is a stack as is the externalisation of U . Thus, we disagree with Coquand and Manna's view of universes as stacks since we find it too general in the respect that $\text{Ob}(U)$ is not required to be a \mathcal{J} -sheaf.

However, Coquand and Manna are working in a univalent metatheory w.r.t. which a sheaf of types is just a sheaf of ∞ -groupoids. They consider the universe U where $U(I)$ is the type of small presheaves A over \mathbb{C}/I such that $\int A \rightarrow Y_{\mathbb{C}}(I)$ is a family of sheaves in $\widehat{\mathbb{C}}$ (in the sense of section 88). For every cover S of I every $A : S \rightarrow U$ has an extension $\bar{a} : Y_{\mathbb{C}}(I) \rightarrow U$ which is unique up to isomorphism and thus up to equality by univalence in the metatheory.

252 Sheafifying Universes

Let $a : \mathcal{E} \rightarrow \mathcal{F}$ be a finite limit preserving left adjoint to a full subcat inclusion $i : \mathcal{F} \hookrightarrow \mathcal{E}$. If $\pi : E \rightarrow U$ is a universe in \mathcal{E} then $a(\pi)$ is a universe in \mathcal{F} .

The geometric morphism $a \dashv i$ extends to a fibered geometric morphism $\Delta \dashv \Gamma : P_a \rightarrow P_{\mathcal{E}}$ whose unit η_{π} at $\pi : E \rightarrow U$ is given by

$$\begin{array}{ccc}
 E & & \\
 \eta_{\pi} \searrow & & \eta_E \searrow \\
 \cdot & \xrightarrow{\quad} & iaE \\
 \Gamma \Delta \pi \downarrow & \lrcorner & \downarrow ia\pi \\
 U & \xrightarrow{\quad} & iaU \\
 & \eta_U &
 \end{array}$$

Factoring $\eta_U = me$ with m monic and e epic allows one to consider

$$\begin{array}{ccccc}
 \cdot & \xrightarrow{\quad} & \cdot & \xrightarrow{\quad} & iaE \\
 \Gamma \Delta \pi \downarrow & \lrcorner & \downarrow m^* ia\pi & & \downarrow ia\pi \\
 U & \xrightarrow{\quad} & \cdot & \xrightarrow{\quad} & iaU \\
 & e & & m &
 \end{array}$$

where $m^* ia\pi$ may be understood as an “extensionalized” version of $\Gamma \Delta \pi$. But both $m^* ia\pi$ and $\Gamma \Delta \pi$ appear as instances of $ia\pi$ which I considered in my paper *Universes in Toposes*.

But the universes $m^* ia\pi$ and $\Gamma \Delta \pi$ are not “univalent” in the sense that isomorphic objects are equal. In this respect it might be more appropriate to consider stacks as suggested by Coquand and Manna which, however, works only if the metatheory itself is univalent. But on the other hand the externalizations of $ia\pi$, $m^* ia\pi$ and $\Gamma \Delta \pi$ also give rise to stacks which are equivalent to the one considered by Coquand and Manna.

253 Σ -continuous objects in $\mathbf{RT}(\mathcal{K}_2)$ (August 2018)

By Schröder's Theorem X is a \mathbf{QCB}_0 -space iff $X \in \mathbf{Mod}(\mathcal{K}_2)$ such that $\eta_X : X \rightarrow \Sigma^{\Sigma^X}$ is a $\neg\neg$ -mono. We know for arbitrary $X \in \mathbf{Asm}(\mathcal{K})$ we have $\Sigma^{\eta_X} \circ \eta_{\Sigma^X} = \text{id}_{\Sigma^X}$. This suggests to define *subspace inclusions* in $\mathbf{Mod}(\mathcal{K}_2)$ as $\neg\neg$ -monos i for which Σ^i is a split epi. Thus $X \in \mathbf{Mod}(\mathcal{K}_2)$ is in \mathbf{QCB}_0 iff η_X is a subspace inclusion.

An object $X \in \mathbf{QCB}_0$ is called Σ -continuous iff it is injective w.r.t. all subspace inclusions in $\mathbf{Mod}(\mathcal{K}_2)$. Thus, for Σ -continuous X the map η_X is a split mono. We now show that this condition is sufficient. Suppose $X \in \mathbf{QCB}_0$ and $e : \Sigma^{\Sigma^X} \rightarrow X$ with $e \circ \eta_X = \text{id}_X$. Suppose $i : Y \rightarrow Z$ is a subspace inclusion in $\mathbf{Mod}(\mathcal{K}_2)$ and $f : Y \rightarrow X$. Then there exists $s : \Sigma^Y \rightarrow \Sigma^Z$ such that $\Sigma^i \circ s = \text{id}_{\Sigma^Y}$. Thus, we have $\Sigma^s \circ \Sigma^{\Sigma^i} = \text{id}_{\Sigma^{\Sigma^Y}}$. Then for $g = e \circ \Sigma^{\Sigma^f} \circ \Sigma^s \circ \eta_Z$ we have

$$\begin{aligned} g \circ i &= e \circ \Sigma^{\Sigma^f} \circ \Sigma^s \circ \eta_Z \circ i = e \circ \Sigma^{\Sigma^f} \circ \Sigma^s \circ \Sigma^{\Sigma^i} \circ \eta_Y = \\ &= e \circ \Sigma^{\Sigma^f} \circ \eta_Y = e \circ \eta_X \circ f = \\ &= f \end{aligned}$$

as desired.

If $X \in \mathbf{QCB}_0$ is a continuous lattice, i.e. injective w.r.t. all subspace inclusions in \mathbf{Sp} , then η_X is a split mono and thus X is Σ -continuous. But X is also Σ -continuous if X is injective just w.r.t. all maps in \mathbf{QCB}_0 which are subspace inclusions in \mathbf{Sp} .

One can characterize Σ -continuous \mathbf{QCB}_0 spaces as retracts of powers of Σ in \mathbf{QCB}_0 . Thus $\Sigma^{\mathbb{N}^{\mathbb{N}}}$ is a Σ -continuous \mathbf{QCB}_0 space which, however, is not a continuous lattice since $\mathbb{N}^{\mathbb{N}}$ is not locally compact. Countably based continuous lattices can be characterized as retracts of $\Sigma^{\mathbb{N}}$.

Finally, we show that all \mathbf{QCB}_0 objects which are continuous lattices are necessarily countably based and thus retracts of $\Sigma^{\mathbb{N}}$. Suppose $X \in \mathbf{QCB}_0$ is a continuous lattice. Then Σ^X is a continuous lattice (i.e. X is core compact) since its \mathbf{QCB}_0 topology is the Scott topology and continuous lattices form a cartesian closed category. But Cor. 6.11 of [ELS04] says that if a core compact space is a quotient of a countably based space then it is itself countably based. Thus X is countably based.

As described in Johnstone *Stone Spaces* the adjunction between topological spaces locales restricts to an equivalence between locales whose underlying complete lattices are continuous and locally compact sober spaces. Thus, since continuous lattices are sober w.r.t. their Scott topology continuous lattices in \mathbf{QCB}_0 are locally compact.

Thus, for every $X \in \mathbf{QCB}_0$ which is sober but not locally compact Σ^X is a Σ -continuous \mathbf{QCB}_0 space which is not a continuous lattice. Typical examples are infinite dimensional separable Banach spaces as e.g. infinite dimensional separable Hilbert space \mathcal{H} .

Finally, notice that Σ -continuous \mathbf{QCB}_0 -spaces are precisely those objects $X \in \mathbf{RT}(\mathcal{K}_2)$ for which $\eta_X : X \rightarrow \Sigma^{\Sigma^X}$ is a split mono. Thus Σ -continuous

QCB₀-spaces are the right notion of continuous lattice within the function realizability topos $\text{RT}(\mathcal{K}_2)$.

254 Applicative Morphisms and Geometric Morphisms

Johnstone suggested to call *partial combinatory algebras* (pca's) *Schönfinkel algebras*. If Λ and M are Schönfinkel algebras then an *applicative morphism* from Λ to M is an entire relation $\theta \subseteq \Lambda \times M$ for which there exists a $\tau \in M$ such that from $\theta(\lambda, \mu)$, $\theta(\lambda', \mu')$ and $\lambda\lambda'$ defined in Λ it follows that $\tau\mu\mu'$ is defined and $\theta(\lambda\lambda', \tau\mu\mu')$. If θ and ϕ are applicative morphisms from Λ to M then $\theta \leq \phi$ iff there exists a $\rho \in M$ such that $\theta(\lambda, \mu)$ implies $\phi(\lambda, \rho\mu)$.

An applicative morphism $\theta \subseteq \Lambda \times M$ is called *computationally dense* or *quasi-surjective* iff there exists a $\rho \in M$ such that for all $\mu \in M$ there exists a $\lambda \in \Lambda$ such that for all $\mu' \in M$ from $\theta(\lambda, \mu')$ it follows that $\mu = \rho\mu'$, i.e. such that there is a map $r : M \rightarrow \Lambda$ such that for all $\mu, \mu' \in M$ from $\theta(r(\mu), \mu')$ it follows that $\mu = \rho\mu'$.

As shown by van Oosten, Hofstra and Johnstone quasi-surjective applicative morphisms $\theta \subseteq \Lambda \times M$ correspond to geometric morphisms from $f : \text{RT}(M) \rightarrow \text{RT}(\Lambda)$ with $f^*(\bar{\Lambda}) = (\Lambda, \theta)$ (where $\bar{\Lambda} = (\Lambda, \Delta(\Lambda))$). If ρ witnesses quasi-surjectivity of θ then f_* is witnessed by the tripos morphism sending $\mu \in M$ to $\{\lambda \in \Lambda \mid \forall \mu' \in M. \theta(\lambda, \mu') \Rightarrow \mu = \rho\mu'\}$. Moreover, if θ corresponds to f and μ to g then $\theta \leq \phi$ iff $f^* \leq g^*$ (notice that f and g are necessarily localic).

As noticed on pp.28-29 of van Oosten's book there is an quasi-surjective applicative morphism γ from $\mathcal{P}\omega$ to \mathcal{K}_2 with $\gamma(A) = \{\alpha \in \mathcal{K}_2 \mid A = \{n \in \omega \mid n+1 \in \text{im}(\alpha)\}\}$ for $A \in \mathcal{P}\omega$ and a quasi-surjective applicative morphism ι from \mathcal{K}_2 to $\mathcal{P}\omega$ with $\iota(\alpha) = \{S(\alpha)\}$ where $S(\alpha) = \{s \in \omega \mid \forall i < |s| s_i = \alpha(i)\}$. One can show that $\gamma\iota \simeq \text{id}_{\mathcal{K}_2}$, i.e. $\gamma\iota \leq \text{id}_{\mathcal{K}_2}$ and $\text{id}_{\mathcal{K}_2} \leq \gamma\iota$, and $\iota\gamma \leq \text{id}_{\mathcal{P}\omega}$. Thus γ gives rise to a full inclusion of $\text{RT}(\mathcal{P}\omega)$ into $\text{RT}(\mathcal{K}_2)$ which has both adjoints where the left adjoint preserves finite limits.¹⁶⁹

¹⁶⁹i.e. an inclusion of $\text{RT}(\mathcal{P}\omega)$ into $\text{RT}(\mathcal{K}_2)$ whose direct image has a right adjoint but typically is not local since this further right adjoint typically is not full and faithful

255 Fibrations in Quasicategories

A *homotopy quasi-pullback* in a model structure is a square

$$\begin{array}{ccc} C & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & I \end{array}$$

such that the mediating arrow $C \rightarrow A \times_I B$ is a weak equivalence. We consider the category \mathbf{sSet} with its classical Quillen model structure.

A map $p : E \rightarrow B$ is a *mid fibration* iff

$$\begin{array}{ccc} E^{\Delta[2]} & \longrightarrow & E^{\Lambda_1[2]} \\ \downarrow & & \downarrow \\ B^{\Delta[2]} & \longrightarrow & B^{\Lambda_1[2]} \end{array}$$

is a homotopy quasi-pullback. This expresses the requirement that composition exists for maps in E above maps in B for which composition exists.

Let $p : E \rightarrow B$ be a map in \mathbf{sSet} . Then $f : y \rightarrow x$ in E is *cartesian* iff

$$\begin{array}{ccc} E/f & \longrightarrow & E/x \\ \downarrow & & \downarrow \\ B/pf & \longrightarrow & B/pf \end{array}$$

is a homotopy quasi-pullback.¹⁷⁰ Informally speaking, this says that for any 2-simplex α in the base B with $\partial_0(\alpha) = pf$ and $h : z \rightarrow x$ with $ph = \partial_1(\alpha)$ there is a 2-simplex β above α with $\partial_1(\beta) = h$ that is unique up to homotopy with this property as illustrated in

from which it is evident in which sense this definition generalizes the usual 1-categorical one. A map $p : E \rightarrow B$ is a *Grothendieck fibration* iff for every $u : j \rightarrow i$ in B and x in E with $px = i$ there is a cartesian $f : y \rightarrow x$ with $pf = u$.

¹⁷⁰Here E/f is an abbreviation for $(E/x)/f$.

A map $p : E \rightarrow B$ in **sSet** is a *discrete fibration* iff

$$\begin{array}{ccc} E^{\Delta[1]} & \xrightarrow{E^{d_0}} & E \\ p^{\Delta[1]} \downarrow & & \downarrow p \\ B^{\Delta[1]} & \xrightarrow{B^{d_0}} & B \end{array}$$

is a homotopy quasi-pullback.

256 Universes of Small Grothendieck Fibrations in **sSet** (21.2.19)

In **sSet** there is a Grothendieck fibration $\pi : E \rightarrow U$ with small fibers such that every Grothendieck fibration $p : Y \rightarrow X$ with small fibers arises as pullback of π along some $f : X \rightarrow U$. But, actually, more is true, namely, that the mapping space $\text{hom}(X, U)$ is equivalent to the non-full subcategory of **sSet**/ X whose objects are small fibrations over X and whose morphisms are **cartesian** functors between them. The reason is that $U([n])$ is the set of small Grothendieck fibrations over $\Delta[n]$. For this reason $\text{hom}(\Delta[1], \text{hom}(X, U)) \cong \text{hom}(\Delta[1] \times X, U)$ for which reason the paths in $\text{hom}(X, U)$ are cartesian functors between Grothendieck fibrations over X .

Points in U correspond to small ∞ -categories and paths in U correspond to ∞ -functors from the end of the path to its starting point. Thus U may be considered as the ∞ -category of small ∞ -categories.

However, for $f, g : X \rightarrow U$ morphisms from $f^*\pi$ to $g^*\pi$ are not necessarily cartesian whereas paths $\Delta[1] \rightarrow \text{hom}(X, U)$ from f to g correspond to cartesian functors from f to g .

257 Hyperconnected Geometric Morphisms

Johnstone proved that every geometric morphism $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ factors as a hyperconnected geometric morphism followed by a localic one where hyperconnected means satisfying one of the equivalent conditions

- (1) F is full and faithful and its image is closed under subquotients
- (2) F is full and faithful and its image is closed under subobjects
- (3) F is full and faithful and its image is closed under quotients
- (4) all units and counits of the adjunction $F \dashv U$ are monos
- (5) U preserves subobject classifiers
- (6) $F_{/I}$ restricts to an equivalence between $\mathbf{Sub}_{\mathcal{S}}(I)$ and $\mathbf{Sub}_{\mathcal{E}}(FI)$.

Johnstone also characterized the geometric morphisms $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ where all $\varepsilon_X : FUX \rightarrow X$ are monic as those geometric morphisms for which the surjective-injective factorization coincides up to equivalence with the hyperconnected-localic one. For all this see A.4.6 of the *Elephant*.

From a fibrational point of view $F \dashv U$ hyperconnected means that the corresponding fibered adjunction $\Delta \dashv \Gamma : P_F \rightarrow P_{\mathcal{S}}$ restricts to an equivalence between $\mathbf{Sub}_{\mathcal{S}}$ and $F^*\mathbf{Sub}_{\mathcal{E}}$.

Condition (4) can be explained from a fibrational point of view as follows. For X over I let $\Gamma_I(X) : G(X) \rightarrow I$. We write $\varphi_X : 1_{G(X)} \rightarrow \coprod_{\Gamma_I(X)} 1_{G(X)}$ for the cocartesian arrow over $\Gamma_I(X)$ whose source is the terminal object in the fiber over $G(X)$. Let $\tilde{\varepsilon}_X : \coprod_{\Gamma_I(X)} 1_{G(X)} \rightarrow X$ be the unique vertical arrow such that $\tilde{\varepsilon}_X \varphi_X$ equals $\varepsilon_X : 1_{GX} \rightarrow X$, the counit of $1 \dashv G$ at X . Condition (4) says that $\tilde{\varepsilon}_X$ is a vertical mono as depicted in

$$\begin{array}{ccc}
 Y & \xrightarrow[\text{cocart.}]{\varphi} & \coprod_{\Gamma_I(X)} Y \\
 \downarrow & & \downarrow \\
 1_{G(X)} & \xrightarrow[\varphi_X]{\text{cocart.}} & \coprod_{\Gamma_I(X)} 1_{G(X)} \xrightarrow{\tilde{\varepsilon}_X} X
 \end{array}$$

i.e. objects over $G(X)$ w.r.t. P_F correspond to vertical morphisms to X which factor through $\tilde{\varepsilon}_X$.

If \mathcal{S} is **Set** then the geometric morphism $\Delta \dashv \Gamma : \mathcal{E} \rightarrow \mathbf{Set}$ is hyperconnected iff Γ preserves subobject classifiers, i.e. iff \mathcal{E} is 2-valued. Recall that a presheaf topos $\mathcal{E} = \widehat{\mathbb{C}}$ is 2-valued iff \mathbb{C} is strongly connected, i.e. all homsets of \mathbb{C} are inhabited. This may explain the terminology “hyperconnected”.

258 Axioms for Cohesion (Lawvere TAC 19, 2007)

Let \mathcal{E} be a topos over a base topos \mathcal{S} as given by a geometric morphism $e : \mathcal{E} \rightarrow \mathcal{S}$ which we assume to be essential and local, i.e. $e_! \dashv e^* \dashv e_* \dashv e^!$ with e^* (and thus also $e^!$) full and faithful.

