

A Model of Type Theory in Simplicial Sets

A brief introduction to Voevodsky's Homotopy Type Theory

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Introduction

As observed in [HS] identity types in intensional type theory endow every type with the structure of a *weak higher dimensional groupoid*. The simplest and oldest notion of weak higher dimensional groupoid is given by *Kan complexes* within the topos **sSet** of simplicial sets. This was observed around 2006 independently by V. Voevodsky and the author.

The aim of this note is to describe how simplicial sets organize into a model of Martin-Löf type theory. Moreover, we explain Voevodsky's *Univalence Axiom* which holds in this model and implements the idea that *isomorphic types are equal* as suggested in [HS]. A full exposition of the theory will be given in a longer article by Voevodsky which is still in preparation, but see [VV]. The current note just gives a first introduction to this circle of ideas.

1 Simplicial Sets and Kan complexes

Due to limitation of space and time we can just give a very brief recap of this classical material (due to D. Kan and D. Quillen from late 1950s and 1960s). An excellent modern reference for this is the first chapter of [GJ].

Let $\mathbf{\Delta}$ be the category of finite nonempty ordinals and monotone maps between them. We write **sSet** for the topos $\mathbf{Set}^{\mathbf{\Delta}^{\text{op}}}$ of *simplicial sets*. We write $[n]$ for the ordinal $n+1 = \{0, 1, \dots, n, n+1\}$ and $\Delta[n]$ for the corresponding representable object in **sSet**. For $0 \leq k \leq n$ we write $i_k^n : \Lambda_k[n] \hookrightarrow \Delta[n]$ for the inclusion of the k -th *horn* $\Lambda_k[n]$ into $\Delta[n]$ which is obtained by removing the interior and the face opposite to vertex k (for $n = 0$ the horn $\Lambda_0[0] = \Delta[0]$). There is an obvious¹ faithful functor $|\cdot|$ from $\mathbf{\Delta}$ into the category **Sp** of spaces

¹With $[n]$ one associates the canonical n -dimensional simplex $\{x \in [0, 1]^{n+1} \mid \sum x_i = 1\}$ endowed with the Euclidean topology. With $\alpha : [n] \rightarrow [m]$ one associates the continuous map $|\alpha|$ from the n -dimensional to the m -dimensional simplex defined as $|\alpha|(x)_j = \sum_{\alpha(i)=j} x_i$.

and continuous maps. This induces the *singular* functor $\mathcal{S} : \mathbf{Sp} \rightarrow \mathbf{sSet}$ sending X to $\mathbf{Sp}(|-, X)$ which has a left adjoint \mathcal{R} called *geometric realization*. The objects in the image of \mathcal{R} are the so-called CW-complexes which can be obtained by glueing simplices in a way as described by some simplicial set. The objects in the image of \mathcal{S} are up to weak equivalence the so called *Kan complexes* as defined in the next paragraph.

On \mathbf{sSet} there is a well known *Quillen model structure* whose class \mathcal{C} of *cofibrations* consists of all monos, whose class \mathcal{W} of *weak equivalences* consists of all maps $f : X \rightarrow Y$ whose geometric realization $\mathcal{R}(f) : \mathcal{R}(X) \rightarrow \mathcal{R}(Y)$ is a homotopy equivalence² and whose class \mathcal{F} of *fibrations* consists of all *Kan fibrations*, i.e. maps $a : A \rightarrow I$ in \mathbf{sSet} with $i_k^n \perp a$ for all $k \leq n$ in \mathbb{N} . Here $f \perp g$ means that for every commuting square $kf = gh$ there is a (typically not unique) diagonal filler, i.e. a map d with $df = h$ and $gd = k$ as in

$$\begin{array}{ccc} \cdot & \xrightarrow{h} & \cdot \\ f \downarrow & \nearrow d & \downarrow g \\ \cdot & \xrightarrow{k} & \cdot \end{array}$$

By definition a simplicial set X is a *Kan complex* iff $X \rightarrow 1$ is a Kan fibration. The extension property w.r.t. the horn inclusions $i_i^2 : \Lambda_i[2] \hookrightarrow \Delta[2]$ expresses that “up to homotopy” morphisms can be composed and every morphism has a left and a right inverse.

