Supplementary material for the paper

Convergence of numerical adjoint schemes arising from optimal boundary control problems of hyperbolic conservation laws

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Proof of Lemma 3.2, Theorem 3.3 and Lemma 3.5.

Proof of Lemma 3.2. The first assertion holds by [7, Lemma 3.1] and (3.6).

To show (ii), set $\bar{v}_j^n = v_j^n, \bar{w}_j^n = w_j^n$ for $j \ge 0$ and $\bar{v}_j^n = \bar{w}_j^n = 0$ for $j < 0$. Then we have by [9] with the scheme (3.5) $\|\mathcal{H}(\bar{v}_\Delta^n) - \mathcal{H}(\bar{w}_\Delta^n)\|_{1,\mathbb{R}} \le \|\bar{v}_\Delta^n - \bar{w}_\Delta^n\|_{1,\mathbb{R}} = \|v_\Delta^n - w_\Delta^n\|_{1,[-\Delta x/2,\infty[}$. We obtain with (3.6) $\|\mathcal{H}_B(v_\Delta^n) - \mathcal{H}_B(w_\Delta^n)\|_{1,[\Delta x/2,\infty[} \le \|v_\Delta^n - w_\Delta^n\|_{1,[-\Delta x/2,\infty[}$ and it follows

$$\|\mathcal{H}_B(v_\Delta^n) - \mathcal{H}_B(w_\Delta^n)\|_{1,[-\Delta x/2,\infty[} \le \|v_\Delta^n - w_\Delta^n\|_{1,[-\Delta x/2,\infty[} + \Delta x|v_0^{n+1} - w_0^{n+1}|.$$

Now, set $v_j^n = v_B^n$ for $j < 0$ and denote by $\tilde{v}_j^{n+1}$ the corresponding grid values obtained by the scheme (3.5). Then $\tilde{v}_j^{n+1} = v_j^{n+1}$ for $j \ge 1$, $\tilde{v}_j^{n+1} = v_B^n$ for $j \le -1$ and by (i) the resulting scheme is monotone. To verify (iii) set $w_\Delta^n = v_\Delta^n(\cdot + \Delta x)$, then by using again $\|\mathcal{H}(v_\Delta^n) - \mathcal{H}(w_\Delta^n)\|_{1,\mathbb{R}} \le \|v_\Delta^n - w_\Delta^n\|_{1,\mathbb{R}}$, see [9], we deduce

$$\|\tilde{v}_\Delta^{n+1}\|_{TV,]\Delta x/2,\infty[} + |v_1^{n+1} - \tilde{v}_0^{n+1}| + |\tilde{v}_0^{n+1} - v_B^n| \le \|v_\Delta^n\|_{TV}$$

and using the triangle inequality we obtain

$$\begin{aligned}
\|v_\Delta^{n+1}\|_{TV} &= \|\tilde{v}_\Delta^{n+1}\|_{TV,]\Delta x/2,\infty[} + |v_1^{n+1} - v_B^{n+1}| \\
&\le \|v_\Delta^n\|_{TV} - |v_1^{n+1} - \tilde{v}_0^{n+1}| - |\tilde{v}_0^{n+1} - v_B^n| + |v_1^{n+1} - v_B^{n+1}| \\
&\le \|v_\Delta^n\|_{TV} - |v_1^{n+1} - v_B^n| + |v_1^{n+1} - v_B^{n+1}| \le \|v_\Delta^n\|_{TV} + |v_B^{n+1} - v_B^n|. \qquad \square
\end{aligned}$$

Proof of Theorem 3.3. First assume that $u_0 \in BV_{loc}(\Omega)$ and $u_B \in BV(0,T)$. Lemma 3.2 yields $y_j^n \in [l - T\|g\|_\infty, r + T\|g\|_\infty]$ and $y_\Delta^n \in BV_{loc}(\Omega)$, since $g \in L^1(0,T;BV(\Omega))$. We extend the boundary data and source by $y_j^n = y_0^n$, $G_j^n = 0$ for $j < 0$ and apply the operator $\mathcal{H}(y_\Delta^n) + \Delta t g_\Delta^n$ with $\mathcal{H}$ in (3.5). For monotone fluxes with Lipschitz constant L_F it is known that $\|\mathcal{H}(y_\Delta^n) - y_\Delta^n\|_1 \le 3\Delta t L_F \|y_\Delta^n\|_{BV}$, see e.g. [9, Prop. 3.5]. We have $y_j^{n+1} = (\mathcal{H}(y_\Delta^n) + \Delta t g_\Delta^n)_j$ for $j \ge 1$ and $\mathcal{H}(y_\Delta^n)_j - y_j^n = 0$ for $j \le -1$. Hence, we obtain $\|y_\Delta^{n+1} - y_\Delta^n\|_{1,[-\Delta x/2,\infty[} \le 3\Delta t L_F \|y_\Delta^n\|_{BV} + \Delta t|u_B^{n+1} - u_B^n| + \Delta t\|g_\Delta^n\|_1$. Since y_Δ^n are uniformly bounded in BV_{loc} and by using the equicontinuity in time, there is a subsequence converging in $L^\infty(0,T;L^1_{loc}(\Omega))$ to a function $y \in L^\infty(0,T;BV_{loc}(\Omega)) \cap C([0,T];L^1_{loc}(\Omega))$. We still have to show that y is an entropy solution of (1.1). For monotone fluxes the discrete entropy inequality holds