For $X \in \mathcal{E}$ consider the map $e^*e_*X \xrightarrow{\varepsilon_X} X \xrightarrow{\eta_X} e^*e_!X$ which is of the form $e^*\theta_X$ for a unique $\theta_X : e_*X \rightarrow e_!X$ since e^* is full and faithful. The ensuing natural transformation $\theta : e^* \rightarrow e_!$ is called the “points-to-pieces” transform. As shown by Johnstone in his TAC 2011 paper the geometric morphism $e : \mathcal{E} \rightarrow \mathcal{S}$ is hyperconnected iff all θ_X are epic which requirement Lawvere refers to as “Nullstellensatz” since it claims that “every piece has a point”.

Lawvere calls \mathcal{E} over \mathcal{S} **precohesive** iff the geometric morphism $e : \mathcal{E} \rightarrow \mathcal{S}$ is essential and local, satisfies the “Nullstellensatz” and $e_!$ preserves finite products. The latter requirement is equivalent to all \mathcal{S}/S being exponential ideals within \mathcal{E}/e^*S via e_S^* .

For every X in \mathcal{E} and S in \mathcal{S} consider the morphism

$$e_!(X^{e^*S}) \times S \xrightarrow{\cong} e_!(X^{e^*S}) \times e_!e^*S \xrightarrow{\cong} e_!(X^{e^*S} \times e^*S) \xrightarrow{e_!\text{ev}} e_!X$$

whose transpose $e_!(X^{e^*S}) \rightarrow (e_!X)^S$ is called the “pieces of powers to powers of pieces” transform which one may require to be an isomorphism in which case the geometric morphism e is called **cohesive**.

A precohesive $e : \mathcal{E} \rightarrow \mathcal{S}$ is called **sufficiently cohesive** iff $e_!\Omega_{\mathcal{E}} \cong 1_{\mathcal{S}}$, i.e. $\Omega_{\mathcal{E}}$ is connected.

A precohesive $e : \mathcal{E} \rightarrow \mathcal{S}$ satisfies SCC (Stable Connected Codiscreteness) iff $\Pi_S \nabla_S : \mathcal{S}/S \rightarrow \mathcal{S}/S$ factors through $\text{Sub}_S(S) \hookrightarrow \mathcal{S}/S$ for all $S \in \mathcal{S}$.

The main result of Lawvere and Menni’s 2015 TAC paper says that a precohesive $e : \mathcal{E} \rightarrow \mathcal{S}$ validating SCC is cohesive iff \mathcal{S} validates IAC in which case $e_* \dashv e^! : \mathcal{S} \hookrightarrow \mathcal{E}$ coincides with $\mathcal{E}_{-\neg} \hookrightarrow \mathcal{E}$.

259 Punctual Local Connectedness (Johnstone TAC 25, 2011)

In his TAC paper *Remarks on Punctual Local Connectedness* from 2011 [Joh11] Johnstone proved that for a base topos \mathcal{S} with a natural numbers object for geometric morphisms $f : \mathcal{E} \rightarrow \mathcal{S}$ the following are equivalent

- (1) f is connected, locally connected and all $\theta_X : f^*X \rightarrow f_!X$ are epic
- (2) f is locally connected, hyperconnected¹⁷¹ and local¹⁷².

Such toposes over \mathcal{S} are called “precohesive”. He also shows that such f are *totally connected*, i.e. $f_! \dashv f^*$ preserves finite limits, iff all $\theta_X : f^*X \rightarrow f_!X$

¹⁷¹Hyperconnected means f is connected, i.e. f^* is full and faithful, and all $\varepsilon_X : f^*f_*X \rightarrow X$ are monic (from a fibrational point of view ε_X monic means that “global elements of X are disjoint” holds in the logic of \mathcal{S}).

¹⁷²i.e. f^* is full and faithful and f_* has a right adjoint $f^!$.

are isos. Moreover, he shows that for precohesive $f : \mathcal{E} \rightarrow \mathcal{S}$ the functor $f_!$ necessarily preserves finite products. As shown a bit later by Lawvere and Menni (TAC 30, 2015) a connected essential geometric morphism $f : \mathcal{E} \rightarrow \mathcal{S}$ is locally connected with $f_!$ preserving finite products if and only if $f_!$ sends pullbacks of arrows with codomain in the image f^* to pullbacks in \mathcal{S} .

259.1 Explaining the Proof of [Joh11]

Let \mathcal{S} and \mathcal{E} be toposes and $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ a bounded geometric morphism which is locally connected and connected. Moreover, we assume that \mathcal{S} and thus also \mathcal{E} has a natural numbers object. As usual unit and counit of $F \dashv U$ are denoted by η and ε , respectively. We write L for the left adjoint of F and denote unit and counit of this adjunction by α and β , respectively. By Cor. 3.3 of [Joh11] the functor U preserves epis iff U has a right adjoint R , i.e. the geometric morphism $F \dashv U$ is local. Unit and counit of $U \dashv R$ are denoted by γ and δ , respectively.

In [Joh11] it is shown that

$$\begin{array}{ccc}
 LFU & \xrightarrow{\beta_U} & U \\
 \downarrow L\varepsilon & & \downarrow U\alpha \\
 L & \xrightarrow{\eta_L} & UFL
 \end{array}$$

commutes. Accordingly, if moreover $U \dashv R$ then

$$\begin{array}{ccc}
 FUR & \xrightarrow{\varepsilon_R} & R \\
 \downarrow F\delta & & \downarrow R\eta \\
 F & \xrightarrow{\gamma_F} & RUF
 \end{array}$$

commutes as well.

Since F is full and faithful both β and η are natural isomorphisms. Let θ be the natural transformation making

$$\begin{array}{ccc}
 LFU & \xrightarrow[\cong]{\beta_U} & U \\
 \downarrow L\varepsilon & \searrow \theta & \downarrow U\alpha \\
 L & \xrightarrow[\cong]{\eta_L} & UFL
 \end{array}$$

commute. Then the diagram

$$\begin{array}{ccc}
 FLFU & \xrightarrow{F\beta_U} & FU \\
 \downarrow FL\varepsilon & \searrow F\theta & \downarrow FU\alpha \\
 FL & \xrightarrow{F\eta_L} & FUFU
 \end{array}$$

commutes as well. From the triangular equalities it follows that

$$\begin{array}{ccc}
 FLFU & \xleftarrow{\alpha_{FU}} & FU \\
 \downarrow FL\varepsilon & \searrow F\theta & \downarrow FU\alpha \\
 FL & \xleftarrow{\varepsilon_{FL}} & FUFU
 \end{array}$$

and thus also

$$F\theta = \alpha\varepsilon$$

by the naturality conditions for α or ε .

Johnstone refers to the requirement that θ is epic as *punctual local connectedness* (PLC). By Cor. 2.5 of [Joh11] it follows from PLC that U preserves epis, i.e. $F \dashv U$ is local.¹⁷³ Since F as a left adjoint preserves epis it follows from

¹⁷³Suppose $e : A \rightarrow B$ is epic in \mathcal{E} . Thus, the map g in

$$\begin{array}{ccc}
 C & \xrightarrow{f} & A \\
 \downarrow g & \lrcorner & \downarrow e \\
 FUB & \xrightarrow{\varepsilon_B} & B
 \end{array}$$

is epic. Since L being a left adjoint preserves epis the composite

$$UC \xrightarrow{\theta_C} FC \xrightarrow{Lg} LFUB \xrightarrow{\beta_{UB}} UB$$

is epic since the counit β is an iso.

Since η is an iso and $\varepsilon_F \circ F\eta = \text{id}_F$ it follows that ε_F is an iso. Thus ε_{FU} is the inverse of $F\eta_U$. Since $U\varepsilon \circ \eta_U = \text{id}_U$ we have $FU\varepsilon \circ F\eta_U = \text{id}_U$ and thus $FU\varepsilon$ is the inverse of $F\eta_U$. Thus $\varepsilon_{FU} = FU\varepsilon$.

Thus we have

$$\begin{aligned}
 \beta_{UB} \circ Lg \circ \theta_C &= \beta_{UB} \circ \theta_{FUB} \circ Ug \\
 &= \beta_{UB} \circ L\varepsilon_{FUB} \circ \beta_{UFUB}^{-1} \circ Ug = \\
 &= \beta_{UB} \circ LFU\varepsilon_B \circ \beta_{UFUB}^{-1} \circ Ug = \\
 &= U\varepsilon_B \circ \beta_{UFUB} \circ \beta_{UFUB}^{-1} \circ Ug
 \end{aligned}$$

PLC that $F\theta$ is epic and thus α is epic. On the other hand if α is epic and U preserves epis then θ is epic since $\eta_L \circ \theta = U\alpha$ is epic and η_L is an iso.

Thus PLC holds iff $F \dashv U$ is local and α is epic.

Suppose now that $F \dashv U$ is local. We discuss now equivalents of PLC under this additional assumption.

First observe that θ_X is epic iff $L\varepsilon_X$ is epic iff $\mathcal{S}(L\varepsilon_X, I)$ is 1-1 for all $I \in \mathcal{S}$ iff $\mathcal{E}(\varepsilon_X, RI)$ is 1-1 for all $I \in \mathcal{S}$. For $f : X \rightarrow RI$ we observe that $U(f)$ is both the transpose of $f\varepsilon_X$ (w.r.t. $F \dashv U$) and the transpose of $\gamma_{FI}f$ (w.r.t. $U \dashv R$). Thus $\mathcal{E}(X, \gamma_{FI})$ is 1-1 iff $\mathcal{E}(\varepsilon_X, RI)$ is 1-1. Accordingly, all θ_X are epic iff all γ_{FI} are monic.¹⁷⁴

Since F and R are full and faithful η and δ are natural isomorphisms. Let ϕ be the natural transformation making

$$\begin{array}{ccc} FUR & \xrightarrow{F\delta} & F \\ \varepsilon_R \downarrow & \phi & \downarrow \gamma_F \\ R & \xrightarrow{R\eta} & RUF \end{array}$$

\cong (on the top and bottom arrows)

commute. Then the diagram

$$\begin{array}{ccc} FURU & \xrightarrow{F\delta_U} & FU \\ \varepsilon_{RU} \downarrow & \phi_U & \downarrow \gamma_{FU} \\ RU & \xrightarrow{R\eta_U} & RUFU \end{array}$$

\cong (on the top and bottom arrows)

commutes as well. From the triangular equalities it follows that

$$\begin{array}{ccc} FURU & \xleftarrow{FU\gamma} & FU \\ \varepsilon_{RU} \downarrow & \phi_U & \downarrow \gamma_{FU} \\ RU & \xleftarrow{RU\varepsilon} & RUFU \end{array}$$

\cong (on the top and bottom arrows)

and thus also

$$\begin{aligned} & \phi_U = \gamma\varepsilon \\ & \hline & = U\varepsilon_B \circ Ug = U(\varepsilon_B \circ g) = U(e \circ f) \\ & = Ue \circ Uf \end{aligned}$$

and thus Ue epic (since $\beta_{UB} \circ Lg \circ \theta_C$ is epic).

¹⁷⁴Recall that we call $X \in \mathcal{E}$ concrete iff $\gamma_X : X \rightarrow RUX$ is monic. Thus, θ is epic iff FI is concrete for all $I \in \mathcal{S}$.

by the naturality conditions for γ or ε .

If ϕ is monic then so is ϕ_U from which it follows that ε is monic. On the other hand if ε is monic then so is ε_R from which it follows that ϕ is monic. Thus ϕ is monic iff ε is monic.

Thus $F \dashv U$ is hyperconnected, i.e. ε is monic, iff $\phi = R\eta^{-1} \circ \gamma_F$ is monic iff γ_F is monic iff $F \dashv U$ validates PLC, i.e. α is epic.¹⁷⁵

Thus, a bounded locally connected local geometric morphism $F \dashv U$ is hyperconnected iff it validates PLC.

Summarizing our considerations we conclude that for bounded geometric morphisms $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ the following two conditions are equivalent

- (1) $F \dashv U$ is connected, locally connected and validates PLC
- (2) $F \dashv U$ is locally connected, hyperconnected and local.

Notice, moreover, that for bounded connected locally connected geometric morphisms $F \dashv U$ satisfying PLC the leftmost adjoint L preserves finite products for which reason F exhibits \mathcal{S} as an exponential ideal in \mathcal{E} .

The geometric morphism from the Sierpiński topos to **Set** is an example of a locally connected local geometric morphism which is not hyperconnected since it is localic and not a geometric equivalence. Actually, for all presheaf toposes over posets with a greatest element and more than one element the geometric morphism to **Set** is locally connected and local but not hyperconnected.

Presheaf toposes over strongly connected sites \mathbb{C} are locally connected and hyperconnected but need not be local. Typical examples of such \mathbb{C} are nontrivial monoids with no nontrivial idempotents as e.g. nontrivial cancellative monoids examples of which are nontrivial groups and monoids like $(\mathbb{N}, +)$, $(\mathbb{R}_0^+, +)$ etc.

¹⁷⁵The argument in [Joh11] is different. His Lemma 2.3 says that ϕ is monic iff U is faithful w.r.t. morphisms with codomain in the image of F iff θ is epic. For seeing this first recall that $\theta = \eta_L^{-1} \circ U\alpha$ and $\phi = R\eta^{-1} \circ \gamma_F$. Now for $f : LX \rightarrow I$ and $g : X \rightarrow FI$ the composites $f \circ \theta_X$ and $\phi_B \circ g$ look as follows

$$\begin{array}{ccc}
 LX & \xrightarrow{f} & I \\
 \eta_{LX}^{-1} \uparrow & & \uparrow \eta_I^{-1} \\
 UFLX & \xrightarrow{UFf} & UFI \\
 U\alpha_X \uparrow & \nearrow U(Ff \circ \alpha_X) & \\
 UX & &
 \end{array}
 \qquad
 \begin{array}{ccc}
 X & \xrightarrow{g} & FI \\
 \gamma_X \downarrow & & \downarrow \gamma_{FI} \\
 RUA & \xrightarrow{RUg} & RUF I \\
 R(\eta_I^{-1} \circ U g) \searrow & & \downarrow R\eta_I^{-1} \\
 & & FI
 \end{array}$$

where the only noninvertible step is applying U to an arrow whose codomain is in the image of F . Thus faithfulness of U on morphisms with codomain in the image of F is equivalent to the respective composition operations being 1-1.

260 Various Axiomatizations of Stably Precohesive Toposes

In their 2015 TAC paper [LM15] Lawvere and Menni show that for an essential connected geometric morphism $p : \mathcal{E} \rightarrow \mathcal{S}$ the following properties are equivalent

- (1) p is locally connected and $p_!$ preserves binary products
- (2) $p_!$ preserves pullbacks of cospans whose common codomain is in the image of p^* .

Obviously, a further equivalent condition is that

- (3) p^* preserves dependent products and $p_!$ preserves binary products.

As is well known p is hyperconnected iff p_* preserves subobject classifiers and p is local iff p_* has a right adjoint $p^!$ and p^* is full and faithful.

In [LM15] *stably precohesive* toposes over a base topos \mathcal{S} have been defined as locally connected hyperconnected local geometric morphisms $p : \mathcal{E} \rightarrow \mathcal{S}$ such that $p_!$ preserves binary products.

In his 2011 TAC paper [Joh11] Johnstone has shown that for a *bounded* locally connected hyperconnected local geometric morphism $p : \mathcal{E} \rightarrow \mathcal{S}$ the functor $p_! : \mathcal{E} \rightarrow \mathcal{S}$ necessarily preserves finite products.

If p is assumed as bounded then

- (i) p_* has a fibered right adjoint $p^!$ iff p is connected and p_* preserves coequalizers
- (ii) p^* has a fibered left adjoint $p_!$ iff p^* preserves dependent products.

In the light of the discussion above one may characterize bounded stably precohesive toposes over \mathcal{S} as bounded geometric morphisms $p : \mathcal{E} \rightarrow \mathcal{S}$ such that

- (1) p_* preserves subobject classifiers
- (2) p_* preserves coequalizers
- (3) p^* preserves dependent products.

That p^* has a fibered left adjoint $p_!$ follows from an appropriate fibered adjoint functor theorem and preservation of binary products by $p_!$ follows from [Joh11]. That p_* has a fibered right adjoint $p^!$ also follows from an appropriate adjoint functor theorem from (2) and preservation of internal sums by p_* (since p hyperconnected implies p connected which is equivalent to p_* preserving internal sums).

In a paper from 2019 with title *The hyperconnected maps that are local* Menni claims/shows that a hyperconnected geometric morphism $p : \mathcal{E} \rightarrow \mathcal{S}$ is local iff $p_* : \mathcal{E} \rightarrow \mathcal{S}$ preserves coequalizers. In Menni's 2017 paper *The construction of π_0 in axiomatic cohesion* he shows in Cor. 3.8 that for a hyperconnected

geometric morphism $p : \mathcal{E} \rightarrow \mathcal{S}$ its inverse image part $p^* : \mathcal{S} \rightarrow \mathcal{E}$ has a left adjoint $p_!$ whenever p^* preserves exponentials. Thus, as stated in Cor.3.9 of *loc.cit.* for a hyperconnected geometric morphism $p : \mathcal{E} \rightarrow \mathcal{S}$ the subcategory $p^* : \mathcal{S} \hookrightarrow \mathcal{E}$ is an exponential ideal iff p^* has a finite product preserving left adjoint $p_!$. Using Prop. 2.7 of [Joh11] he shows in Prop.3.10 of *loc.cit.* that if $p : \mathcal{E} \rightarrow \mathcal{S}$ is hyperconnected local and essential then $p_! : \mathcal{E} \rightarrow \mathcal{S}$ preserves finite products whenever $p^* : \mathcal{S} \rightarrow \mathcal{E}$ preserves exponentials.

Thus a geometric morphism $p : \mathcal{E} \rightarrow \mathcal{S}$ is (stably) precohesive iff p_* preserves subobject classifiers and coequalizers and p^* preserves (dependent) function spaces.

It is an open question whether there exists a hyperconnected local geometric morphism $p : \mathcal{E} \rightarrow \mathcal{S}$ such that p^* preserves exponentials but not dependent function spaces (equivalently not all $(p/I)^* = p^*_{/I} : \mathcal{S}/I \rightarrow \mathcal{E}/p^*I$ preserve exponentials). In Garner and Streicher's TAC 2021 paper it has been shown that there exist local geometric morphisms between Grothendieck toposes whose inverse image part preserves ordinary exponentials but not dependent function spaces. Thus the question is whether the additional assumption of hyperconnectedness makes a difference, i.e. the inverse image part preserves dependent function spaces whenever it preserves ordinary exponentials.