It is shown in [GJ] that \mathcal{S} factors through the full subcategory of Kan complexes and every Kan complex X is weakly equivalent to a singular complex via $\eta_X : X \rightarrow \mathcal{S}(\mathcal{R}(X))$. Moreover, one can show that a map $f : X \rightarrow Y$ between Kan complexes is a weak equivalence iff f is a homotopy equivalence, i.e. there is a map $g : Y \rightarrow X$ such that $gf \sim \text{id}_X$ and $fg \sim \text{id}_Y$.³

In \mathbf{sSet} one can develop a fair amount of homotopy theory and as shown in [GJ] inverting weak equivalences in \mathbf{sSet} gives rise to the same *homotopy category* as inverting weak equivalences in \mathbf{Sp} . Thus, from a homotopy point of view \mathbf{sSet} and \mathbf{Sp} are different ways of speaking about the same thing. However, the “combinatorial” topos \mathbf{sSet} is in many respects much nicer than the “geometric” category \mathbf{Sp} . This we will exploit when interpreting intensional Martin-Löf type theory in \mathbf{sSet} .

2 Homotopy Model for Type Theory

For basic information about type theory and its semantics see [Ho, S91, S93]. Type theory is the basis of interactive theorem provers like Coq as described in

²i.e. there exists a continuous map $g : \mathcal{R}(Y) \rightarrow \mathcal{R}(X)$ such that both composita are homotopy equivalent to the identities $\text{id}_{\mathcal{R}(X)}$ and $\text{id}_{\mathcal{R}(Y)}$, respectively

³For $f, g : A \rightarrow B$ we write $f \sim g$ iff there is a map $h : \Delta[1] \times A \rightarrow B$ with $h(0, -) = f$ and $h(1, -) = g$.

[BC]. Since \mathbf{sSet} is a topos and thus locally cartesian closed it provides a model of extensional type theory since \mathbf{sSet} contains also a natural numbers object N .

In order to obtain a non-extensional interpretation of identity types we restrict families of types to be *Kan fibrations*. Accordingly, types are Kan complexes, i.e. weak higher dimensional groupoids. In this respect the simplicial sets model appears as a natural generalization of the groupoid model of [HS] which was our main motivation for introducing it.

Evidently, the class \mathcal{F} contains all isomorphism and is closed under composition and pullbacks along arbitrary morphisms in \mathbf{sSet} . Using the fact that trivial cofibrations are stable under pullbacks along Kan fibrations (referred to as *right proper* in the literature) one easily establishes that

Theorem 2.1 *Kan fibrations are closed under Π , i.e. whenever $a : A \rightarrow I$ and $b : B \rightarrow A$ are in \mathcal{F} then $\Pi_a(b)$ is in \mathcal{F} , too.*

For interpreting equality on X we factor the diagonal $\delta_X : X \rightarrow X \times X$ as

$$\begin{array}{ccc} X & \xrightarrow{r_X} & \text{Id}(X) \\ & \searrow \delta_X & \downarrow p_X \\ & & X \times X \end{array}$$

with $p_X \in \mathcal{F}$ and $r_X \in \mathcal{C} \cap \mathcal{W}$ which is possible since $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a Quillen model structure. The Kan fibration p_X will serve as interpretation of

$$x, y : X \vdash \text{Id}_X(x, y)$$

as suggested in [AW].⁴ For families of types as given by a Kan fibration $a : A \rightarrow I$ one factors the fibrewise diagonal $\delta_a : A \rightarrow A \times_I A$ in an analogous way. However, there is a problem since such factorisations are in general not stable under pullbacks. To overcome this problem we will introduce *universes* à la Martin-Löf.

As described in [VV] a universe in \mathbf{sSet} is a Kan fibration $p_U : \tilde{U} \rightarrow U$. We write \mathcal{D}_U for the class of Kan fibrations which can be obtained as pullbacks of p_U along some map in \mathbf{sSet} . In [VV] Voevodsky has shown how such a universe induces a contextual category $CC[p_U]$ which interprets dependent sums if \mathcal{D}_U is closed under composition and which interprets dependent products if \mathcal{D}_U is closed under Π .

Some time ago M. Hofmann and the author observed how to lift a Grothendieck universe \mathcal{U} in \mathbf{Set} to a type theoretic universe $p_U : \tilde{U} \rightarrow U$ in a presheaf topos $\hat{\mathcal{C}} = \mathbf{Set}^{\mathcal{C}^{\text{op}}}$. The object U is defined as

$$U(I) = \mathcal{U}^{(\mathcal{C}/I)^{\text{op}}} \quad U(\alpha) = \mathcal{U}^{\Sigma_\alpha^{\text{op}}}$$

where for $\alpha : J \rightarrow I$ the functor $\Sigma_\alpha : \mathcal{C}/J \rightarrow \mathcal{C}/I$ is postcomposition with α . As common we write α^* for $U(\alpha)$. Notice that with $A \in U(I)$ one may associate