$$U_k(y_j^{n+1}) \le U_k(y_j^n) - \frac{\Delta t}{\Delta x}\Delta^+ Q_k(y_{j-1}^n, y_j^n) + \Delta t U_k'(y_j^{n+1}) G_j^n \tag{4.1}$$

with $U_k(u) = |u - k|$, $Q_k(u,v) = F([u,k]_+, [v,k]_+) - F([u,k]_-, [v,k]_-)$, $U_k'(u) = \operatorname{sgn}(u-k)$, for any $k \in \mathbb{R}$ and $[\alpha,\beta]_+ := \max\{\alpha,\beta\}$ and $[\alpha,\beta]_- := \min\{\alpha,\beta\}$. In fact, since H is monotone increasing w.r.t. all arguments and $k = H(k,k,k)$, we thus have

$$\begin{aligned}
[y_j^{n+1}, k]_+ &\le H([y_{j-1}^n, k]_+, [y_j^n, k]_+, [y_{j+1}^n, k]_+) + \Delta t \mathbf{1}_{\{y_j^{n+1} > k\}} G_j^n, \\
[y_j^{n+1}, k]_- &\ge H([y_{j-1}^n, k]_-, [y_j^n, k]_-, [y_{j+1}^n, k]_-) + \Delta t \mathbf{1}_{\{y_j^{n+1} < k\}} G_j^n.
\end{aligned}$$

Taking the difference yields the discrete entropy inequality (4.1). Let $\phi_j^n \geq 0$ be grid values of a test function $\phi \in C_c^1([0,T[\times[0,\infty[)$, $\phi \geq 0$. We obtain

$$\sum_{j\geq 1, n\geq 0} \phi_j^n \Delta x \left(U_k(y_j^{n+1}) - U_k(y_j^n) - \Delta t U_k'(y_j^{n+1}) G_j^n + \lambda \Delta^+ Q_k(y_{j-1}^n, y_j^n)\right) \leq 0$$

and summation by parts yields

$$\begin{aligned} &\sum_{j\geq 1, n\geq 0} \Delta x \left(U_k(y_j^{n+1})(\phi_j^n - \phi_j^{n+1}) - \lambda \Delta^+ \phi_j^n Q_k(y_j^n, y_{j+1}^n) - \Delta t \phi_j^n U_k'(y_j^{n+1}) G_j^n\right) \\ &- \sum_{j\geq 1} \Delta x U_k(y_j^0)\phi_j^0 - \sum_{n\geq 0} \Delta t Q_k(u_B^n, y_1^n)\phi_1^n \leq 0. \end{aligned} \tag{4.2}$$

Doing the same for the original scheme instead of the discrete entropy inequality yields

$$\begin{aligned} &\sum_{j\geq 1, n\geq 0} \Delta x \left(y_j^{n+1}(\phi_j^n - \phi_j^{n+1}) - \lambda \Delta^+ \phi_j^n F(y_j^n, y_{j+1}^n) - \Delta t \phi_j^n G_j^n\right) \\ &- \sum_{j\geq 1} \Delta x y_j^0 \phi_j^0 - \sum_{n\geq 0} \Delta t F(u_B^n, y_1^n)\phi_1^n = 0. \end{aligned} \tag{4.3}$$