As shown in section 284 a local geometric morphism $p : \mathcal{E} \rightarrow \mathcal{S}$ is hyperconnected iff all the components of the natural transformation $\phi : p^* \rightarrow p^!$ are monic (where $\phi_I : p^*I \rightarrow p^!I$ is the transpose of the inverse of the isomorphism $\eta_I : I \rightarrow p_*p^*I$). For stably precohesive $p : \mathcal{E} \rightarrow \mathcal{S}$ one can show that for $\Pi_u v : P \rightarrow I$ in \mathcal{S} the dependent function space $p^*\Pi_u v$ is given by $p^!\Pi_u v \circ \phi_P$. In Hemelaer and Rogers's 2021 APCS paper it was shown that this is not the case in general for hyperconnected local geometric morphisms since their inverse image part need not even preserve ordinary exponentials.

261 “Strongly Localic” Toposes

Toposes of the form $\text{Sh}(B)$ for a cBa B can be characterized as those toposes \mathcal{E} over \mathbf{Set} such that every epi e in \mathcal{E} is split, i.e. $es = \text{id}$ for some s . Let $\Delta \dashv \Gamma : \mathcal{E} \rightarrow \mathbf{Set}$ be the structural geometric morphism of \mathcal{E} which is localic. Thus, every $X \in \mathcal{E}$ appears as subquotient of some $\Delta(I)$, i.e. there is an epi $e : C \twoheadrightarrow A$ for some subobject $m : C \twoheadrightarrow \Delta(I)$. Let $s : A \rightarrow C$ with $es = \text{id}_A$ then $ms : A \twoheadrightarrow \Delta(I)$. Thus, every object A of \mathcal{E} appears as subobject of $\Delta(I)$ for some I in \mathbf{Set} .

We call a geometric morphism $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ **strongly localic** iff every object A of \mathcal{E} appears as subobject of FI for some I in \mathcal{S} . Suppose $a : A \rightarrow FI$ then $m : A \twoheadrightarrow FJ$ for some J in \mathcal{S} and thus we have

$$\begin{array}{ccccc}
 A & \xlongequal{\quad} & A & & \\
 \downarrow a & & \downarrow \langle a, m \rangle & \swarrow m & \\
 FI & \xleftarrow{F\pi} & F(I \times J) & \xrightarrow{F\pi'} & FJ
 \end{array}$$

i.e. a appears as subobject of Fu in \mathcal{E}/FI for some $u : J \rightarrow I$.

Do there exist strongly localic toposes \mathcal{E} over \mathbf{Set} , i.e. every object A of \mathcal{E} appears as subobject of $\Delta(I)$ for some set I , where not every epi splits? No, in Hofstra’s Thesis it has been shown in Prop. 6.3.5 that for Grothendieck toposes \mathcal{E} over \mathbf{Set} it holds that every epi in \mathcal{E} is split iff $\Delta \dashv \Gamma : \mathcal{E} \rightarrow \mathbf{Set}$ is strongly localic, i.e. 1 is a *strong bound*¹⁷⁶.

¹⁷⁶a strong bound for a geometric morphism $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ is an object S in \mathcal{E} such that every object A in \mathcal{E} appears as subobject of $S \times FI$ for some I in \mathcal{S}

262 Triposes and Implicative Algebras

A “philosophical” tripos over a topos \mathcal{S} is a regular¹⁷⁷ functor F from \mathcal{S} to a topos \mathcal{E} such that every object A of \mathcal{E} appears as subquotient of FI for some I in \mathcal{S} . A “mathematical” tripos over \mathcal{S} is a “philosophical” tripos $F : \mathcal{S} \rightarrow \mathcal{E}$ such that $F^*\mathbf{Sub}_{\mathcal{E}}$ admits a generic family, i.e. there is a $t : T \rightarrow F\Sigma$ such that every $m : P \rightarrow FI$ fits into a pullback square

$$\begin{array}{ccc} P & \longrightarrow & T \\ \downarrow m & \lrcorner & \downarrow t \\ FI & \xrightarrow{Fp} & F\Sigma \end{array}$$

for some $p : I \rightarrow \Sigma$ typically not unique.

A morphism of “philosophical” triposes from $F_1 : \mathcal{S} \rightarrow \mathcal{E}_1$ to $F_2 : \mathcal{S} \rightarrow \mathcal{E}_2$ is a regular functor $G : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ with $F_2 = GF_1$. Notice that G is necessarily a “philosophical” tripos. But there is no reason why G should be a “mathematical” tripos even if F_1 and F_2 are¹⁷⁸. Thus, for “mathematical” triposes F_1 and F_2 one defines a morphism of “mathematical” triposes from F_1 to F_2 as a regular functor G with $F_2 = GF_1$ such that $G^*\mathbf{Sub}_{\mathcal{E}_2}$ admits a generic family. Notice that such a G necessarily is a “mathematical” tripos itself.

Miquel has shown that every tripos over \mathbf{Set} is induced by an implicative algebra. But Miquel’s theorem generalizes to arbitrary base toposes \mathcal{S} . Thus, morphisms of “mathematical” triposes G from F_1 to F_2 correspond to implicative algebras internal to \mathcal{E}_1 .

We do not know whether “mathematical” triposes $F_1, F_2 : \mathbf{Set} \rightarrow \mathcal{E}$ are necessarily equivalent as triposes. But for “philosophical” triposes we know that this is wrong since for every natural number $n > 0$ the functor $F_n : \mathbf{Set} \rightarrow \mathbf{Set} : S \mapsto S^n$ is a philosophical tripos but F_n and F_m are equivalent if and only if $n = m$.

For $i=1,2$ let $F_i : \mathcal{S} \rightarrow \mathcal{E}_i$ be the constant objects functor for the tripos induced by an implicative algebra \mathcal{A}_i in \mathcal{S} , i.e. $\mathcal{E}_i = \mathcal{S}[\mathcal{A}_i]$. Regular functors $G : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ with $F_2 = G \circ F_1$ correspond to cartesian morphism $g : F_1^*\mathbf{Sub}_{\mathcal{E}_1} \rightarrow F_2^*\mathbf{Sub}_{\mathcal{E}_2}$ preserving regular logic, i.e. finite limits and existential quantification. Obviously, such g are uniquely determined by $h = g_{\mathcal{A}_1}(\text{id}_{\mathcal{A}_1}) : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ since $g_I(\varphi : I \rightarrow \mathcal{A}_1) = h \circ \varphi$.

Accordingly, a morphism of implicative algebras from \mathcal{A}_1 to \mathcal{A}_2 is a function $h : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ such that

- (1) the cartesian $g : F_1^*\mathbf{Sub}_{\mathcal{E}_1} \rightarrow F_2^*\mathbf{Sub}_{\mathcal{E}_2}$ given by $g_I(\varphi : I \rightarrow \mathcal{A}_1) = h \circ \varphi$ preserves regular logic, i.e. finite limits and existential quantification, and

¹⁷⁷i.e. preserves finite limits and regular epis

¹⁷⁸though it is if G has a right adjoint since then $G \dashv V$ is a localic geometric morphism

- (2) for the ensuing regular functor $G : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ the presheaf $G^*\mathbf{Sub}(\mathcal{E}_2)$ appears as quotient of a representable presheaf, i.e. there is a $t : T \rightarrow G\Sigma$ such that every $m : P \rightarrow GI$ fits into a pullback diagram

$$\begin{array}{ccc}
 P & \longrightarrow & T \\
 \downarrow m & \lrcorner & \downarrow t \\
 GI & \xrightarrow{Gp} & G\Sigma
 \end{array}$$

for some $p : I \rightarrow \Sigma$

where $F_i : \mathcal{S} \rightarrow \mathcal{E}_i$ is the “mathematical” tripos over \mathcal{S} induced by \mathcal{A}_i .

There is a problem for base toposes different from **Set** !

The problem is that although from a mathematical tripos $F : \mathcal{S} \rightarrow \mathcal{E}$ we can construct a fibered poset $F^*\mathbf{Sub}_{\mathcal{E}}$ equivalent to $\mathcal{S}(-, \Sigma)$ the collection

$$\Phi = \{p : 1 \rightarrow \Sigma \mid p^*t \dashv \vdash \top_1\} = \{p : 1 \rightarrow \Sigma \mid Fp \text{ factors through } t\}$$

lives in **Set** and **not** in \mathcal{S} . But there need not exist a subobject $s : S \rightarrow \Sigma$ such that $p \in \Phi$ if and only if p factors through s nor need such an s be unique up to isomorphism in case it exists!

263 Toposes of Graphs and of Reflexive Graphs

Let \mathcal{E} be the topos of reflexive graphs and \mathcal{F} be the topos of graphs. The inclusion i of the site for \mathcal{F} into the site for \mathcal{E} induces an essential geometric morphism $i_! \dashv i^* \dashv i_* : \mathcal{F} \rightarrow \mathcal{E}$.

For B in \mathcal{E} let $\mathcal{S}(B)$ be the full subcategory of \mathcal{E}/B on morphisms $p : E \rightarrow B$ which (not only preserve) but also reflect distinguished loops. The inclusion functor $I_B : \mathcal{S}(B) \hookrightarrow \mathcal{E}/B$ not only preserves finite limits but also has a right adjoint R_B which restricts $p : E \rightarrow B$ to the subgraph E' obtained from E by removing all edges which are sent by p to distinguished loops but are not distinguished loops themselves.

One readily checks that $p : E \rightarrow B$ is an object of $\mathcal{S}(B)$ iff

$$\begin{array}{ccc} \Delta\Gamma E & \xrightarrow{\varepsilon_E} & E \\ \Delta\Gamma p \downarrow & (\dagger) & \downarrow p \\ \Delta\Gamma B & \xrightarrow{\varepsilon_B} & B \end{array}$$

is a pullback.¹⁷⁹ From this observation it follows that pullback along $u : B' \rightarrow B$ sends $\mathcal{S}(B)$ to $\mathcal{S}(B')$ since in

$$\begin{array}{ccccc} \Delta\Gamma E' & \xrightarrow{\varepsilon_{E'}} & E' & \xrightarrow{v} & E \\ \Delta\Gamma p' \downarrow & & \downarrow p' & \lrcorner & \downarrow p \\ \Delta\Gamma B' & \xrightarrow{\varepsilon_{B'}} & B' & \xrightarrow{u} & B \end{array}$$

the left square is a pullback since the outer rectangle is a pullback because

$$\begin{array}{ccccc} \Delta\Gamma E' & \xrightarrow{\Delta\Gamma v} & \Delta\Gamma E & \xrightarrow{\varepsilon_E} & E \\ \Delta\Gamma p' \downarrow & \lrcorner & \downarrow \Delta\Gamma p & \lrcorner & \downarrow p \\ \Delta\Gamma B' & \xrightarrow{\Delta\Gamma u} & \Delta\Gamma B & \xrightarrow{\varepsilon_B} & B \end{array}$$

since Γ and Δ preserve finite limits and thus pullbacks.

Thus, the collection \mathcal{S} of maps $p : E \rightarrow B$ with (\dagger) a pullback is stable under pullbacks in \mathcal{E} .

¹⁷⁹Thus the geometric morphism $\Gamma \dashv \nabla$ is not locally connected!

264 Countably Presented Locales are Spatial

R. Heckmann (2015) has shown that countably presented locales are spatial and up to isomorphism are the subspaces of $\mathcal{P}\omega$ (with Scott topology) which appear as countable intersections of UCO subspaces of $\mathcal{P}\omega$. Here UCO stands for “union of an open and a closed subset”, i.e. sets of the form $U \Rightarrow V = \{x \in \mathcal{P}\omega \mid x \in U \Rightarrow x \in V\} = \mathbb{C}U \cup V$.

de Brecht and Schröder refer to countably presented locales as “**quasi-Polish**” spaces. They can be alternatively characterized as equalizers of pairs of endomaps on $\mathcal{P}\omega$ taken in the category **Sp** of spaces and continuous maps. As shown in Heckmann’s paper these quasi-Polish spaces are closed under finite limits (and, actually, even closed under countable products).

Let \mathbb{C} be the small category of quasi-Polish spaces and \mathcal{J} be the open cover topology on \mathbb{C} . The ensuing “gros” topos $\mathcal{F} = \mathbf{Sh}(\mathbb{C}, \mathcal{J})$ is a model of Brouwerian intuitionistic mathematics as shown by M. Fourman in his old paper on *Continuous Truth* though he did not emphasize that \mathbb{C} is wellpointed since this requires bar induction on the meta level.

Since Baire space \mathcal{B} is Polish and thus also quasi-Polish the monoid of continuous endomaps of \mathcal{B} is a full subcategory of \mathbb{C} . Moerdijk in his Thesis investigated the topos of continuous actions of this topological monoid as a model for Brouwer’s theory of choice sequences.

Following considerations by Awodey and Bauer in their paper *Sheaf Toposes for Realizability* one may consider functors

$$Y : \mathbf{RT}(\mathcal{A}) \rightarrow \mathcal{F} : X \mapsto \mathbf{RT}(\mathcal{A})(J(-), X)$$

induced by the embedding $J : \mathbb{C} \hookrightarrow \mathbf{RT}(\mathcal{A})$ when \mathcal{A} is the pca $\mathcal{P}\omega$ or \mathcal{K}_2 . In both cases J factors through the inclusion of \mathbf{QCB}_0 into the respective realizability toposes since \mathbb{C} is a full subcategory of \mathbf{QCB}_0 . Notice, however, that in both cases Y is not full and faithful.

265 $\exists_r s^* \dashv \forall_s r^*$

Let \mathcal{E} be a topos and $\langle r, s \rangle : R \rightarrow I \times J$. For predicates φ on I the predicate $\forall_s r^* \varphi$ on J is equivalent to $\forall i: I. iRj \rightarrow \varphi(i)$ for $j \in J$. Obviously, a left adjoint of $\forall_s r^*$ is given by $\exists_r s^*$ which sends a predicate ψ on J to the predicate on I which is given by $\exists j: J. iRj \wedge \psi(j)$ for $i \in I$.

In less categorical language this may be rephrased as $\exists j: J. iRj \wedge \psi(j) \vdash \varphi(i)$ iff $iRj \wedge \psi(j) \vdash \varphi(i)$ iff $\psi(j) \vdash iRj \rightarrow \varphi(i)$ iff $\psi(j) \vdash \forall i: I. iRj \rightarrow \varphi(i)$.

One may write this as $\diamond_{R^{\text{op}}} \dashv \square_R$ when defining $\diamond_R \varphi$ as $\exists i: I. iRj \wedge \varphi(i)$ and $\square_R \varphi$ as $\forall i: I. iRj \rightarrow \varphi(i)$ as common in Kripke semantics for modal logic. But $\diamond_R \dashv \square_R$ will fail in general unless R is e.g. a symmetric relation on I .

266 An Essential Local Geometric Morphism which is Not Locally Connected though its Inverse Image Part is an Exponential Ideal (Jan. 2021)

In [BP80] the authors introduced and studied a property of geometric morphisms they called “molecular” which, however, nowadays is usually referred to as “locally connected”. One of the various characterizations of this property is that the inverse image part F of the geometric morphism $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ preserves dependent products, i.e. right adjoints to pullback functors. As described below this requirement is tantamount to F having a fibered or “indexed” left adjoint. In *loc.cit* they also consider the property that F has an enriched left adjoint which amounts to F preserving (ordinary) exponentials and having a left adjoint.

In this note we will exhibit a geometric morphism $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ which is not locally connected though its inverse image part F preserves exponentials. Moreover, the functor F will be full and faithful and have a left adjoint which preserves finite products and the functor U will have a right adjoint R .

266.1 Preliminaries

A geometric morphism $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ is called *locally connected* iff F has a left adjoint L which is fibered or indexed over \mathcal{S} , i.e.

$$\begin{array}{ccc}
 B & \xrightarrow{f} & A \\
 \downarrow b & \lrcorner & \downarrow a \\
 FJ & \xrightarrow{Fu} & FI
 \end{array}
 \quad \text{implies} \quad
 \begin{array}{ccc}
 LB & \xrightarrow{Lf} & LA \\
 \downarrow \hat{b} & \lrcorner & \downarrow \hat{a} \\
 J & \xrightarrow{u} & I
 \end{array}$$

as discussed e.g. in [BP80, Jo02, Str20].

For I (and thus also FI) terminal this condition boils down to the requirement that $L \dashv F$ validates *Frobenius reciprocity*, i.e. $\langle \widehat{\pi}_1, L\pi_2 \rangle : L(FI \times A) \rightarrow I \times LA$ is an isomorphism whenever $\pi_1 : FI \times A \rightarrow FI$ and $\pi_2 : FI \times A \rightarrow FI$ form a limiting cone. One easily checks that $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ is locally connected iff $L_I = \varepsilon_I \circ L_{/FI} \dashv F_{/I} = F_I$ validates Frobenius reciprocity for all $I \in \mathcal{S}$. By A.1.5.8 of [Jo02] an adjunction between cartesian closed categories validates Frobenius reciprocity iff its right adjoint preserves exponentials.

Thus, for a geometric morphism $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ the following conditions are equivalent

- (1) $F \dashv U$ is locally connected
- (2) F preserves dependent products (i.e. right adjoints to pullback functors)
- (3) $F_{/I}$ preserves exponentials for all $I \in \mathcal{S}$

as formulated in Prop. C.3.3.1 of [Jo02].

Notice that the above observations remain valid under the weaker assumption that $F : \mathcal{S} \rightarrow \mathcal{E}$ preserves finite limits but not necessarily has a right adjoint. For obvious reasons (A.1.5.8 of [Jo02]) under such weaker assumptions we refer to requirement (3) above as *stably Frobenius*. Stably Frobenius adjunctions $L \dashv F$ with full and faithful right adjoint F are sometimes called *semi-left-exact reflections* because they are reflections whose left adjoint preserves pullbacks of cospans where one of the arrows is in the subcategory as given by F .