⁴Already for ordinary groupoids (see [HS]) the diagonal is hardly ever a fibration.

via the Grothendieck construction a morphism $P_A : \mathbf{Elts}(A) \rightarrow \mathbf{Y}_C(I)$ which exhibits $\mathcal{U}^{(\mathcal{C}/I)^{\text{op}}}$ as equivalent to the full subcategory of $\widehat{\mathcal{C}}/\mathbf{Y}_C(I)$ on those maps whose fibres are small in the sense of \mathcal{U} . By Yoneda sections of P_A correspond to elements of $A(\text{id}_I)$. Moreover, for $a \in A(\text{id}_I)$ and $\alpha : J \rightarrow I$ in \mathcal{C} we have

$$\begin{array}{ccc}
\mathbf{Y}_C(J) & \xrightarrow{\mathbf{Y}_C(\alpha)} & \mathbf{Y}_C(I) \\
\alpha^* a \downarrow & \lrcorner & \downarrow a \\
\mathbf{Elts}(\alpha^* A) & \longrightarrow & \mathbf{Elts}(A) \\
P_{\alpha^* A} \downarrow & \lrcorner & \downarrow P_A \\
\mathbf{Y}_C(J) & \xrightarrow{\mathbf{Y}_C(\alpha)} & \mathbf{Y}_C(I)
\end{array}$$

with a and $\alpha^* a$ sections of P_A and $P_{\alpha^* A}$, respectively. One easily checks that $\alpha^* a = A(\alpha \xrightarrow{\alpha} \text{id}_I)(a)$. These observations suggest the following definition of the presheaf \widetilde{U}

$$\widetilde{U}(I) = \{\langle A, a \rangle \mid A \in \mathcal{U}(I) \text{ and } a \in A(\text{id}_I)\}$$

for $I \in \mathcal{C}$ and

$$\widetilde{U}(\alpha)(\langle A, a \rangle) = \langle U(\alpha)(A), A(\alpha \xrightarrow{\alpha} \text{id}_I)(a) \rangle$$

for $\alpha : J \rightarrow I$ in \mathcal{C} . The map $p_U : \widetilde{U} \rightarrow U$ sends $\langle A, a \rangle$ to A . One easily checks that p_U is *generic* for maps with fibres small in the sense of \mathcal{U} , i.e. these maps are up to isomorphism precisely those which can be obtained as pullback of p_U along some map in $\widehat{\mathcal{C}}$.

Now for $\mathcal{C} = \mathbf{\Delta}$ we adapt this idea in such a way that p_U is generic for Kan fibrations with fibres small in the sense of \mathcal{U} . For this purpose we redefine U as

$$U([n]) = \{A \in \mathcal{U}^{(\mathbf{\Delta}/[n])^{\text{op}}} \mid P_A \text{ is a Kan fibration}\}$$

where $P_A : \mathbf{Elts}(A) \rightarrow \mathbf{\Delta}[n]$ is obtained from A by the Grothendieck construction. For maps α in $\mathbf{\Delta}$ we can define $U(\alpha)$ as above which makes sense since Kan fibrations are stable under pullbacks. We also define \widetilde{U} and p_U using the same formulas as above but understood as restricted to U in its present form.

Theorem 2.2 *The simplicial set U is a Kan complex.*

This has been shown in [VV] for a different construction of the universe. A simpler proof of Theorem 2.2 for the above construction of U has been found recently by A. Joyal (see [KLV]).

Theorem 2.3 *The map $p_U : \widetilde{U} \rightarrow U$ is universal for Kan fibrations which are small in the sense of \mathcal{U} .*

Proof: For showing that p_U is a Kan fibration suppose

$$\begin{array}{ccc} \Lambda_k[n] & \xrightarrow{a} & \tilde{U} \\ i_k^n \downarrow & & \downarrow p_U \\ \Delta[n] & \xrightarrow{A} & U \end{array}$$

commutes. Since the pullback of p_U along A is the Kan fibration $P_A : \text{Elts}(A) \rightarrow \Delta[n]$ there exists a diagonal filler $\bar{a} : \Delta[n] \rightarrow \tilde{U}$ making

$$\begin{array}{ccc} \Lambda_k[n] & \xrightarrow{a} & \tilde{U} \\ i_k^n \downarrow & \nearrow \bar{a} & \downarrow p_U \\ \Delta[n] & \xrightarrow{A} & U \end{array}$$

commute.