Now $y_0^n = u_B^n$ and with $\bar{u} = y_0^n - \frac{\Delta t}{\Delta x}(F(y_0^n, y_1^n) - F(y_0^n, y_0^n))$ we obtain as above

$$U_k(\bar{u}) \leq U_k(y_0^n) - \frac{\Delta t}{\Delta x}(Q_k(y_0^n, y_1^n) - Q_k(y_0^n, y_0^n))$$

and by the convexity of U_k (note that $\operatorname{sgn}(u-k) \in \partial U_k(u)$)

$$U_k(\bar{u}) \geq U_k(y_0^n) - \frac{\Delta t}{\Delta x}\operatorname{sgn}(y_0^n - k)(F(y_0^n, y_1^n) - F(y_0^n, y_0^n)).$$

By combining the last two inequalities, we obtain

$$Q_k(y_0^n, y_1^n) - Q_k(y_0^n, y_0^n) - \operatorname{sgn}(y_0^n - k)(F(y_0^n, y_1^n) - F(y_0^n, y_0^n)) \leq 0.$$

Inserting this in (4.2) yields with $F(y_0^n, y_0^n) = f(y_0^n)$

$$\begin{aligned} &\sum_{j\geq 1, n\geq 0} \Delta x \left(U_k(y_j^{n+1})(\phi_j^n - \phi_j^{n+1}) - \lambda \Delta^+ \phi_j^n Q_k(y_j^n, y_{j+1}^n) - \Delta t \phi_j^n U_k'(y_j^{n+1}) G_j^n\right) \\ &- \sum_{j\geq 1} \Delta x U_k(y_j^0)\phi_j^0 - \sum_{n\geq 0} \Delta t \left(Q_k(y_0^n, y_0^n) + \operatorname{sgn}(y_0^n - k)(F(y_0^n, y_1^n) - f(y_0^n))\right)\phi_1^n \leq 0. \end{aligned} \tag{4.4}$$

Taking test functions $\max\{0, 1 - x/\delta\}\phi(t)$ with $\phi \in C_c^1(]0,T[)$, $\phi \geq 0$ in (4.3) yields for $\Delta x = \Delta t/\lambda \to 0$ and $\delta \searrow 0$

$$\int_{[0,T[} f(y(t, 0+))\phi(t)\,\mathrm{d}t = \lim_{\Delta x \to 0} \int_{[0,T[} F(u_{B,\Delta}(t), y_\Delta(t, \Delta x))\phi(t)\,\mathrm{d}t.$$

Which verifies the first statement (3.9). By continuity this also holds for test functions $\phi \in L^1(0,T)$. Inserting this in (4.4) with the same test functions such that for all k

the set $\{u_B = k\}$ has measure 0 yields

$$\begin{aligned}\int_{[0,T[} &(\operatorname{sgn}(y(t,0+)-k)(f(y(t,0+))-f(k)) - \operatorname{sgn}(u_B(t)-k)(f(u_B(t))-f(k)))\\ &\qquad - \operatorname{sgn}(u_B(t)-k)(f(y(t,0+))-f(u_B(t)))\phi(t)\,\mathrm{d}t \le 0,\end{aligned}$$

and thus

$$\int_{[0,T[} (\operatorname{sgn}(y(t,0+)-k) + \operatorname{sgn}(k-u_B(t)))(f(y(t,0+))-f(k))\phi(t)\,\mathrm{d}t \le 0. \tag{4.5}$$

This is equivalent to

$$\min_{k\in I(y(t,0+),u_B(t))} \operatorname{sgn}(k-y(t,0+))(f(y(t,0+))-f(k)) = 0,$$

thus the limit function satisfies the boundary condition in the BLN sense. Now, let $\phi \in C_c^1([0,T[\times[0,\infty[)$, $\phi \ge 0$ be arbitrary. Using test functions $\min\{1, x/\delta\}\phi(t,x)$ in (4.2) the limit $\Delta x = \Delta t/\lambda \to 0$ and $\delta \searrow 0$ results in

$$\begin{aligned}&\int_{\Omega_T} (-|y-k|\phi_t - \operatorname{sgn}(y-k)(f(y)-f(k))\phi_x - \phi\operatorname{sgn}(y-k)g)\,\mathrm{d}x\,\mathrm{d}t\\ &- \int_{\Omega} \operatorname{sgn}(y(t,0+)-k)(f(y(t,0+))-f(k))\phi(t,0)\,\mathrm{d}t - \int_{\Omega} |u_0(x)-k|\phi(0,x)\,\mathrm{d}x \le 0.\end{aligned}$$