Moreover, by A.4.3.1 of [Jo02] a full reflective subcategory of a cartesian closed category is an exponential ideal iff the left adjoint preserves finite products. Thus, for a locally connected geometric morphism $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ the left adjoint L of F preserves finite products iff F is full and faithful and the corresponding full subcategory is an exponential ideal. As shown in [LM15] for essential connected geometric morphisms $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ the left adjoint L of F preserves pullbacks of cospans whose common codomain is in the image of F iff it is locally connected and L preserves binary products.¹⁸⁰

266.2 The Counterexample

For an adjunction $L \dashv F : \mathbb{A} \rightarrow \mathbb{B}$ let $L_! \dashv L^* : \widehat{\mathbb{A}} \rightarrow \widehat{\mathbb{B}}$ be the adjunction where $L_!$ is the left Kan extension of $Y_{\mathbb{A}} \circ L$ along $Y_{\mathbb{B}}$ and L^* is change of base along L . Notice, moreover, that L^* is naturally isomorphic to $F_!$, the left Kan extension of $Y_{\mathbb{B}} \circ F$ along $Y_{\mathbb{A}}$, which also preserves representable objects.

Thus, if $L_! \dashv L^*$ is stably Frobenius, i.e. the geometric morphism $L^* \dashv L_*$ is locally connected, then $L \dashv F$ is stably Frobenius as well since $L^* \cong F_!$ and both $L_!$ and $F_!$ preserve representable objects.

If \mathbb{B} is cartesian closed and $L \dashv F : \mathbb{A} \rightarrow \mathbb{B}$ is an adjunction where F is the inclusion of an exponential ideal then so is $L_! \dashv L^* : \widehat{\mathbb{A}} \rightarrow \widehat{\mathbb{B}}$ since L preserves finite products and this property extends to $L_!$ since \times preserves (small) colimits (in each argument).

Lemma 266.1 *Suppose \mathbb{B} is locally cartesian closed and $L \dashv F : \mathbb{A} \rightarrow \mathbb{B}$ is an adjunction where F is full and faithful and L preserves binary products. If the connected geometric morphism $L^* \dashv L_* : \widehat{\mathbb{A}} \rightarrow \widehat{\mathbb{B}}$ is locally connected then \mathbb{A} is locally cartesian closed.*

Proof: Suppose that $L^* \dashv L_*$ is locally connected, i.e. $L_! \dashv L^*$ is stably Frobenius. Thus, since $L_!$ and $L^* \cong F_!$ preserve representable objects the adjunction $L \dashv F$ is also stably Frobenius. Since F by assumption is full and faithful the

¹⁸⁰As shown in Prop. 2.7 of [Jo11] if $L \dashv F \dashv U \dashv R : \mathcal{S} \rightarrow \mathcal{E}$ with F (and thus also R) full and faithful then \mathcal{S} is an exponential ideal in \mathcal{E} (via F) whenever F preserves exponentials and all components of the canonical transformation $\theta : U \rightarrow L$ are epic. Here as in [Jo11, LM15] $\theta_A : UA \rightarrow LA$ is the unique morphism whose image under F is $FUA \xrightarrow{\varepsilon_A} A \xrightarrow{\eta_A} FLA$.

As also shown in [Jo11] for locally connected hyperconnected and local geometric morphisms $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ the left adjoint L of F necessarily preserves finite products.

adjunction $L \dashv F$ is semi-left-exact. Then by Lemma 4.4 of [GL12] it follows that \mathbb{A} is locally cartesian closed. \square

In [GL12] the authors consider the following situation. Let \mathbb{B} be a small category equivalent to the category of finite reflexive graphs and morphisms between them and $F : \mathbb{A} \rightarrow \mathbb{B}$ be the inclusion of the full subcategory of finite preorders. In Example 6.4 of [GL12] it is shown that \mathbb{A} is an exponential ideal in \mathbb{B} and thus L preserves finite products. Moreover, as also shown in Example 6.4 of *loc.cit.* the category \mathbb{A} is not locally cartesian closed. Thus, it follows by Lemma 266.1 above that the geometric morphism $L^* \dashv L_*$ is not locally connected.

Another class of examples arises from realizability models of Synthetic Domain Theory as considered in [LS97] where one instantiates the reflective adjunction $L \dashv F : \mathbb{A} \rightarrow \mathbb{B}$ as follows. For \mathbb{B} one takes for the category $\mathbf{Mod}(\mathcal{A})$ of modest sets over some partial combinatory algebra \mathcal{A} and for \mathbb{A} the full reflective subcategory of *complete extensional Σ -spaces* where the object Σ (intuitively corresponding to the so-called Sierpiński space $\{\perp < \top\}$) in $\mathbf{Mod}(\mathcal{A})$ is induced by a *divergence structure* on \mathcal{A} . The category $\mathbf{Mod}(\mathcal{A})$ is locally cartesian closed and the full subcategory of complete extensional Σ -spaces is an exponential ideal but not locally cartesian closed.

As a summary of our considerations we obtain the following theorem.

Theorem 266.1 *The geometric morphism $L^* \dashv L_* : \widehat{\mathbb{A}} \rightarrow \widehat{\mathbb{B}}$ is essential and local but not locally connected, i.e. L^* does not preserve dependent products, although the left adjoint $L_!$ of L^* preserves finite products, i.e. the full subcategory of $\widehat{\mathbb{A}}$ as given by L^* is an exponential ideal.*

Alas, the functor L_* does not preserve subobjects classifiers, i.e. the geometric morphism $L^* \dashv L_*$ is not hyperconnected. But in [HR20] one finds an example of an essential hyperconnected and local geometric morphism which fails to be locally connected. Though their and our counterexample are quite different in nature together they seem to point into the direction that essential hyperconnected local geometric morphisms need not be locally connected even if their inverse image part preserves exponentials. This would provide a (negative) answer to the question raised in the last paragraph of section 10 of [LM15].

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267 Stably Frobenius (bis) (6.1.21)

Let $F \dashv U : \mathbb{B} \rightarrow \mathbb{A}$ be an adjunction between categories with finite limits. For I in \mathbb{B} let $U_I = U_{/I} : \mathbb{B}/I \rightarrow \mathbb{A}/UI$ with left adjoint $F_I = \Sigma_{\varepsilon_I} \circ F_{/UI}$.

Consider the following pullbacks

$$\begin{array}{ccc}
 P & \xrightarrow{q} & UB \\
 \downarrow p & \lrcorner & \downarrow Ub \\
 A & \xrightarrow{a} & UI
 \end{array}
 \qquad
 \begin{array}{ccc}
 \tilde{P} & \xrightarrow{\tilde{q}} & B \\
 \downarrow \tilde{p} & \lrcorner & \downarrow b \\
 FA & \xrightarrow{\hat{a}} & I
 \end{array}$$

Further consider

$$\begin{array}{ccc}
 P & \xrightarrow{\alpha} & U\tilde{P} & \xrightarrow{U\tilde{q}} & UB \\
 \downarrow p & \lrcorner & \downarrow U\tilde{p} & \lrcorner & \downarrow Ub \\
 A & \xrightarrow{\eta_A} & UFA & \xrightarrow{U\hat{a}} & UI
 \end{array}
 \qquad
 \begin{array}{ccc}
 FP & \xrightarrow{\hat{\alpha}} & \tilde{P} & \xrightarrow{\tilde{q}} & B \\
 \downarrow Fp & \lrcorner & \downarrow \tilde{p} & \lrcorner & \downarrow b \\
 FA & \xrightarrow{\hat{a}} & FA & \xrightarrow{\hat{a}} & I
 \end{array}$$

where $U\tilde{q} \circ \alpha = q$ and $\tilde{q} \circ \hat{\alpha} = \tilde{q}$.

Frobenius reciprocity for $F_I \dashv U_I$ (in the second formulation of section 79) requires \hat{a} to be an isomorphism, i.e.

$$\begin{array}{ccc}
 FP & \xrightarrow{\hat{q}} & B \\
 \downarrow Fp & & \downarrow b \\
 FA & \xrightarrow{\hat{a}} & I
 \end{array}$$

to be a pullback, which is equivalent to the rectangle

$$\begin{array}{ccccc}
 FP & \xrightarrow{Fq} & FUB & \xrightarrow{\varepsilon_B} & B \\
 \downarrow Fp & & \downarrow FUb & \searrow F_I U_I b & \downarrow b \\
 FA & \xrightarrow{Fa} & FUI & \xrightarrow{\varepsilon_I} & I
 \end{array}$$

being a pullback as required by Frobenius reciprocity for $F_I \dashv U_I$ (in the first formulation of section 79).

268 Ordinary Frobenius from Stably Frobenius

(6.1.21)

Let $F \dashv U : \mathbb{B} \rightarrow \mathbb{A}$ be an adjunction between categories with finite limits. This adjunction is *stably Frobenius* iff

$$\begin{array}{ccc}
 \begin{array}{ccc} P & \xrightarrow{q} & UB \\ \downarrow p & \lrcorner & \downarrow Ub \\ A & \xrightarrow{a} & UI \end{array} & \text{implies} & \begin{array}{ccc} FP & \xrightarrow{\widehat{q}} & B \\ \downarrow Fp & \lrcorner & \downarrow b \\ FA & \xrightarrow{\widehat{a}} & I \end{array}
 \end{array}$$

We obtain the ordinary Frobenius property instantiating I by 1

$$\begin{array}{ccc}
 \begin{array}{ccc} A \times UB & \xrightarrow{\pi_2} & UB \\ \downarrow \pi_1 & \lrcorner & \downarrow \\ A & \longrightarrow & U1 \end{array} & \text{implies} & \begin{array}{ccc} F(A \times UB) & \xrightarrow{\widehat{\pi_2}} & B \\ \downarrow F\pi_1 & \lrcorner & \downarrow b \\ FA & \longrightarrow & 1 \end{array}
 \end{array}$$

which since $U1$ is terminal amounts to the requirement that

$$F(A \times UB) \xrightarrow{\langle F\pi_1, \widehat{\pi_2} \rangle} FA \times B$$

is an isomorphism (where $\widehat{\pi_2} = \varepsilon_B \circ F(\pi_2)$).

269 Locally Connected Geometric Morphisms

A geometric morphism $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ is called *locally connected* or *molecular* iff one of the following equivalent conditions holds

- (1) the fibered functor $\Delta : P_{\mathcal{S}} \rightarrow P_F$ has a fibered left adjoint Π
- (2) the functor F preserves dependent products
- (3) $F/I : \mathcal{S}/I \rightarrow \mathcal{E}/FI$ preserves exponentials for all I in \mathcal{S}

where Δ is given by applying F to commuting squares in \mathcal{S} . For a proof of the equivalences see C3.3.1 of Johnstone's *Elephant*.

The equivalence of (1) and (2) follows from the appropriate fibered adjoint functor theorem whereas the equivalence of (2) and (3) is a consequence of the fact that for a finite limit category all pullback functors have right adjoints iff all its slices are cartesian closed.

Theorem 2 of Barr and Paré’s paper *Molecular Toposes* shows that for a geometric morphism $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ the inverse image part F preserves exponentials iff F has an enriched left adjoint L , i.e. $L \dashv F$ such that $U((FI)^A) \cong I^{LA}$ naturally in $I \in \mathcal{S}$ and $A \in \mathcal{E}$.

Thus, a geometric morphism $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ is molecular, i.e. its inverse image part has a fibered left adjoint, iff all its slices satisfy the weaker requirement that their inverse image part has an enriched left adjoint.

Garner and Streicher in their 2021 TAC paper have come up with an example of a(n even local) geometric morphism which is not molecular though its inverse image part preserves ordinary exponentials.

270 Locally Connected Sites

By C3.3.10 of Johnstone’s *Elephant* a bounded topos $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ is locally connected iff \mathcal{E} is given by site $(\mathbb{C}, \mathcal{J})$ internal to \mathcal{S} such that every $S \in \mathcal{J}(I)$ is connected and inhabited as a full subcategory of \mathbb{C}/I .

As observed by Moerdijk in the appendix of his 1986 paper *Continuous fibrations and inverse limits of toposes* this condition on a site $(\mathbb{C}, \mathcal{J})$ internal to \mathcal{S} is equivalent to the requirement that $\Delta : \mathcal{S} \rightarrow \mathcal{S}^{\text{C}^{\text{op}}}$ factors through the inclusion $\text{Sh}_{\mathcal{S}}(\mathbb{C}, \mathcal{J}) \hookrightarrow \mathcal{S}^{\text{C}^{\text{op}}}$, i.e. every constant presheaf over \mathbb{C} is a \mathcal{J} -sheaf.

271 Sites for stably precohesive toposes

Local hyperconnected toposes over a base topos \mathcal{S} are induced by sites $(\mathbb{C}, \mathcal{J})$ internal to \mathcal{S} where \mathbb{C} has a terminal object and is hyperconnected in the sense that all hom-sets of \mathbb{C} are inhabited, i.e. \mathbb{C} has a terminal object and all objects of \mathbb{C} have a global element.

Stably precohesive toposes over \mathcal{S} , i.e. locally connected local and hyperconnected toposes over \mathcal{S} , are induced by sites $(\mathbb{C}, \mathcal{J})$ internal to \mathcal{S} where \mathbb{C} has a terminal object, all objects of \mathbb{C} have a global element and all covers $S \in \mathcal{J}(I)$ are connected as subcategories of \mathbb{C}/I as follows from Theorem C3.3.10 of Johnstone’s *Elephant*.

What we lack is a characterization of sites $(\mathbb{C}, \mathcal{J})$ internal to \mathcal{S} where \mathbb{C} has a terminal object, is hyperconnected and, moreover, the inverse image part of the geometric morphism $\text{Sh}_{\mathcal{S}}(\mathbb{C}, \mathcal{J}) \rightarrow \mathcal{S}$ preserves exponentials. If we had such a characterization then we could try to prove from this that all covers in \mathcal{J} are connected.

272 Every topos appears as subtopos of a totally connected one (J. Funk, April 2021)

Let $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ be a geometric morphism. Then we may consider the string of adjunctions $P_F \dashv 1 \dashv G : \mathcal{E} \downarrow F \rightarrow \mathcal{S}$. Since P_F preserves finite limits and has right adjoint right inverse 1 the geometric morphism $1 \dashv G : \mathcal{E} \downarrow F \rightarrow \mathcal{S}$ is totally connected, i.e. locally connected and P_F preserves finite limits. Thus, this geometric morphism is also connected (since P_F preserves terminal objects).

The identification of \mathcal{E} with the fiber of P_F over $1_{\mathcal{S}}$ is the direct image part of an injective geometric morphism whose inverse image part is given by $\partial_0 : \mathcal{E} \downarrow F \rightarrow \mathcal{E}$

$$\begin{array}{ccc}
 \mathcal{E} & \xleftarrow{\partial_0} & \mathcal{E} \downarrow F \\
 & \searrow F & \uparrow 1 \\
 & & \mathcal{S}
 \end{array}$$

which observation appears (in a less conceptual way) in the proof of C3.3.14 of Johnstone's *Elephant*.

One may try to characterize when F preserves exponentials. It certainly is the case if $\partial_0 : \mathcal{E} \downarrow F \rightarrow \mathcal{E}$ does since 1 preserves even dependent products because the geometric morphism $1 \dashv G$ is locally connected. However, we can't see any reason why $\partial_0 : \mathcal{E} \downarrow F \rightarrow \mathcal{E}$ should preserve exponentials whenever F does. But it does whenever $\partial_0 : \mathcal{E} \downarrow F \rightarrow \mathcal{E}$ is the inverse image part of an open inclusion, i.e. $\partial_0 : \mathcal{E} \downarrow F \rightarrow \mathcal{E}$ is isomorphic to $(V \rightarrow F1)^*$ for some subterminal V in \mathcal{E} .

273 Hyperconnected Geometric Morphisms whose inverse image part preserves exponentials

Suppose $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ is a hyperconnected geometric morphism such that F preserves exponentials. Menni has shown that then F has a left adjoint L . We further assume that L preserves finite products which is equivalent to the full subcategory \mathcal{S} of \mathcal{E} as given by F being an exponential ideal.

The counit of $F \dashv U$ at $\Omega_{\mathcal{E}}$ is a subobject of $\Omega_{\mathcal{E}}$ such that

$$\begin{array}{ccc} FU1_{\mathcal{E}} & \longrightarrow & 1_{\mathcal{E}} \\ FU\top_{\mathcal{E}} \downarrow & & \downarrow \top_{\mathcal{E}} \\ FU\Omega_{\mathcal{E}} & \xrightarrow{\varepsilon_{\Omega_{\mathcal{E}}}} & \Omega_{\mathcal{E}} \end{array}$$

is a pullback. By hyperconnectedness $U\top_{\mathcal{E}}$ is canonically isomorphic to $\top_{\mathcal{S}}$ and thus

$$\begin{array}{ccc} F1_{\mathcal{S}} & \longrightarrow & 1_{\mathcal{E}} \\ F\top_{\mathcal{S}} \downarrow & & \downarrow \top_{\mathcal{E}} \\ F\Omega_{\mathcal{S}} & \xrightarrow{\varepsilon_{\Omega_{\mathcal{E}}}} & \Omega_{\mathcal{E}} \end{array}$$

is a pullback for which reason $FU\top_{\mathcal{E}}$ classifies \mathcal{S} -definable subobjects as follows from

$$\begin{array}{ccccc} P & \longrightarrow & FQ & \longrightarrow & F1_{\mathcal{S}} \\ m \downarrow \lrcorner & & Fn \downarrow \lrcorner & & \downarrow F\top_{\mathcal{S}} \\ A & \xrightarrow{\eta_A} & FLA & \xrightarrow{Fq} & F\Omega_{\mathcal{S}} \end{array}$$

where q is the upper transpose of the unique map $A \rightarrow F\Omega_{\mathcal{S}}$ classifying m and n is the subobject classified by q .

Since L preserves finite products we moreover have

$$(F\Omega_{\mathcal{S}})^A \cong (F\Omega_{\mathcal{S}})^{FLA}$$

naturally in $A \in \mathcal{E}$ since we have

$$\frac{\frac{B \rightarrow (F\Omega_{\mathcal{S}})^A}{B \times A \rightarrow F\Omega_{\mathcal{S}}}}{L(B \times A) \cong LB \times LA \rightarrow \Omega_{\mathcal{S}}}}{\frac{LB \rightarrow \Omega_{\mathcal{S}}^{LA}}{B \rightarrow F(\Omega_{\mathcal{S}}^{LA}) \cong (F\Omega_{\mathcal{S}})^{FLA}}}}$$

naturally in $A, B \in \mathcal{E}$.