For showing that p_U is universal suppose that $a : A \rightarrow I$ is a Kan fibration small in the sense of \mathcal{U} . Then one gets a as pullback of p_U along the morphism $A : I \rightarrow U$ sending $x \in I([n])$ to a \mathcal{U} -valued presheaf over $\mathbf{\Delta}/[n]$ which via the Grothendieck construction is isomorphic to x^*a . \square

Thus p_U provides us with a universe in \mathbf{sSet} which is closed under dependent sums and products. The constant presheaf $N = \Delta(\mathbb{N})$ over $\mathbf{\Delta}$ with value \mathbb{N} is a natural numbers object in \mathbf{sSet} . Since N is a small Kan complex the universe U also hosts the natural numbers object N .

For interpreting identity types in this universe we consider the fibrewise diagonal $\delta_{\tilde{U}} : \tilde{U} \rightarrow \tilde{U} \times_U \tilde{U}$ with $\pi_i \circ \delta_{\tilde{U}} = \text{id}$ for $i = 0, 1$ where

$$\begin{array}{ccc} \tilde{U} \times_U \tilde{U} & \xrightarrow{\pi_1} & \tilde{U} \\ \pi_0 \downarrow & \lrcorner & \downarrow p_U \\ \tilde{U} & \xrightarrow{p_U} & U \end{array}$$

and a factorisation

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{r_{\tilde{U}}} & \text{Id}_{\tilde{U}} \\ & \searrow \delta_{\tilde{U}} & \downarrow p_{\tilde{U}} \\ & & \tilde{U} \times_U \tilde{U} \end{array}$$

with $p_{\tilde{U}} \in \mathcal{F}$ and $r_{\tilde{U}} \in \mathcal{C} \cap \mathcal{W}$. For small types A , i.e. types in U , the interpretation of Id_A and r_A is obtained by pulling back along the morphism into U interpreting A .

For interpreting the eliminator J for Id-types we pull back the whole situation along the projection p from the generic context

$$\Gamma \equiv A : U, C : (\Pi x, y:A)\text{Id}_A(x, y) \rightarrow U, d : (\Pi x:A)C(x, x, r_A(x))$$

to the context $A : U$. Since p is a Kan fibration and pullbacks along Kan fibrations preserve weak equivalences we have $p^*r_{\tilde{U}} \in \mathcal{C} \cap \mathcal{W}$. Let $q : \tilde{C} \rightarrow p^*\text{Id}_{\tilde{U}}$ be the interpretation of the type family $\Gamma, x, y:A, z:\text{Id}_A(x, y) \vdash C(x, y, z)$ and $d : p^*\tilde{U} \rightarrow \tilde{C}$ be the interpretation of $\Gamma, x, y:A, z:\text{Id}_A(x, y) \vdash d(x) : C(x, y, z)$. Obviously, we have $q \circ d = p^*r_{\tilde{U}}$. Since q is a Kan fibration and $p^*r_{\tilde{U}} \in \mathcal{C} \cap \mathcal{W}$ by the defining properties of Quillen model structures there is a map J making the diagram

$$\begin{array}{ccc} \tilde{C} & \xrightarrow{q} & p^*\text{Id}_{\tilde{U}} \\ d \uparrow & \swarrow J & \parallel \\ p^*\tilde{U} & \xrightarrow{p^*r_{\tilde{U}}} & p^*\text{Id}_{\tilde{U}} \end{array}$$

commute. This map J serves as interpretation of the eliminator for identity types associated with types in the universe U .

NB Since we factor $\delta_{\tilde{U}}$ and d relative to the generic context Γ and interpret occurrences of r and J as pullbacks of this generic situation *these interpretations are stable under substitution*, i.e. the *Beck-Chevalley condition* holds for them. However, since in general trivial cofibrations are not stable under arbitrary pullbacks the instantiations of $r_{\tilde{U}}$ are not guaranteed to be trivial cofibrations. This problem, however, can be avoided when choosing $r_{\tilde{U}}$ as the canonical map $\tilde{U} \rightarrow \tilde{U}^{\Delta[1]}$ in the fibre over U because such maps are stable under arbitrary pullbacks.

If one starts from the universe $\mathcal{U} = \{\emptyset, \{\emptyset\}\}$ one obtains a universe $p_U : \tilde{U} \rightarrow U$ where $U([n])$ is the set of those subobjects $m : P \hookrightarrow \Delta[n]$ which are Kan fibrations. One easily shows by induction over n that such subobjects are trivial in the sense that m is an isomorphism whenever P is not initial.⁵ Thus p_U is obtained by restricting $\top : 1 \rightarrow \Omega_{\mathbf{sSet}}$ along the mono $2 \rightarrow \Omega_{\mathbf{sSet}}$. When interpreting \mathbf{Prop} by this p_U one obtains a boolean, 2-valued proof-irrelevant interpretation of \mathbf{Coq} .