Including (4.5) yields the weak formulation of the entropy inequality

$$\begin{aligned}&\int_{\Omega_T} (-|y-k|\phi_t - \operatorname{sgn}(y-k)(f(y)-f(k))\phi_x - \phi\operatorname{sgn}(y-k)g)\,\mathrm{d}x\,\mathrm{d}t\\ &- \int_{\Omega} \operatorname{sgn}(u_B(t)-k)(f(y(t,0+))-f(k))\phi(t,0)\,\mathrm{d}t - \int_{\Omega} |u_0(x)-k|\phi(0,x)\,\mathrm{d}x \le 0\end{aligned}$$

see also [3, 20]. The limit $\lim_{t\searrow 0}\|y(t,\cdot)-u_0\|_{L^1_{loc}(\Omega)} = 0$ follows from the equicontinuity in time and $u_{0,\Delta} \to u_0$ in $L^1_{loc}(\Omega)$. Consequently the limit function y is the entropy solution of (1.1). By a subsequence-subsequence argument the convergence holds for the whole sequence Δ. Using an approximation argument as in [9] the same holds for controls in $u \in (L^1\cap L^\infty)(\Omega)\times(L^1\cap L^\infty)(0,T)$ without the additional BV-bound. □

Proof of Lemma 3.5. In [30] the assertion was proved for Cauchy problems, see also [28]. We proceed similarly, but take the boundary data into account. Let without restriction $\tau_0 = 0$. Define $\ell_j^n := \frac{y_{j+1}^n - y_j^n}{\Delta x}$. We analyze first ℓ_0^n, $n = 1,\dots,N_T$, i.e., the behavior at the boundary. By assumption, $f'(u_B) \ge f'(\gamma) = \beta > 0$ on $[0,T]$ holds for some $\gamma > \sigma$, hence we have $y_0^n > \sigma$ for all $n = 1,\dots,N_T$. Moreover, $g = 0$ for $x \le \varepsilon_g$ and thus $G_1^n = 0$. We have to distinguish the following cases:

Case 1: $y_1^n \ge \sigma$, $y_2^n \ge \sigma$. The mEO-scheme reads $y_1^{n+1} = y_1^n - \lambda(f(y_1^n) - f(y_0^n))$, which yields $y_1^{n+1} - y_0^{n+1} = y_0^n - y_0^{n+1} + y_1^n - y_0^n - \lambda(f(y_1^n) - f(y_0^n))$. Now, since $f'' \ge m_{f''} > 0$ holds, we obtain

$$f(y_1^n) - f(y_0^n) \ge f'(y_0^n)(y_1^n - y_0^n) + \frac{m_{f''}}{2}(y_1^n - y_0^n)^2.$$

Hence,

$$\begin{aligned}y_1^{n+1} - y_0^{n+1} &\le y_0^n - y_0^{n+1} + (y_1^n - y_0^n)\left(1 - \lambda f'(y_0^n) - \frac{\lambda m_{f''}}{2}(y_1^n - y_0^n)\right)\\ &\le y_0^n - y_0^{n+1} + (y_1^n - y_0^n)_+(1 - \lambda f'(y_0^n)).\end{aligned}$$

Thus, we have

$$y_1^{n+1} - y_0^{n+1} \le y_0^n - y_0^{n+1} + (y_1^n - y_0^n)_+(1 - \lambda f'(\gamma)).$$

Case 2: $y_1^n \ge \sigma,\ y_2^n < \sigma$. Then the mEO-scheme reads $y_1^{n+1} = y_1^n - \lambda(f(y_1^n) + f(y_2^n) - f(\sigma) - f(y_0^n))$. This yields

$$y_1^{n+1} - y_0^{n+1} \le y_0^n - y_0^{n+1} + y_1^n - y_0^n - \lambda(f(y_1^n) - f(y_0^n))$$

and we can proceed as in Case 1.

Case 3: $y_1^n < \sigma,\ y_2^n \ge \sigma$. The mEO-scheme is given by $y_1^{n+1} = y_1^n - \lambda(f(\sigma) - \max\{f(y_0^n), f(y_1^n)\})$ and we obtain

$$y_1^{n+1} - y_0^{n+1} = y_0^n - y_0^{n+1} + y_1^n - y_0^n - \lambda(f(\sigma) - \max\{f(y_0^n), f(y_1^n)\}).$$

The grid points y_j^n are bounded, so f has a Lipschitz constant L_f. Therefore

$$-\lambda(f(\sigma) - \max\{f(y_0^n), f(y_1^n)\} \le \lambda L_f \max\{y_0^n - \sigma, \sigma - y_1^n\} \le \lambda L_f(y_0^n - y_1^n).$$