274 Glueing Change of Base along Distributors gives rise to Presheaf Toposes

A **distributor** from \mathbb{A} to \mathbb{B} is a functor $\phi : \mathbb{B}^{\text{op}} \times \mathbb{A} \rightarrow \mathbf{Set}$ or equivalently a functor $\phi : \mathbb{A} \rightarrow \widehat{\mathbb{B}}$ which in both cases is denoted as $\phi : \mathbb{A} \dashrightarrow \mathbb{B}$. We write $\phi_! : \widehat{\mathbb{A}} \rightarrow \widehat{\mathbb{B}}$ for left Kan extension of ϕ along the Yoneda functor $Y_{\mathbb{A}} : \mathbb{A} \rightarrow \widehat{\mathbb{A}}$ and $\phi^* : \widehat{\mathbb{B}} \rightarrow \widehat{\mathbb{A}}$ for its right adjoint sending $Y \in \widehat{\mathbb{B}}$ to $\phi^*Y = \widehat{\mathbb{B}}(\phi(-), Y) \in \widehat{\mathbb{A}}$. We write **Dist** for the bicategory whose objects are small categories and whose hom-category $\mathbf{Dist}(\mathbb{A}, \mathbb{B}) = \mathbf{Set}^{\mathbb{B}^{\text{op}} \times \mathbb{A}}$. By the considerations above we have

$$\mathbf{CoCont}(\widehat{\mathbb{A}}, \widehat{\mathbb{B}}) \simeq \mathbf{Dist}(\mathbb{A}, \mathbb{B}) \simeq \mathbf{Cont}(\widehat{\mathbb{B}}, \widehat{\mathbb{A}})$$

where $\phi : \mathbb{A} \dashrightarrow \mathbb{B}$ corresponds to the cocontinuous functor $\phi_! : \widehat{\mathbb{A}} \rightarrow \widehat{\mathbb{B}}$ which in turn corresponds to the continuous $\phi^* : \widehat{\mathbb{B}} \rightarrow \widehat{\mathbb{A}}$.

For $\phi : \mathbb{A} \dashrightarrow \mathbb{B}$ its **collage** is the category $\mathbf{C}(\phi)$ which is obtained from the disjoint union of \mathbb{A} and \mathbb{B} by adding $\mathbf{C}(\phi)(B, A) = \phi(B, A)$ where for $f \in \phi(B, A)$, $u : A \rightarrow A'$ and $v : B' \rightarrow B$ the composite $u \circ f \circ v$ is given by $\phi(v, u)(f)$. One easily can see that

$$\widehat{\mathbf{C}(\phi)} \simeq \widehat{\mathbb{A}} \downarrow \phi^*$$

since a presheaf C on $\mathbf{C}(\phi)$ is given by $X \in \widehat{\mathbb{A}}$ (the restriction of C to \mathbb{A}), $Y \in \widehat{\mathbb{B}}$ (the restriction of C to \mathbb{B}) and $\alpha(B, A) : \phi(B, A) \rightarrow Y(B)^{X(A)}$ satisfying the appropriate coherence conditions which amount to $\alpha_A : X(A) \rightarrow \widehat{\mathbb{B}}(\phi(A), Y)$ naturally in $A \in \mathbb{A}$ and thus $\alpha : X \Rightarrow \phi^*Y$.

For $F : \mathbb{A} \rightarrow \mathbb{B}$ we may consider the distributor $\phi_F = Y_{\mathbb{B}} \circ F : \mathbb{A} \rightarrow \widehat{\mathbb{B}}$, i.e. $\phi_F(B, A) = \mathbb{B}(B, FA)$, and obtain

$$\widehat{\mathbf{C}(\phi_F)} \simeq \widehat{\mathbb{A}} \downarrow F^*$$

since ϕ_F^* is equivalent to F^* .

We write $I_{\mathbb{A}}$ and $I_{\mathbb{B}}$ for the obvious inclusions of \mathbb{A} and \mathbb{B} into $\mathbf{C}(\phi_F)$, respectively. The inclusion $I_{\mathbb{B}}$ has a right adjoint $C_{\mathbb{B}}$ whose counit at $I \in \mathbb{A}$ is the morphism $\varepsilon_I : FI \rightarrow I$ in $\mathbf{C}(\phi_F)$ corresponding to $\text{id}_{FI} \in \phi_F(FI, I) = \mathbb{B}(FI, FI)$. Thus, every morphism $f : J \rightarrow I$ in $\mathbf{C}(\phi_F)$ factors as $f = \varepsilon_I \circ v$ for a unique $v : J \rightarrow FI$ in \mathbb{B} . Obviously, we have $F = C_{\mathbb{B}} \circ I_{\mathbb{A}}$ and thus also $F^* = I_{\mathbb{A}}^* C_{\mathbb{B}}^*$.

Using the notation introduced in the previous paragraph the equivalence $\widehat{\mathbf{C}(\phi_F)} \simeq \widehat{\mathbb{A}} \downarrow F^*$ can be explicitated as follows. A presheaf $C \in \widehat{\mathbf{C}(\phi_F)}$ gives rise to the presheaves $A = I_{\mathbb{A}}^* C \in \widehat{\mathbb{A}}$ and $B = I_{\mathbb{B}}^* C \in \widehat{\mathbb{B}}$ and a natural transformation $\alpha : A \Rightarrow F^*B$ with $\alpha_I = C(\varepsilon_I)$ for $I \in \mathbb{A}$. Vice versa given $A \in \widehat{\mathbb{A}}$, $B \in \widehat{\mathbb{B}}$ and $\alpha : A \Rightarrow F^*B$ this induces a unique $C \in \widehat{\mathbf{C}(\phi_F)}$ with $A = I_{\mathbb{A}}^* C$, $B = I_{\mathbb{B}}^* C$ and $\alpha_I = C(\varepsilon_I)$ for $I \in \mathbb{A}$.

275 Toposes over the Sierpiński Topos

Let \mathcal{S} be the Sierpiński topos $\widehat{\mathbb{2}}$. A category internal to the Sierpiński topos \mathcal{S} corresponds to a split fibration C over $\mathbb{2}$, i.e. a functor $C(0 \xrightarrow{\alpha} 1) : C(1) \rightarrow C(0)$. The corresponding externalization \underline{C} of C to a split fibration over \mathcal{S} sends an $X \in \mathcal{S}$ to the category $\mathbf{Sp}(\mathbb{2})(X, C)$ considering X as a split discrete fibration over $\mathbb{2}$.

Cartesian functors from \underline{C} (considered as a fibered category over \mathcal{S}) to $P_{\mathcal{S}}$ correspond to split cartesian functors from \underline{C} to $Sp(P_{\mathcal{S}})$, the right adjoint splitting of the fundamental fibration of \mathcal{S} . The latter is equivalent to the externalization of the **big** category $\mathbf{Set}(\mathbb{2})$ internal to \mathcal{S} where $\mathbf{Set}(\mathbb{2})(1) = \mathcal{S}$, $\mathbf{Set}(\mathbb{2})(0) = \mathbf{Set}$ and $\mathbf{Set}(\mathbb{2})(\alpha)$ is 0^* , i.e. the restriction of presheaves over $\mathbb{2}$ to the fiber over 0.

The category $\mathcal{S}^{C^{op}}$ has as objects internal functors F from C^{op} to $\mathbf{Set}(\mathbb{2})$, i.e.

$$\begin{array}{ccc} C(1)^{op} & \xrightarrow{F(1)} & \mathbf{Set}^{2^{op}} \\ C(\alpha)^{op} \downarrow & & \downarrow \mathbf{Set}^{0^{op}} \\ C(0)^{op} & \xrightarrow{F(0)} & \mathbf{Set}^{1^{op}} \end{array}$$

where $i : \mathbb{1} \rightarrow \mathbb{2}$ picks the object i in $\mathbb{2}$. Such F correspond to natural transformations $\alpha_F = F(1)(-)(\alpha)$ from $F_1 = \mathbf{Set}^{1^{op}} \circ F(1)$ to $F_0 \circ C(\alpha)^{op} = \mathbf{Set}^{0^{op}} \circ F(1)$ where we write F_0 for $F(0)$. A morphism $\tau : F \Rightarrow G$ in $\mathcal{S}^{C^{op}}$ is given by a pair of natural transformations $\tau_1 : F_1 \Rightarrow G_1$ and $\tau_0 : F_0 \Rightarrow G_0$ making the diagram

$$\begin{array}{ccc} F_1 & \xrightarrow{\tau_1} & G_1 \\ \alpha_F \Downarrow & & \Downarrow \alpha_G \\ F_0 C(\alpha)^{op} & \xrightarrow{\tau_0 C(\alpha)^{op}} & G_0 C(\alpha)^{op} \end{array}$$

commute. Thus, the topos $\mathcal{S}^{C^{op}}$ is equivalent to $\widehat{C(1) \downarrow C(\alpha)^*}$, i.e. the glueing of the finite limit preserving change of base functor $C(\alpha)^* : C(0) \rightarrow C(1)$, which by the observation at the end of section 274 in turn is equivalent to $\mathbf{Set}^{C(\phi_{C(\alpha)})^{op}}$.

In the light of this latter observation the geometric morphism $\Delta_C \dashv \Gamma_C : \mathcal{S}^{C^{op}} \rightarrow \mathcal{S}$ can be explicitated as follows. Let $U : \mathbf{C}(\phi_{C(\alpha)}) \rightarrow \mathbb{2}$ be the forgetful functor sending $C(i)$ to the object i and all morphisms from objects in $C(0)$ to objects in $C(1)$ to α . Then Δ_C is given by U^* and Γ_C is given by its right adjoint U_* sending a presheaf F over $\mathbf{C}(\phi_{C(\alpha)})$ to the map $\mathbf{Set}^{C(\phi_{C(\alpha)})^{op}}(1, F) \rightarrow \mathbf{Set}^{C(0)^{op}}(1, F_0)$ induced by change of base along the inclusion of $C(0)$ into $\mathbf{C}(\phi_{C(\alpha)})$.

Thus, bounded geometric morphisms to the Sierpiński topos $\mathcal{S} = \mathbf{Set}^{2^{op}}$ correspond to subtoposes of $\mathbf{Set}^{C(\phi_{C(\alpha)})^{op}}$ for some category C internal to \mathcal{S} .

For this purpose one has to explicitate what are Grothendieck topologies on $\mathbf{C}(\phi_{C(\alpha)})$ and, thus, before what are sieves on objects in $\mathbf{C}(\phi_{C(\alpha)})$.

A sieve on $I_0 \in C(0)$ in $\mathbf{C}(\phi_{C(\alpha)})$ is nothing but a sieve on I_0 in the category $C(0)$ whereas a sieve S on $I_1 \in C(1)$ in $\mathbf{C}(\phi_{C(\alpha)})$ is of the form $S = S_1 \cup_{\varepsilon_{I_1}} S_0$ where S_1 is a sieve on I_1 in $C(1)$ and S_0 is a sieve on $C(\alpha)(I_1)$ in $C(0)$ such that $C(\alpha)(u) \in S_0$ for every $u \in S_1$, i.e. $S_1 \subseteq C(\alpha)^{-1}[S_0]$.¹⁸¹ Since the components of ε are all monic the subobject classifier Ω in $\mathbf{Set}^{\mathbf{C}(\phi_{C(\alpha)})^{\text{op}}}$ can be described as follows. The restriction of Ω to $C(0)$ is just the subobject classifier of $\widehat{C(0)}$. For $I_1 \in C(1)$ the elements of $\Omega(I_1)$ are pairs (S_0, S_1) such that S_0 is a sieve in $C(0)$ on $C(\alpha)(I_1)$ and S_1 is a sieve in $C(1)$ on I_1 such that $S_1 \subseteq C(\alpha)^{-1}[S_0]$. For $u_1 : I'_1 \rightarrow I_1$ in $C(1)$ we have $u_1^*(S_0, S_1) = (C(\alpha)(u_1)^*S_0, u_1^*S_1)$ and $\varepsilon_{I'_1}^*(S_0, S_1) = S_0$. The corresponding object in $\widehat{C(1)} \downarrow C(\alpha)^*$ is $(I_{C(1)}^* \Omega, \pi, I_{C(0)}^* \Omega)$ with $\pi_{I_1}(S_0, S_1) = S_0$.

A sheaf subtopos of $\mathbf{Set}^{\mathbf{C}(\phi_{C(\alpha)})^{\text{op}}}$ is given by a subobject D of Ω which is upwards closed. Such a D is a Grothendieck topology if, moreover, for any S in $D(I)$ a sieve T on I is in $D(I)$ whenever $u^*T \in D(J)$ for all $u : J \rightarrow I$ in S . As usual Grothendieck topologies correspond via their characteristic map to natural transformations $j : \Omega \rightarrow \Omega$ which are idempotent and preserve \top (maximal sieves) and \wedge (intersection of sieves).

Instantiating the General Scheme

Let $C : \mathcal{P}^{\text{op}} \rightarrow \mathbf{Cat}$ be the unique monoid homomorphism $\mathbb{1} \rightarrow \mathbb{M}$ where \mathbb{M} is the monoid of continuous endomaps on \mathbb{N}_∞ , the one point compactification of the discrete space \mathbb{N} . The category $\mathbf{C}(\phi_{C(\alpha)})$ for which we write \mathbb{C} has two objects 0 and 1 with $\mathbb{C}(0, 1) = \mathbb{M} = \mathbb{C}(0, 0)$, $\mathbb{C}(1, 1)$ containing only id_1 and $\mathbb{C}(1, 0) = \emptyset$ and composition is inherited from \mathbb{M} . The presheaf topos $\widehat{\mathbb{C}}$ is equivalent to $\mathbf{Set} \downarrow U$ where $U : \widehat{\mathbb{M}} \rightarrow \mathbf{Set}$ is the forgetful functor isomorphic to $\widehat{\mathbb{M}}(\mathbb{Y}_{\mathbb{M}}(*), -)$ sending an action of \mathbb{M} on a set to this very set.

Let \mathcal{S} be the Sierpiński topos $\widehat{\mathbb{2}}$. The category C internal to \mathcal{S} induces the geometric morphism $\Delta_C \dashv \Gamma_C : \mathcal{S}^{C^{\text{op}}} \rightarrow \mathcal{S}$ which can be explicitated as follows. The functor $\Delta_C : \mathcal{S} \rightarrow \mathbf{Set} \downarrow U$ sends an object $f : X_1 \rightarrow X_0$ in \mathcal{S} to $(\Delta_{\mathbb{M}}(X_0), f)$ and the functor $\Gamma_C : \mathbf{Set} \downarrow U \rightarrow \mathcal{S}$ sends an object $(X, g : I \rightarrow U(X))$ in $\mathbf{Set} \downarrow U$ to the corestriction of g to global elements of X , i.e. the pullback of g along the inclusion $\Gamma(X) \hookrightarrow U(X)$. Moreover, since $\mathcal{S}^{C^{\text{op}}}$ is a presheaf topos over \mathcal{S} the geometric morphism $\Delta_C \dashv \Gamma_C$ is also locally connected, i.e. Δ_C has a fibered left adjoint Π_C sending an object $(X, g : I \rightarrow U(X))$ in $\mathbf{Set} \downarrow U$ to $q_X \circ g$ where $q_X : U(X) \rightarrow U(X)/\sim_X$ is the quotient map for the equivalence relation \sim_X on the underlying set of X induced by the action of \mathbb{M} .

As is well known Johnstone's topological topos \mathcal{T} appears as subtopos of $\widehat{\mathbb{M}}$ via an injective geometric morphism $a \dashv i : \mathcal{T} \hookrightarrow \widehat{\mathbb{M}}$. Moreover, the unique geometric morphism $\Delta_{\mathcal{T}} \dashv \Gamma_{\mathcal{T}} : \mathcal{T} \rightarrow \mathbf{Set}$ is local, i.e. $\Gamma_{\mathcal{T}}$ has a full and faithful right adjoint $\nabla_{\mathcal{T}}$.

¹⁸¹here $\varepsilon_{I_1} : C(\alpha)(I_1) \rightarrow I_1$ is the morphism in $\mathbf{C}(\phi_{C(\alpha)})$ corresponding to $\text{id}_{C(\alpha)(I_1)} \in \phi_{C(\alpha)}(C(\alpha)(I_1), I_1) = C(0)(C(\alpha)(I_1), C(\alpha)(I_1))$ as discussed in section 274

276 A more abstract characterization of Grothendieck toposes over the Sierpiński topos

Grothendieck toposes over the Sierpiński topos $\mathcal{S} = \mathbf{Set}^{2^{\text{op}}}$ correspond to bounded geometric morphism $\mathcal{E} \rightarrow \mathcal{S}$. As is well known inverse image parts of bounded geometric morphisms from \mathcal{E} to \mathcal{S} correspond via left Kan extension along the Yoneda functor Y_2 to finite limit preserving functors from $\mathbf{2}$ to \mathcal{E} which in turn correspond to subterminals in \mathcal{E} .

Let U be a subterminal object in \mathcal{E} then the direct image part of the corresponding bounded geometric morphism is given by $\mathcal{E}(!_U, -) : \mathcal{E}(1, -) \rightarrow \mathcal{E}(U, -)$. The corresponding inverse image part sends an object $u : J \rightarrow I$ in \mathcal{S} to the object $\prod_{i \in I} S_{u^{-1}(i)}$ where for a set K the object S_K in \mathcal{E} is the K -fold sum of $1_{\mathcal{E}}$ amalgamated over $U \mapsto 1_{\mathcal{E}}$. One may construct S_K as the quotient of $\Delta(K) = \prod_{k \in K} 1_{\mathcal{E}}$ by the equivalence relation $E : \Delta(K) \times \Delta(K) \rightarrow \Omega_{\mathcal{E}}$ where¹⁸² $E \circ \langle \iota_{k_1}, \iota_{k_2} \rangle$ is $\top_{\mathcal{E}}$ if $k_1 = k_2$ and the characteristic predicate $\chi_U : 1 \rightarrow \Omega_{\mathcal{E}}$ of $U \mapsto 1_{\mathcal{E}}$ otherwise.