Finally we want to emphasize that the model sketched in this section implements the idea that types are *weak higher dimensional groupoids* which are here realized as Kan complexes. Moreover, it keeps the interpretation of \mathbf{Prop} from the naive model in \mathbf{Set} .

⁵It is an open question, however, whether for any Kan fibration $p : E \rightarrow B$ its image is a union of connected components of B .

3 Voevodsky’s Univalence Axiom

We now give the formulation of Voevodsky’s Univalence Axiom which as shown in [VV, KLV] holds in the model described in the previous section.

For this purpose we first introduce a few abbreviations

$$\begin{aligned} \text{iscontr}(X:U) &= (\Sigma x:X)(\Pi y:Y) \text{Id}_X(x, y) \\ \text{hfiber}(X, Y:U)(f:X \rightarrow Y)(y:Y) &= (\Sigma x:X) \text{Id}_Y(f(x), y) \\ \text{isweq}(X, Y:U)(f:X \rightarrow Y) &= (\Pi y:Y) \text{iscontr}(\text{hfiber}(X, Y, f, y)) \\ \text{Weq}(X, Y:U) &= (\Sigma f:X \rightarrow Y) \text{isweq}(X, Y, f) \end{aligned}$$

where $\text{iscontr}(X)$ says that X is *contractible*, $\text{hfiber}(f)(y)$ is the *homotopy fiber* of f at y and $\text{isweq}(f)$ says that f is a *weak equivalence*.

Using the eliminator J for identity types one constructs canonical maps

$$\text{eqweq}(X, Y:U) : \text{Id}_U(X, Y) \rightarrow \text{Weq}(X, Y)$$

which Voevodsky’s *Univalence Axiom*⁶ claims to be all weak equivalences. From a type theoretic point of view this amounts to postulating a new constant

$$\text{UnivAx} : (\Pi X, Y:U) \text{isweq}(\text{eqweq}(X, Y))$$

which, alas, doesn’t seem to have any computational meaning.

Notice, moreover, that $\text{isweq}(X, Y)(f)$ is equivalent to

$$\text{isiso}(X, Y)(f) \equiv (\Sigma g:Y \rightarrow X)((\Pi x:X)\text{Id}_X(g(fx), x)) \times ((\Pi y:Y)\text{Id}_Y(f(gy), y))$$

which formally says that f is an isomorphism but due to the interpretation of identity types in **sSet** rather claims that f is a *homotopy equivalence*. This equivalence is provable in type theory without the Univalence Axiom (see [VV] for a Coq file containing a machine checked proof). It is in accordance with the fact that in **sSet** morphisms to Kan complexes are weak equivalences iff they are homotopy equivalences. The type theoretic argument may be seen as an example for a “synthetic” version of a theorem in homotopy theory.

A surprising consequence of the Univalence Axiom is the function extensionality principle

$$((\Pi x:X) \text{Id}_Y(fx, gx)) \rightarrow \text{Id}_{X \rightarrow Y}(f, g)$$

for $f, g : X \rightarrow Y$ in U (see [VV] for a Coq file containing a machine checked proof). Thus, in presence of the Univalence Axiom a map in U is a weak equivalence iff there is a map in the opposite direction such that both composita are *propositionally* equal to the respective identities. This, however, does not mean that such maps are actually isomorphisms in the external sense since after all there are plenty of homotopy equivalences which are not isomorphisms.

⁶The name “univalent” insinuates that the universe U contains up to propositional equality only one representative for each class of weakly equivalent types.

4 Conclusion

The model of intensional type theory in **sSet** realizes the idea from [HS] that propositional equality of types should coincide with isomorphism. It is not clear so far what are the benefits of this coincidence for the formalization of category theory. In particular, it does not render unnecessary *coherence* conditions for monoidal or indexed categories.

However, one may use type theory as an internal language for developing homotopy theory synthetically. The basic idea is that the type of paths from x to y in X is given *a priori* by $\text{Id}_X(x, y)$. For information on synthetic homotopy theory consult the blog <http://homotopytypetheory.org>.

From recent work by Gepner and Kock [GK] it follows that all *right proper Cisinski model structures* (where cofibrations are the monomorphisms) give rise to models of type theory validating the Univalence Axiom. This shows that independently from its type theoretic status univalence is quite ubiquitous.

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