277 Hyperconnected Toposes over the Sierpiński Topos

Let \mathcal{S} be the Sierpiński topos $\widehat{\mathbf{2}}$. Then hyperconnected geometric morphisms $\mathcal{E} \rightarrow \mathcal{S}$ exist iff \mathcal{E} is 3-valued in which case they are unique up to isomorphism. Such \mathcal{E} arise as subtoposes of toposes of the form $\mathcal{S}^{C^{\text{op}}}$ where C is a category internal to \mathcal{S} such that both $C(0)$ and $C(1)$ are categories with a terminal object where every object has a point and $C(\alpha) : C(1) \rightarrow C(0)$ preserves terminal objects.

Does there exist a local hyperconnected geometric morphism whose inverse image part preserves exponentials but is not locally connected?

One might try to take for \mathcal{E} the topos of presheaves over the category \mathbb{C} of finite ordinals and monotone maps. The inclusion $i : \mathbf{2} \hookrightarrow \mathbb{C}$ has a left adjoint r sending 0 to 0 and all other finite ordinals to 1. By change of base this lifts to a geometric morphism $F = r^* \dashv i^* = U : \mathcal{E} \rightarrow \mathcal{S}$ which is essential and local since F has a left adjoint $L = r_!$ and U has a right adjoint $R = i_*$. Moreover, this geometric morphism is hyperconnected since i^* preserves subobject classifiers as one easily checks.

One might hope that F preserves exponentials but not dependent products but, actually, the functor F is equivalent to $\mathbf{Fam}(\Delta : \mathbf{Set} \rightarrow \mathbf{sSet})$ which is the inverse image part of a locally connected, hyperconnected and local geometric morphism obtained by applying \mathbf{Fam} to the locally connected, hyperconnected and local geometric morphism $\Delta \dashv \Gamma : \mathbf{sSet} \rightarrow \mathbf{Set}$.

¹⁸²here we implicitly employ the canonical isomorphism $\Delta(K \times K) \cong \Delta(K) \times \Delta(K)$

278 Hyperconnected and Local Geometric Morphisms to the Sierpiński Topos

Let $G : \mathbb{C}_1 \rightarrow \mathbb{C}_0$ be a functor between small categories then the change-of-base functor $G^* : \widehat{\mathbb{C}}_0 \rightarrow \widehat{\mathbb{C}}_1$ has a left adjoint $G_!$ and a right adjoint G_* .

We write \mathcal{E} for the topos $\widehat{\mathbb{C}}_1 \downarrow G^*$. Let \mathcal{S} be the Sierpiński topos $\widehat{\mathbb{2}}$ where $\mathbb{2}$ is the category with objects 0 and 1 and a single non-identity morphism $\alpha : 0 \rightarrow 1$.

Let $F : \mathcal{S} \rightarrow \mathcal{E}$ be the functor sending $A \in \mathcal{S}$ to the object $(\Delta_{\mathbb{C}_1}(A_1), \tau_A, \Delta_{\mathbb{C}_0}(A_0))$ in \mathcal{E} where the components of τ_A are given by $A(\alpha) : A(1) \rightarrow A(0)$.

The functor F has a right adjoint U sending an object $X = (X_1, \tau_X, X_0)$ in \mathcal{E} to $U(X) = (U(X)_1, \alpha_{U(X)}, U(X)_0)$ in \mathcal{S} where $U(X)_0 = \Gamma_{\mathbb{C}_0}(X_0)$, $U(X)_1 = \{(s_0, s_1) \in \Gamma_{\mathbb{C}_0}(X_0) \times \Gamma_{\mathbb{C}_1}(X_1) \mid (\tau_X)_I(s_1(I)) = s_0(G(I)) \text{ for all } I \in \mathbb{C}_1\}$ and $\alpha_{U(X)} : (s_0, s_1) \mapsto s_0$.

We assume that both \mathbb{C}_0 and \mathbb{C}_1 have terminal objects which are preserved by G , i.e. the category internal to \mathcal{S} as given by $G : \mathbb{C}_1 \rightarrow \mathbb{C}_0$ has a terminal object. Then we have a simpler construction of a right adjoint to F namely by sending an object $X = (X_1, \tau : X_1 \rightarrow G^*X_0, X_0) \in \mathcal{E}$ to $U(X) = \tau_1 : X_1(1) \rightarrow X_0(G(1)) = X_0(1)$.

The functor U has a right adjoint R sending $A = (A_1, \alpha_A, A_0) \in \mathcal{S}$ to the object $R(A)$ in \mathcal{E} where $R(A)_0(I_0) = A_0^{\mathbb{C}_0(1, I_0)}$, $R(A)_1(I_1) = \{(a_0, a_1) \in A_0^{\mathbb{C}_0(1, G(I_1))} \times A_1^{\mathbb{C}_1(1, I_1)} \mid \alpha_A \circ a_1 = a_0 \circ G_{1, I_1}\}$ and $\tau_{R(A)} : R(A)_1 \Rightarrow G^*R(A)_0$ is given by first projection in each component. Functoriality and naturality conditions for these data follow from the fact that

$$\begin{array}{ccccc}
 \mathbb{C}_1(1, J_1) & \xrightarrow{\mathbb{C}_1(1, u_1)} & \mathbb{C}_1(1, I_1) & \xrightarrow{a_1} & A_1 \\
 \downarrow G_{1, J_1} & & \downarrow G_{1, I_1} & & \downarrow \alpha_A \\
 \mathbb{C}_1(1, G(J_1)) & \xrightarrow{\mathbb{C}_0(1, G(u_1))} & \mathbb{C}_0(1, G(I_1)) & \xrightarrow{a_0} & A_0
 \end{array}$$

commutes for all $u_1 : J_1 \rightarrow I_1$.

We further assume that in \mathbb{C}_0 and \mathbb{C}_1 all objects have global elements, i.e. the category internal to \mathcal{S} as given by $G : \mathbb{C}_1 \rightarrow \mathbb{C}_0$ is hyperconnected, i.e. all its hom-sets are inhabited.

279 Local Toposes over the Sierpiński Topos

Let $P : \mathbb{C} \rightarrow \mathbb{2}$ be a fibration over the ordinal $\mathbb{2}$ considered as a category $\mathbb{2}$ with two objects 0 and 1 and a single non-identity morphism $\alpha : 0 \rightarrow 1$. Using a bit of choice this is tantamount to a functor $G = \alpha^* : \mathbb{C}_1 \rightarrow \mathbb{C}_0$. We further assume that in \mathbb{C}_0 and \mathbb{C}_1 all objects have a global element, i.e. the category internal to \mathcal{S} as given by $G : \mathbb{C}_1 \rightarrow \mathbb{C}_0$ is hyperconnected, i.e. all its hom-sets are inhabited.

We further assume that P is a fibration of categories with terminal objects, i.e. P has a right adjoint right inverse 1 picking an object 1_i in \mathbb{C}_i and a cartesian morphism $1_\alpha : 1_1 \rightarrow 1_0$ over α .

This gives rise to a geometric morphism $F \dashv U : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{2}}$ where F is given by P^* and U by 1^* . This geometric morphism is local since P^* is full and faithful and 1^* has a right adjoint R as given by 1_* . Moreover, the geometric morphism is locally connected since it exhibits $\widehat{\mathbb{C}}$ as a presheaf topos over base topos $\widehat{\mathbb{2}}$ and presheaf toposes are necessarily locally connected over arbitrary base toposes. Moreover, the geometric morphism $F \dashv U : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{2}}$ is hyperconnected iff in the fibers of P all hom-sets are inhabited iff in the fibers of P all objects have global elements.

Maybe for obtaining a counterexample separating the conceptually correct fibered notion and Lawvere's weak notion of precohesive geometric morphism it suffices to consider the case where the base topos \mathcal{S} is the Sierpiński topos $\widehat{\mathbb{2}}$. Let $P : \mathbb{C} \rightarrow \mathbb{2}$ be a fibration of categories with a terminal object where \mathbb{C}_1 is the terminal category. The induced topos \mathcal{E} over \mathcal{S} is given by the geometric morphism $P^* \dashv 1^* : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{2}}$.¹⁸³

For explicating the further adjoints we employ the fact that $\widehat{\mathbb{C}}$ is equivalent to $\mathbf{Set} \downarrow \Gamma$ where $\Gamma = \widehat{\mathbb{C}}_0(1, -) : \widehat{\mathbb{C}}_0 \rightarrow \mathbf{Set}$. Then the left adjoint L to F sends an object $X(\alpha) : X(1) \rightarrow \Gamma(X(0))$ in $\mathbf{Set} \downarrow \Gamma$ to the map $L(X) = \theta_{X(0)} \circ X(\alpha) : X(1_1) \rightarrow \Pi(X(0))$ where $\theta_{X(0)} : \Gamma(X(0)) \rightarrow \Pi(X(0))$ sends points to the unique connected component in which they are contained. The right adjoint R to U sends an object S in $\widehat{\mathbb{2}}$ to the map $R(S) : \widehat{\mathbb{2}}(1, S) \rightarrow \Gamma \nabla(S(0)) = S(0) : (s_0, s_1) \mapsto s_0$ where $\nabla(S) \in \widehat{\mathbb{C}}_0$ is given by $\nabla(S)(I) = S^{\mathbb{C}_0(1, I)}$.

¹⁸³Notice that \mathcal{E} is equivalent to the scoping of $\widehat{\mathbb{C}}_0$ since $\mathbf{Set} \downarrow \Gamma$ with $\Gamma = \widehat{\mathbb{C}}_0(1, -) : \widehat{\mathbb{C}}_0 \rightarrow \mathbf{Set}$ is equivalent to $\widehat{\mathbb{C}}$.

280 Toposes over the Object Classifier $\mathbf{Set}^{\mathbf{FinSet}}$ and the Boolean Algebra Classifier $\mathbf{Set}^{\mathbf{FinSet}^{\text{op}}}$

J. Hemelaer has shown in November 2021 that geometric morphisms to the Sierpiński topos are molecular whenever their inverse image part preserves ordinary exponentials. Thus, for exhibiting an essential local hyperconnected geometric morphism $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ with F preserving ordinary but not dependent function spaces one has to consider base toposes \mathcal{S} different from the Sierpiński topos.

Let \mathcal{S} be the Grothendieck topos $\mathbf{Set}^{\mathbf{FinSet}}$. Inverse image parts of geometric morphisms from Grothendieck toposes \mathcal{E} to \mathcal{S} correspond (via left Kan extension along $Y_{\mathbf{FinSet}^{\text{op}}} : \mathbf{FinSet}^{\text{op}} \rightarrow \mathbf{Set}^{\mathbf{FinSet}}$) to finite limit preserving functors $G : \mathbf{FinSet}^{\text{op}} \rightarrow \mathcal{E}$ which in turn are determined up to isomorphism by $A = G(1)$ as follows. For every object A of \mathcal{E} there is a unique up to isomorphism finite limit preserving functor $G_A : \mathbf{FinSet}^{\text{op}} \rightarrow \mathcal{E}$ with $A \cong G_A(1)$, namely $G_A = A^{\Delta_{\mathcal{E}}}$ where $\Delta_{\mathcal{E}} : \mathbf{FinSet} \rightarrow \mathcal{E}$ preserves finite sums and terminal objects. The left Kan extension F_A of G_A along $Y_{\mathbf{FinSet}^{\text{op}}}$ sends $X \in \mathbf{Set}^{\mathbf{FinSet}}$ to the colimit of $\text{Elts}(X) \xrightarrow{f^X} \mathbf{FinSet} \xrightarrow{G_A} \mathcal{E}$. The right adjoint to F_A is given by $U_A : \mathcal{E} \rightarrow \mathcal{S}$ sending X to $\mathcal{E}(G_A, X) : \mathbf{FinSet} \rightarrow \mathbf{Set}$.

Let \mathcal{S} be the Grothendieck topos $\mathbf{Set}^{\mathbf{FinSet}^{\text{op}}}$. Inverse image parts of geometric morphisms from Grothendieck toposes \mathcal{E} to \mathcal{S} correspond (via left Kan extension along $Y_{\mathbf{FinSet}} : \mathbf{FinSet} \rightarrow \mathbf{Set}^{\mathbf{FinSet}^{\text{op}}}$) to finite limit preserving functors $A : \mathbf{FinSet} \rightarrow \mathcal{E}$, i.e. boolean algebras A in \mathcal{E} . The left Kan extension of A along $Y_{\mathbf{FinSet}}$ sends $X \in \mathcal{S}$ to the colimit of $\text{Elts}(X) \xrightarrow{f^X} \mathbf{FinSet} \xrightarrow{A} \mathcal{E}$. Its right adjoint sends X in \mathcal{E} to $\mathcal{E}(A, X) : \mathbf{FinSet}^{\text{op}} \rightarrow \mathbf{Set}$.

281 Grothendieck topology on \mathbb{C} with terminal object corresponding to $\Gamma \dashv \nabla : \mathbf{Set} \leftrightarrow \widehat{\mathbb{C}}$

is given by the subobject J of $\Omega_{\widehat{\mathbb{C}}}$ as given by

$$\begin{array}{ccc}
 J & \longrightarrow & \nabla \Gamma 1_{\widehat{\mathbb{C}}} \\
 \downarrow \lrcorner & & \downarrow \nabla \Gamma \top_{\widehat{\mathbb{C}}} \\
 \Omega_{\widehat{\mathbb{C}}} & \xrightarrow{\varepsilon_{\Omega_{\widehat{\mathbb{C}}}}} & \nabla \Gamma \Omega_{\widehat{\mathbb{C}}}
 \end{array}$$

consisting of all sieves $S \subseteq Y_{\mathbb{C}}(I)$ with $\text{hom}_{\mathbb{C}}(1, I) \subseteq S$ (since for presheaves X over \mathbb{C} the counit $\varepsilon_X : X \rightarrow \nabla \Gamma X$ sends $x \in X(I)$ to $\lambda i \in \mathbb{C}(1, I).X(i)(x)$).

282 Linear Categories à la Lawvere

A category \mathbb{C} is **linear** iff it has finite sums and products such that

- (1) the unique morphism $0 \rightarrow 1$ is an isomorphism
- (2) the morphism $[\langle \text{id}_A, 0_{A,B} \rangle, \langle 0_{B,A}, \text{id}_B \rangle] : A+B \rightarrow A \times B$ is an isomorphism for all objects A and B in \mathbb{C}

where we write $0_{A,B}$ for $A \rightarrow 1 \rightarrow 0 \rightarrow B$.

Typical examples of linear categories are the category of abelian groups and, more generally, categories of modules over a commutative ring R .

283 Levels in Toposes and “Aufhebung”

A *level* in a category \mathbb{A} is a functor $P : \mathbb{A} \rightarrow \mathbb{B}$ having full and faithful left and right adjoints L and R , respectively, with $PL = \text{Id}_{\mathbb{B}} = PR$. It is an *Aufhebung* of a level $\ell \dashv p \dashv r : \mathbb{C} \hookrightarrow \mathbb{A}$ iff both ℓ and r factor through R .

284 Characterizing Local Geometric Morphisms which are Hyperconnected

Let $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ be a local geometric morphism between toposes, i.e. $F \dashv U$ is a geometric morphism such that F is full and faithful and U has a right adjoint R which necessarily is also full and faithful. We write η and ε for unit and counit of $F \dashv U$, respectively, and γ and δ for unit and counit of $U \dashv R$, respectively.

Since F is full and faithful the unit $\eta_I : I \rightarrow UFI$ is an isomorphism and thus has an inverse $\eta_I^{-1} : UFI \rightarrow I$. Its transpose w.r.t. $U \dashv R$ is given by $\phi_I = R\eta_I^{-1} \circ \gamma_{FI} : FI \rightarrow RI$.

Since R is full and faithful the counit $\delta_I : URI \rightarrow I$ is an isomorphism and thus has an inverse $\delta_I^{-1} : I \rightarrow URI$. Its transpose w.r.t. $F \dashv U$ is given by $\psi_I = \varepsilon_{RI} \circ F\delta_I^{-1} : FI \rightarrow RI$.

The transpose of ψ_I w.r.t. $U \dashv R$ is $\delta_I \circ U\psi_I = \delta_I \circ U\varepsilon_{RI} \circ UF\delta_I^{-1}$. Since $\delta_I \circ U\varepsilon_{RI} \circ UF\delta_I^{-1} \circ \eta_I = \delta_I \circ U\varepsilon_{RI} \circ \eta_{URI} \circ \delta_I^{-1} = \delta_I \circ \delta_I^{-1} = \text{id}_I = \eta_I^{-1} \circ \eta_I$ and η_I is an isomorphism we conclude that the transpose of ψ_I w.r.t. $U \dashv R$ is η_I^{-1} and thus $\phi_I = \psi_I$.

If $F \dashv U$ is hyperconnected, i.e. all ε_I are monic, then all ψ_I and thus all ϕ_I are monic.

Suppose that all ϕ_I are monic. Then all ψ_I are monic and thus all ε_{RUX} are monic. Thus, since $U\gamma_X$ is split monic, the map $\varepsilon_{RUX} \circ FU\gamma_X = \gamma_X \circ \varepsilon_X$ is monic as well from which it follows that all ε_X are monic, i.e. that the geometric morphism $F \dashv U$ is hyperconnected.

Thus, we have shown that a local geometric morphism $F \dashv U$ is hyperconnected if and only if all $\phi_I : FI \rightarrow RI$ are monic.

285 Which Limits does a Fibration Have? (21/2/22)

This question brought up by M. Anel and J. Weinberger though sounding most reasonable at first sight turns out as partly ill-posed as we will try to explain. However, reasonable answers for certain particular reformulations are definitely possible.

For a fibration $P : \mathbb{X} \rightarrow \mathbb{B}$ one may ask the question which limits it has when thinking of it as a “large” category over \mathbb{B} as we have learnt from Jean Bénabou¹⁸⁴. Already more than 40 years ago he has given a very clear answer to the question what it means that P has all small limits namely that P has internal products and finite limits. For saying precisely what it means to have internal products one has to require the base to have finite limits since otherwise one cannot formulate the “Chevalley” condition expressing how products behave under change of base.¹⁸⁵ Also for expressing that P has finite limits it is most beneficial to assume that the base category \mathbb{B} has finite limits since under this moderate assumption it amounts to the requirement that \mathbb{X} has and P preserves finite limits.

However, it is a more delicate question whether a fibration $P : \mathbb{X} \rightarrow \mathbb{B}$ has limits of a particular shape. We first discuss why such a notion of “shape” can be understood in two different ways when working over base toposes different from **Set**.

Firstly, let \mathbb{D} be a small category. Then one may consider the fibration $P^{(\mathbb{D})}$ obtained as pullback of $P^{\mathbb{D}} : \mathbb{X}^{\mathbb{D}} \rightarrow \mathbb{B}^{\mathbb{D}}$ along the functor $\mathbb{B} \rightarrow \mathbb{B}^{\mathbb{D}}$ sending an object I of \mathbb{B} to the constant functor with value I . There is an obvious fibered functor $\Delta_{\mathbb{D}} : P \rightarrow P^{(\mathbb{D})}$ sending an object X over I to the constant functor from \mathbb{D} to \mathbb{X}_I . The fibration P is said to have \mathbb{D} -limits iff $\Delta_{\mathbb{D}}$ has a fibered right adjoint $\text{Lim}_{\mathbb{D}}$. Requiring this for all small external \mathbb{D} is certainly possible in which case one might say that P **has all small external limits**.

Secondly, let C be a category internal to the base \mathbb{C} . As explained by Bénabou this may be considered as a split fibration \underline{C} over \mathbb{B} . There is an obvious fibered functor $\Delta_C : P \rightarrow P^{\underline{C}}$ sending objects to “constant functors from C to that object”.¹⁸⁶ The fibration P “has C -limits” iff Δ_C has a fibered right adjoint Lim_C . As shown in the respective chapter of F. Borceux’s “Handbook of Categorical Algebra” a fibration P has C -limits iff P has all internal products and all finite limits. This is the text book version of results by Bénabou contained in his 1980 Louvain-la-Neuve lectures to which Borceux refers but I think these results can be already found in Benabou’s 1974 Montreal lectures.

This **having all small (internal) limits** can be expressed in a most convenient way when the base category has finite limits. In any case this notion appears as much more appropriate and relevant than the more naive one of

¹⁸⁴who passed away a few days ago on Friday 11.2.2022 a few months before his 90th birthday

¹⁸⁵Clearly P has internal products iff the dual fibration P^{op} over \mathbb{B} (sic!) has internal sums. As convincingly explained by Bénabou in the early 1970s a fibration P over \mathbb{B} has internal sums iff P is a cofibration where cocartesian arrows are stable under pullbacks along cartesian arrows.

¹⁸⁶For an explanation of undefined notions see e.g. my notes on Fibered Categories.

“having all small external limits” as discussed previously.

But notice that up to now we just have discussed the question whether a fibration has limits for **all** diagrams of a certain shape (be it external or internal). This is certainly sufficient for most purposes but in contrast to ordinary non-fibered category theory where one rather (at least at first) discusses the question whether a particular diagram has a limit. When working over arbitrary base categories \mathbb{B} this appears a bit problematic since they are typically not well-pointed even when they are toposes.

For sake of simplicity let us discuss the case of binary products. If X_0 and X_1 are objects in \mathbb{X} over 1 in \mathbb{B} then I would say their product exists w.r.t. P iff it exists in \mathbb{X} and can be chosen to lie in the fiber over 1. But what if $X_i \in P(I_i)$ for $i = 0, 1$? Well, it should be a product cone in \mathbb{X} which by P is sent to a product cone in \mathbb{B} . Obviously, the answer to the general question is in accordance with the answer to the particular question.

But, what if the base \mathbb{B} does not have binary products? This was the case considered by Bénabou first under the additional (necessary) assumption that X_0 and X_1 both live in the fiber over I . Then their product cone is given by vertical maps $\pi_0 : Y \rightarrow X_0$ and $\pi_1 : Y \rightarrow X_1$ such that for $f_0 : Z \rightarrow X_0$ and $f_1 : Z \rightarrow X_1$ in \mathbb{X} with $P(f_0) = P(f_1)$ there exists a unique $g : Z \rightarrow Y$ in \mathbb{X} with $\pi_i g = f_i$ for $i = 0, 1$. As shown by Bénabou (and reported in Borceux’s book and my notes) this latter requirement is equivalent to the former one under the assumption that \mathbb{B} has finite products.

It might be an informative(?) exercise trying to formulate what is the limit in P of $D \in P^{\mathcal{C}}(I)$ for some category C internal to \mathbb{B} . **But is this really necessary and for which purposes?**

286 Are Fibrations Indispensible? (19/2/22)

When doing category theory over general bases in the fibrational style we have learnt from Bénabou one has to be sort of inventive, i.e. to guess the general pattern from analyzing the particular case when the base is **Set**. Experience tells us that this is possible in most cases.

But what if we want to do it systematically? I would start with a base topos \mathcal{S} and postulate one or more universes in it to have a notion or some notions of smallness as is common in ordinary category theory when done in a careful way following the tradition of SGA4 which, however, with a few exceptions (Joyal, Makkai and maybe a few others) has been abandoned by most people for disputable ideological reasons. Notice that one may split a universe $\pi : E \rightarrow U$ in a way first suggested by Voevodsky: employing axiom of choice for classes on the meta-level one chooses a pullback cone for any map $I \rightarrow U$.

Armed with these assumptions we may naively try to formulate category theory in the internal language of \mathcal{S} and look how far we can get. Actually, basic things should be expressible in a straightforward way. But problems would crop up when considering the notion of *equivalence* of categories. There is a weak one as given by full and faithful functors which are essentially surjective. And there is a strong one as given by functors F having a “quasi-inverse”, i.e. a functor G in the opposite direction, such that both composites are isomorphic to the respective identity functors. But for showing that the weak and the strong notion of equivalence coincide one needs an axiom of choice for the universe(s) which typically is not available in sufficiently general base toposes \mathcal{S} .

What are ways out of this problem? Well, one may freely invert weak equivalences between split fibrations over \mathcal{S} but as shown by Bénabou this leads from **Sp**(\mathcal{S}) to **Fib**(\mathcal{S}) (see p.13 of my notes on Fibered Categories). From this point of view an **external approach via Grothendieck fibrations seems to be unavoidable!**

286.0.1 Does HoTT provide a way out of this dilemma?

A much more recent attempt of addressing this problem has been suggested within HoTT (Homotopy Type Theory). It essentially amounts to live with weak equivalences only and change the logic in such a way that they **appear** as strong. The point is that when considering a weak equivalence as a relation between categories the reverse relation gives rise to something which certainly is **not** a functor but appears as a “functional relation” from the point of view how logic is interpreted in HoTT simply because weak equivalences reflect isomorphisms.

Whether such a trick is sufficient for doing category theory over arbitrary bases has to be seen! In any case, more objectively and freed from quasi-religious beliefs, it amounts to the question whether one can do “relative category” using split fibrations only avoiding to give an ontological status to quasi-inverses of split cartesian functors.

287 Does Precohesive Imply Stably Precohesive?

(24/2/22)

Let $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ be a hyperconnected and local geometric morphism, i.e. U preserves subobject classifiers and has a full and faithful right adjoint R . Such a geometric morphism is called **precohesive** iff F has a left adjoint L preserving binary products. As already shown in Johnstone's 2011 TAC paper it is equivalent to require that F has a left adjoint and F preserves (ordinary) exponentials. As shown by M. Menni in 2017 it suffices to just require that F preserves ordinary exponentials since this guarantees the existence of a left adjoint L to F provided the geometric morphism is assumed as hyperconnected and local.

But, actually, as already shown in Barr and Paré's 1980 paper introducing molecular, i.e. locally connected, geometric morphisms the inverse image part of a geometric morphism to \mathcal{S} preserves ordinary exponentials if and only if it has a left adjoint enriched over \mathcal{S} .

A geometric morphism $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ is called **stably precohesive** iff all its slices are precohesive, i.e. $F/I : \mathcal{S}/I \rightarrow \mathcal{E}/FI$ is the inverse image part of a precohesive geometric morphism for all I in \mathcal{S} . Since the properties hyperconnected and local are stable under slicing a geometric morphism is stably precohesive iff it is locally connected, hyperconnected and local.

In Johnstone's 2011 TAC paper it has been shown that a local geometric morphism $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ is hyperconnected iff U is faithful w.r.t. morphisms whose codomain is in the image of F , i.e. for $f, g : A \rightarrow FI$ from $Uf = Ug$ it follows that $f = g$. But I don't see how this fact could help in settling Lawvere and Menni's question (from their 2015 TAC paper) whether all precohesive geometric morphisms are already stably precohesive.

Maybe it helps to look at things in terms of internal sites, i.e. to further assume that $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ is bounded, i.e. \mathcal{E} is equivalent to $\text{Sh}_{\mathcal{S}}(\mathbb{C}, \mathcal{J})$ where \mathcal{J} is a Grothendieck topology on a category \mathbb{C} internal to \mathcal{S} . If \mathcal{E} is hyperconnected and local over \mathcal{S} via $F \dashv U$ one may choose \mathbb{C} as having a terminal object and all objects of \mathbb{C} having a global element internally to \mathcal{S} . One knows from Theorem C.3.3.10 of the *Elephant* that \mathcal{E} is locally connected over \mathcal{S} via $F \dashv U$ iff the internal site $(\mathbb{C}, \mathcal{J})$ can be chosen as locally connected, i.e. every $S \in \mathcal{J}(I)$ is connected as a full (internal) subcategory of \mathbb{C}/I .

But how can one reformulate the requirement that F preserves ordinary exponentials in terms of requirements on the (internal) site $(\mathbb{C}, \mathcal{J})$? Answering Lawvere and Menni's question positively (under the assumption of boundedness of $F \dashv U$) would amount to showing that these requirements are equivalent to all $S \in \mathcal{J}(I)$ being connected as full (internal) subcategories of \mathbb{C}/I . But I do not see how one could show this nor do I have a counterexample!

288 Precohesive Toposes over a Base Topos \mathcal{S}

(2/1/23)

are given by geometric morphisms $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ such that

- (1) U has a right adjoint R and
- (2) for every I in \mathcal{S} the functor $F/I : \mathcal{S}/I \rightarrow \mathcal{E}/FI$
 - (a) preserves (ordinary) exponentials and
 - (b) restricts to a 1-1-correspondence between $\text{Sub}_{\mathcal{S}}(I)$ and $\text{Sub}_{\mathcal{E}}(FI)$.

Notice that (2a) is equivalent to the requirement that $F \dashv U$ is molecular, i.e. F has a left adjoint L fibered over \mathcal{S} . Moreover, condition (2b) amounts to the requirement that $F \dashv U$ is hyperconnected, i.e. U preserves subobject classifiers. But inverse image parts of hyperconnected geometric morphisms are necessarily full and faithful for which reason the geometric morphism $F \dashv U$ is also local.

From Prop. 7 of Johnstone's paper *Remarks on Punctual Local Connecteness* (TAC 2011) it follows that if the inverse image part of an essential hyperconnected local geometric morphism preserves exponentials then its left adjoint preserves binary products, i.e. the inverse image part is an exponential ideal.

Moreover, as already shown in Barr and Paré's 1980 JPAA paper *Molecular Toposes* the inverse image part of a geometric morphism has a left adjoint enriched over \mathcal{S} iff it preserves ordinary exponentials.

Thus, for precohesive geometric morphisms $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ for all I in \mathcal{S} the left adjoint L_I of F/I necessarily preserves binary products, i.e. $F/I : \mathcal{S}/I \hookrightarrow \mathcal{E}/FI$ is an exponential ideal.

289 Characterization of (Stably) Precohesive Geometric Morphisms as Particular Molecular Geometric Morphisms (10/2/23)

From section 284 we know that a local geometric morphism $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ is hyperconnected iff for all $I \in \mathcal{S}$ the morphism $\phi_I = FI \rightarrow RI$ (obtained as transpose of $\eta_I^{-1} : UFI \rightarrow I$ w.r.t. $U \dashv R$) is monic.

A geometric morphism $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ is stably precohesive iff it is local and hyperconnected and the functor F preserves dependent function types, i.e. $F \dashv U$ is molecular.

Thus, alternatively, we may characterize (stably) precohesive geometric morphisms as local geometric morphisms $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ such that F preserves (dependent) function types and U preserves subobject classifiers.

290 An observation on connected molecular geometric morphisms

Let $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ be a connected molecular geometric morphism, i.e. F is full and faithful and $F/I : \mathcal{S}/I \rightarrow \mathcal{E}/FI$ preserves exponentials for all $I \in \mathcal{S}$. Then F has a left adjoint L and for every $I \in \mathcal{S}$ the functor F/I has a left adjoint $L_I = \Sigma_{\varepsilon_I} \circ L/FI$ where ε is the counit of $L \dashv F$.

Let us further assume that all F/I are exponential ideals, i.e. all L_I preserve binary products. Since F is full and faithful all ε_I are isos and thus all L/FI preserve binary products, i.e. L sends pullbacks of cospans in \mathcal{E} whose common codomain is in the image of F to pullbacks in \mathcal{S} .

For this reason L sends subobjects of FI to subobjects of LFI which can be seen as follows. Suppose $m : P \rightarrow FI$ is monic then

$$\begin{array}{ccc} P & \xlongequal{\quad} & P \\ \parallel & & \downarrow m \\ P & \xrightarrow{\quad m \quad} & FI \end{array}$$

is a pullback and thus

$$\begin{array}{ccc} LP & \xlongequal{\quad} & LP \\ \parallel & & \downarrow Lm \\ LP & \xrightarrow{\quad Lm \quad} & LFI \end{array}$$

is a pullback, too, from which it follows that Lm is monic.

291 Pulling back $\Delta \dashv \Gamma$ along a finite limit preserving functor F

Let $F : \mathbb{A} \rightarrow \mathbb{B}$ and $G : \mathbb{B} \rightarrow \mathbb{C}$ finite limit preserving functors between categories with finite limits. Then pulling back $\Delta : P_{\mathbb{B}} \rightarrow P_G$ along F one obtains $F^*\Delta : P_F = F^*P_{\mathbb{B}} \rightarrow F^*P_G \cong P_{GF}$ sending $a : A \rightarrow FI$ to $Ga : GA \rightarrow GFI$.

Suppose G has a right adjoint V . Then Δ has a fibered right adjoint Γ over \mathbb{B} . Pulling back $\Delta \dashv \Gamma$ along F we obtain $F^*\Delta \dashv F^*\Gamma : P_{GF} \cong F^*P_G \rightarrow F^*P_{\mathbb{B}} = P_F$. The fibered left adjoint sends $a : A \rightarrow FI$ to $Ga : GA \rightarrow GFI$ and its fibered right adjoint sends $c : C \rightarrow GFJ$ to $\eta_{FJ}^* Vc$ (where η is the unit of

$G \dashv V$). The following diagram

$$\begin{array}{ccc}
 GA & \xrightarrow{f} & C \\
 \downarrow Ga & & \downarrow c \\
 GFI & \xrightarrow{GFu} & GFJ
 \end{array}
 \quad \text{vs.} \quad
 \begin{array}{ccc}
 A & \xrightarrow{\check{f}} & VC \\
 \downarrow a & & \downarrow Vc \\
 FI & \xrightarrow{Fu} & FJ \xrightarrow{\eta_{FJ}} VGFJ
 \end{array}$$

illustrates the natural 1-1-correspondence between maps from $\Delta(a)$ to c over u and maps from a to $\eta_{FJ}^* Vc$ over u establishing the adjunction $F^* \Delta \dashv F^* \Gamma$.

292 Are hyperconnected local geometric morphisms molecular?

A negative answer has been given by Hemelaer and Rogers in their 2021 ACS paper.

Let $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ be a local and hyperconnected geometric morphism. We write R for the right adjoint of U and $\phi_I : FI \rightarrow RI$ for the lower transpose of $\varepsilon_I^{-1} : UFI \rightarrow I$. From section 284 it follows that the ϕ_I are all monic.

Let $\Delta \dashv \Gamma : P_F \rightarrow P_{\mathcal{S}}$ be the fibered geometric morphism induced by the ordinary geometric morphism $F \dashv U$. Then Γ has a fibered right adjoint ∇ where ∇_I is given by $\phi_I^* \circ R_{/I}$. Since fibered right adjoints preserve internal products for $u : J \rightarrow I$ and $v : K \rightarrow J$ in \mathcal{S} we have $\nabla_I(\Pi_u v) \cong \Pi_{Fu}(\nabla_J v)$. From

$$\begin{array}{ccc}
 FK & & \\
 \swarrow m & \searrow \phi_K & \\
 & \cdot & RK \\
 \downarrow Fv & \downarrow \nabla_J v & \downarrow Rv \\
 FJ & \xrightarrow{\phi_J} & RJ
 \end{array}$$

it follows that $\Pi_{Fu} m : \Pi_{Fu} Fv \rightarrow \Pi_{Fu}(\nabla_J v) \cong \nabla_I(\Pi_u v)$

But the point of Hemelaer and Rogers result is that the canonical morphism $F(\Pi_u v) \rightarrow \Pi_{Fu} Fv$ need not be an isomorphism in general!

293 Presheaf Toposes over Presheaf Toposes

First we observe that a category internal to a presheaf topos $\mathcal{S} = \widehat{\mathbb{B}}$ is nothing but a split fibration $C : \mathbb{C} \rightarrow \mathbb{B}$ (which we may identify with the corresponding presheaf of categories $\frac{dC}{d\mathbb{B}} : \mathbb{B}^{\text{op}} \rightarrow \mathbf{Cat}$). The presheaf topos $\mathcal{S}^{C^{\text{op}}}$ is given by $\widehat{\mathbb{C}}$ and the inverse image part of its structural geometric morphism to $\widehat{\mathbb{B}}$ is given by $C^* : \widehat{\mathbb{B}} \rightarrow \widehat{\mathbb{C}}$, i.e. change of base along C , which has a right adjoint C_* and a left adjoint $C_!$. Moreover, the left adjoint $C_!$ is fibered over $\widehat{\mathbb{B}}$ since presheaf toposes over arbitrary base toposes are known to be molecular.

The category C internal to $\widehat{\mathbb{B}}$ has a terminal object iff C is a fibration of categories with terminal objects, i.e. has a right adjoint right inverse 1_C , in which case the right adjoint C_* is given by 1_C^* and thus itself has a right adjoint $C^! = (1_C)_*$ which is full and faithful and thus so is C^* . If C is a fibration of categories with terminal objects then the geometric morphism $C^* \dashv C_*$ is hyperconnected iff the internal category C is hyperconnected, i.e. every object of $C(I)$ has a global element.

Grothendieck toposes over \mathcal{S} are given by subtoposes $i : \mathcal{E} \hookrightarrow \mathcal{S}^{C^{\text{op}}}$ for some category C internal to \mathcal{S} . The structural geometric morphism from \mathcal{E} to \mathcal{S} is given by $i^* \circ C^* \dashv C_* \circ i_*$. As observed by Moerdijk the topos \mathcal{E} over \mathcal{S} is molecular iff one may choose C and i in such a way that C^* factors through i_* .

294 Why Lawvere has abandoned fibered categories? (10/2/23)

As Martin Hyland suggested presumably because of Lawvere's notion of "tiny object".

If \mathcal{E} is a topos and $F : \mathcal{E} \rightarrow \mathcal{E}$ has a fibered and thus enriched right adjoint U then we have $(UB)^A \cong B^{FA}$ and thus $UB \cong B^{F1}$. Thus, if F preserves terminal objects the right adjoint $U \cong \text{Id}_{\mathcal{E}}$ and thus also $F \cong \text{Id}_{\mathcal{E}}$.

An object A in \mathcal{E} is tiny or atomic iff $(-)^A$ has a right adjoint $(-)_A$ over **Set**. This, of course, can be generalized to toposes \mathcal{E} over a base topos \mathcal{S} by requiring that the functor $(-)^A : \mathcal{E} \rightarrow \mathcal{E}$ fibered over \mathcal{S} has a right adjoint $(-)_A$ fibered over \mathcal{S} and not over \mathcal{E} . This is reminiscent of the definition of a geometric morphism $F \dashv U : \mathcal{E} \rightarrow \mathcal{S}$ being local where one requires U to have a right adjoint fibered over \mathcal{S} and not over \mathcal{E} .

I think the real reason is that Lawvere got "allergic" to fibered categories because he considered their most prominent proponent Bénabou as "dangerous" for category theory. But presumably less so than most other North American category theorists as Marta Bunge thought.

295 Licata, Orton, Pitts and Spitters on Universe Construction Using the Amazing Right Adjoint (2018)

They consider universe constructions in presheaf toposes $\widehat{\mathbb{C}}$ where \mathbb{C} is a category with finite products. Typically \mathbb{C} contains an interval object \mathbb{I} . We write \wp for $(-)^{\mathbb{I}} : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ and $\sqrt{(-)}$ for its right adjoint $(-)^{\mathbb{I}}$.

Let $\pi_0 : E_{\mathbb{I}0} \rightarrow U_{\mathbb{I}0}$ and $\pi_1 : E_{\mathbb{I}1} \rightarrow U_{\mathbb{I}1}$ be universes in $\widehat{\mathbb{C}}$ such that $!_{U_{\mathbb{I}0}} : U_{\mathbb{I}0} \rightarrow 1$ can be obtained as pullback of π_1 and π_0 can be obtained as pullback of π_1 along some mono $U_{\mathbb{I}0} \rightarrow U_{\mathbb{I}1}$.

Let $C : \wp U_{\mathbb{I}0} \rightarrow U_{\mathbb{I}1}$ be a “fibration structure” on $U_{\mathbb{I}0}$ and $\check{C} : U_{\mathbb{I}0} \rightarrow \sqrt{U_{\mathbb{I}1}}$ be its transpose w.r.t. $\wp \dashv \sqrt{(-)}$. Consider

$$\begin{array}{ccc}
 U & \xrightarrow{\text{pr}_1} & \sqrt{E_{\mathbb{I}1}} \\
 \text{pr}_0 \downarrow \lrcorner & & \downarrow \sqrt{\pi_1} \\
 U_{\mathbb{I}0} & \xrightarrow{\check{C}} & \sqrt{U_{\mathbb{I}1}}
 \end{array}$$

then by transposition w.r.t. $\wp \dashv \sqrt{(-)}$ maps $\Phi : \Gamma \rightarrow U$ correspond to pairs $(A : \Gamma \rightarrow U_{\mathbb{I}0}, \alpha : \wp \Gamma \rightarrow E_{\mathbb{I}1})$ where $A = \text{pr}_0 \circ \Phi$ and $\alpha = \widehat{\text{pr}_1} \circ \Phi$ as indicated in the following diagram

$$\begin{array}{ccc}
 \wp \Gamma & & \Gamma \\
 \swarrow \alpha & & \swarrow \check{\alpha} \\
 \wp U & \xrightarrow{\widehat{\text{pr}_1}} & E_{\mathbb{I}1} \\
 \swarrow \wp A & \searrow \wp \text{pr}_0 & \downarrow \pi_1 \\
 \wp U_{\mathbb{I}0} & \xrightarrow{C} & U_{\mathbb{I}1}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \Gamma & & \Gamma \\
 \swarrow \Phi & & \swarrow \check{\Phi} \\
 U & \xrightarrow{\text{pr}_1} & \sqrt{E_{\mathbb{I}1}} \\
 \swarrow A & \searrow \text{pr}_0 & \downarrow \sqrt{\pi_1} \\
 U_{\mathbb{I}0} & \xrightarrow{\check{C}} & \sqrt{U_{\mathbb{I}1}}
 \end{array}$$

296 Reflections with Stable Units

A reflection is an adjunction whose right adjoint is full and faithful. The following terminology was introduced in the 1985 paper *Reflective Subcategories, localizations and factorization systems* by Cassidy, Hébert and Kelly. Recall that a localization is a reflection between finite limits categories whose left adjoint preserves finite limits.

A reflection $L \dashv F : \mathbb{B} \rightarrow \mathbb{A}$ between finite limit categories is called *semi-left-exact* iff L sends pullbacks of the form

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ \downarrow b & & \downarrow a \\ FJ & \xrightarrow{Fu} & FI \end{array} \quad (\dagger)$$

to pullbacks and it *has stable units* iff L sends pullbacks of the form

$$\begin{array}{ccc} P & \xrightarrow{q} & C \\ \downarrow p & & \downarrow c \\ A & \xrightarrow{\eta_A} & FLA \end{array}$$

to pullbacks.

We first show that having stable units entails semi-left-exact.

Proof: Suppose (\dagger) is a pullback. Consider a pullback

$$\begin{array}{ccc} P & \xrightarrow{p_0} & J \\ \downarrow p_1 \lrcorner & & \downarrow u \\ LA & \xrightarrow{\hat{a}} & I \end{array} \quad (\ddagger)$$

which is preserved by the right adjoint F . Let $q : Q \rightarrow FP$ be the unique arrow with $Fp_0 \circ q = b$ and rendering the diagram

$$\begin{array}{ccccc} B & \xrightarrow{q} & FP & \xrightarrow{Fp_0} & FJ \\ \downarrow f & & \downarrow Fp_1 \lrcorner & & \downarrow Fu \\ A & \xrightarrow{\eta_A} & FLA & \xrightarrow{F\hat{a}} & FI \end{array}$$

commutative. Thus the left square is a pullback which L sends to a pullback since $L \dashv F$ is assumed to have stable units. The functor L sends the right square to a square which is isomorphic to (\ddagger) and thus a pullback. Accordingly, the functor L sends (\ddagger) to a pullback. Thus we have shown that $L \dashv F$ is semi-left-exact. \square

Next we show that a reflection $L \dashv F$ has stable units iff L sends pullbacks of cospans with common codomain in the image of F to pullbacks.

Proof: The backward direction is obvious.

For the forward direction suppose $F \dashv U$ has stable units. We have shown already that then $L \dashv F$ is semi-left-exact. Suppose $a : A \rightarrow FI$ and $b : B \rightarrow FI$. Then their pullback is given by

$$\begin{array}{ccccc}
 P & \xrightarrow{p_1} & Q & \xrightarrow{q_1} & B \\
 \downarrow p_0 & \lrcorner & \downarrow q_0 & \lrcorner & \downarrow b \\
 A & \xrightarrow{\eta_A} & FLA & \xrightarrow{F\hat{a}} & FI
 \end{array}$$

since $a = F\hat{a} \circ \eta_A$. Since $L \dashv F$ has stable units L sends the left pullback to a pullback and since $L \dashv F$ is also semi-left-exact L sends the right pullback to a pullback. Thus L sends the pullback of a and b to a pullback. \square

But one may show the forward direction also directly as follows. A pullback

$$\begin{array}{ccc}
 Q & \xrightarrow{q_1} & LB \\
 \downarrow q_0 & \lrcorner & \downarrow \hat{b} \\
 LA & \xrightarrow{\hat{a}} & I
 \end{array}$$

in \mathbb{B} is sent by the right adjoint F to a pullback in \mathbb{A} . Thus a pullback of $a = \hat{a} \circ \eta_A$ and $b = \hat{b} \circ \eta_B$ may be decomposed as follows

$$\begin{array}{ccccc}
 P & \xrightarrow{\quad} & \bullet & \xrightarrow{\quad} & B \\
 \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \eta_B \\
 \bullet & \xrightarrow{\quad} & FQ & \xrightarrow{Fq_1} & FLB \\
 \downarrow & \lrcorner & \downarrow Fq_0 & \lrcorner & \downarrow F\hat{b} \\
 A & \xrightarrow{\eta_A} & FLA & \xrightarrow{F\hat{a}} & FI
 \end{array}$$

The functor L sends the rightmost lowermost square to a pullback. Moreover, since $L \dashv F$ has stable units and L inverts units L inverts also all pullbacks of units. Thus L inverts all maps in the leftmost uppermost square of the above diagram from which it follows that L sends the outer square above to a pullback.

297 Cavallo and Sattler’s Disjunctive Cube Category

Let \square_{\vee} be the algebraic theory of (join-)semilattices with a least element 0 and a greatest element 1. Morphisms from n to m in $\square_{\vee}^{\text{op}}$ are m -tuples of (join-)semilattice morphisms from 2^n to 2 preserving 1.

In their paper *Relative Elegance and Cartesian Cubes with One Connection* from November 2022 the authors show that presheaves over \square_{\vee} , i.e. “cubical sets with one connection”, can be endowed with a model structure which is Quillen equivalent to the canonical one on (spaces or) simplicial sets.

298 Relating Cartesian and Dedekind Cubical Sets via an Essential Geometric Morphism

Let \mathbb{C} the category of finite lattices and monotone maps between them which may be chosen as small, e.g. as retracts of $\mathcal{P}(2^n)$ where $n \in \mathbb{N}$. Let \mathbb{D} be a small category whose objects are (all) finite sets (up to isomorphism) and whose morphisms from J to I are maps $f : I \rightarrow 2 + J$. Notice that \mathbb{D} is the opposite of the Kleisli category of the monad on finite sets corresponding to the algebraic theory with two constants 0 and 1.

Let $F : \mathbb{D} \rightarrow \mathbb{C}$ be the functor sending $u : I \rightarrow 2 + J$ in \mathbb{D} to the map $F(u) : \mathcal{P}(J) \rightarrow \mathcal{P}(I)$ in \mathbb{C} where for $i \in I$ and $S \in \mathcal{P}(J)$ we have $i \in F(u)(S)$ iff $u(i) \in S$ or $u(i) = 1 \in 2$. Thus, if $u(i) \in J$ then $i \in F(u)(S)$ iff $u(i) \in S$ and if $u(i) \in 2 = \Omega$ then $F(u)(S)(i) = u(i)$ for all $S \in \mathcal{P}(J)$.

The topos of Dedekind cubical sets is $\widehat{\mathbb{C}}$ and the topos of cartesian cubical sets is $\widehat{\mathbb{D}}$. Then $F^* : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{D}}$ is the inverse image part of an essential geometric morphism from $\widehat{\mathbb{D}}$ to $\widehat{\mathbb{C}}$ whose direct image part is given by F_* right adjoint to F^* and whose further left adjoint is given by $F_!$ left adjoint to F^* .

D. Frieberg has asked whether $F_!$ preserve monos?

299 Grothendieck Universes, Samuel Galaxies and Zermelo Universes

A *Grothendieck universe* (as defined in SGA4) is a set U such that

- (U1) U is transitive, i.e. $y \in x \in U$ implies $y \in U$
- (U2) U is closed under pairing, i.e. $x, y \in U$ implies $\{x, y\} \in U$
- (U3) U is closed under powersets, i.e. $x \in U$ implies $\mathcal{P}(x) \in U$
- (U4) U is closed under unions, i.e. $a \in U$ and $f : a \rightarrow U$ implies $\bigcup_{i \in a} f(i) \in U$.

One, actually, should also require as an axiom (U0) that the set ω of natural numbers is an element of U . From (U2) it follows that $\{x\} \in U$ whenever $x \in U$. Thus (U2) and (U4) imply that for $a \in U$ and $f : a \rightarrow U$ the set $\{f(i) \mid i \in a\} \in U$, i.e. that U validates a strong form of the replacement axiom.

P. Samuel (in an unpublished paper) has suggested to weaken the notion of a Grothendieck universe to that of a *galaxy* by dropping the requirement (U4).

It is not clear that galaxies are closed under binary products. Moreover, it does not seem to be possible to show that galaxies G are closed under big unions, i.e. $\bigcup a \in G$ whenever $a \in G$. But adding these two requirements gives rise to the notion of a *Zermelo universe*. Notice that ZFC proves that V_λ is a Zermelo universe for all limit ordinals $\lambda > \omega$.

If one looks at MacLane's alternative but equivalent definition of Grothendieck universe in his book *Categories for the Working Mathematician* and drops his requirement that for $a \in U$ and $f : a \rightarrow U$ its image $\{f(x) \mid x \in a\} \in U$ one obtains the notion of a Zermelo universe.

I think it is indispensable to require closure under binary products. But closure under \bigcup appears as less demanding since for $b \in G$ and $a \subseteq \mathcal{P}(b)$ we have $\mathcal{P}(b)$ in G and since $\bigcup a \in \mathcal{P}(b)$ also $\bigcup a \in G$. Thus, maybe a good definition of galaxy is a transitive set which closed under pairing, binary cartesian products and powersets and contains ω as an element.

300 Patariaia's Conjecture

claims that every Heyting algebra is isomorphic to the lattice of subobjects of 1 of some elementary topos.

Let \mathcal{A} be a Heyting algebra and $X_{\mathcal{A}}$ be its Esakia space. The underlying set of $X_{\mathcal{A}}$ is the set of all lattice homomorphisms $\varphi : \mathcal{A} \rightarrow 2$ where 2 is the ordinal $0 < 1$. For every $a \in \mathcal{A}$ and $i \in 2$ let $U_a^{(i)} = \{\varphi : \mathcal{A} \rightarrow 2 \mid \varphi(a) = i\}$. Obviously, for $a, b \in \mathcal{A}$ and $i \in 2$ we have $U_{a \wedge b}^{(i)} = U_a^{(i)} \cap U_b^{(i)}$ and $U_{a \vee b}^{(i)} = U_a^{(i)} \cup U_b^{(i)}$. Moreover $U_1^{(1)} = |X_{\mathcal{A}}|$ and $U_0^{(1)} = \emptyset$. The topology on $X_{\mathcal{A}}$ is the least one containing all $U_a^{(i)}$ as basic opens. Notice that the complement of $U_a^{(i)}$ is $U_a^{(1-i)}$. Moreover, we may order the underlying set of $X_{\mathcal{A}}$ by subset inclusion, i.e. $\varphi \leq \psi$ iff $\varphi^{-1}(1) \subseteq \psi^{-1}(1)$.

Then $h : \mathcal{A} \rightarrow \mathcal{O}(X_{\mathcal{A}}) : a \mapsto U_a^{(1)}$ is a 1-1 map which preserves finite meets and joins and, moreover, reflects the order. But notice that for this purpose we could have defined the topology on $X_{\mathcal{A}}$ as the one generated by sets of the form $U_a = U_a^{(1)}$. From now on we consider $X_{\mathcal{A}}$ as endowed with this topology. We write $\mathfrak{B}_{\mathcal{A}}$ for $\{U_a \mid a \in \mathcal{A}\}$.

Thus $\mathbf{Sh}(X_{\mathcal{A}})$ is a spatial and thus localic topos. Thus, in $\mathbf{Sh}(X_{\mathcal{A}})$ every object appears as subquotient of a small sum of terminal objects, i.e. as quotient of a small sum of subterminals. Let \mathcal{E} be the full subcategory of $\mathbf{Sh}(X_{\mathcal{A}})$ on those objects A whose support $\text{supp}(A)$ is in $\mathfrak{B}_{\mathcal{A}}$.

One can show that \mathcal{E} is closed under finite limits taken in $\mathbf{Sh}(X_{\mathcal{A}})$. Let $\Omega_{\mathcal{E}}$ be the subobject of $\Omega_{\mathbf{Sh}(X_{\mathcal{A}})}$ where $\Omega_{\mathcal{E}}(U) = \{U_a \mid U_a \subseteq U\}$. Then for $A \in \mathcal{E}$ and $m : P \rightarrow A$ in $\mathbf{Sh}(X_{\mathcal{A}})$ the object P is in \mathcal{E} iff the characteristic map χ for m factors through $\Omega_{\mathcal{E}}$. Since $\top : 1 \rightarrow \Omega_{\mathbf{Sh}(X_{\mathcal{A}})}$ factors through $\Omega_{\mathcal{E}} \rightarrow \Omega_{\mathbf{Sh}(X_{\mathcal{A}})}$ the support of $\Omega_{\mathcal{E}}$ is 1 and thus $\Omega_{\mathcal{E}}$ is in \mathcal{E} . Thus for all X in $\mathbf{Sh}(X_{\mathcal{A}})$ the map $\top^X : 1 \cong 1^X \rightarrow \Omega_{\mathcal{E}}^X$ witnesses that $\Omega_{\mathcal{E}}^X$ has support 1 . This would mean that \mathcal{E} is a topos.

But, alas, this does not work since $\Omega_{\mathcal{E}}$ is not a sheaf!

301 Jibladze's Suggestion on the Lawvere-Menni Problem

Let \mathbb{C} be a small category and $A : \mathbb{C} \rightarrow \mathbf{Set}$ then 2^A is a boolean algebra within $\widehat{\mathbb{C}} = \mathbf{Set}^{\mathbb{C}^{\text{op}}}$ which, however, is not complete.

If $A : \mathbb{C} \rightarrow \mathbf{Cat}$ is a presheaf of (small) categories then $\mathbf{Set}^A : \mathbb{C}^{\text{op}} \rightarrow \mathbf{Cat}$ is a presheaf of toposes sending maps in \mathbb{C} to functors having both adjoints, i.e. inverse image parts of essential geometric morphisms between (presheaf) toposes.