Hence, under the CFL-condition $\lambda L_f \le 1$ it holds

$$y_1^{n+1} - y_0^{n+1} \le y_0^n - y_0^{n+1} + (y_0^n - y_1^n)(-1 + \lambda L_f) \le y_0^n - y_0^{n+1}.$$

Case 4: $y_1^n < \sigma,\ y_2^n < \sigma$. The mEO-scheme reads $y_1^{n+1} = y_1^n - \lambda(f(y_2^n) - \max\{f(y_0^n), f(y_1^n)\})$. This yields

$$y_1^{n+1} - y_0^{n+1} \le y_0^n - y_0^{n+1} + y_1^n - y_0^n - \lambda(f(\sigma) - \max\{f(y_0^n), f(y_1^n)\})$$

and we proceed as in Case 3. Altogether, we obtain

$$y_1^{n+1} - y_0^{n+1} \le y_0^n - y_0^{n+1} + (y_1^n - y_0^n)_+(1 - \lambda f'(\gamma)). \tag{4.6}$$

Let $\tilde{C}_0 = \max\{0, \frac{y_1^0 - y_0^0}{\Delta x}\}$. Since the boundary data generate by assumption no rarefaction centers, we find a constant $C_B \ge 0$ with $\lambda \frac{y_0^n - y_0^{n+1}}{\Delta t} \le C_B$ for $n = 0, \dots, N_T - 1$. We define a sequence a_n by $a_{n+1} \le C_B + m a_n$ with $a_0 \le \tilde{C}_0$ and $m = 1 - \lambda f'(\gamma) \in]0, 1[$. This yields $a_n \le m^n \tilde{C}_0 + C_B \sum_{i=0}^{n-1} m^i \le m^n \tilde{C}_0 + \frac{C_B}{1-m}$ and with (4.6) we deduce

$$\ell_0^n \le m^n \tilde{C}_0 + \frac{C_B}{1-m} =: C_n, \quad n = 0, \dots, N_T. \tag{4.7}$$

Let $L_g \ge 0$ be the Lipschitz constant of g with respect to x. By extending results of [30, Lem. 6.5.2] for the EO-scheme to the mEO-scheme, see below, we obtain with some $\nu > 0$ for all $0 < c \le \nu$ the estimate

$$\ell_j^{n+1} \le \ell_{j,1}^{n,+} - \Delta t c(\ell_{j,1}^{n,+})^2 + \Delta t L_g, \quad \ell_{j,1}^{n,+} := \max_{k=-1,0,1}\{(\ell_{j+k}^n)_+\},\ j \ge 1. \tag{4.8}$$

Set $\psi(\ell) := \ell - \Delta t c \ell^2 + \Delta t L_g$. We derive now an upper bound for ℓ_j^n. We observe that $\ell_j^n \leq 2M_y/\Delta x + C_B/(1-m)$ for $j \geq 0$. We clearly find a maximal $0 < c \leq \nu$ such that it holds for all $0 < \Delta \leq \Delta_0$

$$\psi'(\ell) = 1 - 2c\Delta t \ell \geq m \quad \forall\, \ell \in \left[0, \max\{2M_y/\Delta x + 2C_B/(1-m), \sqrt{L_g/c}\}\right]. \tag{4.9}$$

The latter interval contains all $\ell_{j,1}^{n,+}$, $j \geq 1$ and all C_n and ψ has the unique fixed point $\ell = \sqrt{L_g/c}$ on the interval. With (4.7), (4.8) we obtain $\ell_j^{n+1} \leq \max\{C_{n+1}, \psi(\ell_{j,1}^{n,+})\}$ for all $j \geq 0$.

Define $M_n := \sup_{j\geq 1} \max\{\ell_j^n, C_n\}$, then the monotonicity of ψ yields $\ell_j^{n+1} \leq \max\{C_{n+1}, \psi(M_n)\}$ and thus

$$M_{n+1} \leq \max\{C_{n+1}, \psi(M_n)\}. \tag{4.10}$$

Now define

$$\bar{M}_{n+1} := \max\{\tilde{C}_{n+1}, \psi_{\tilde{c}}(\bar{M}_n)\}, \quad \bar{M}_0 = M_0$$

then the monotonicity of $\psi_{\tilde{c}}$ yields $\bar{M}_n \geq M_n$ for all $n \geq 0$. We consider two cases.

Case 1: $\bar{M}_0 \leq \sqrt{L_g/c}$. Then by (4.8) and the fact that $\sqrt{L_g/c}$ is the unique fixed point on the interval in (4.8), we have $\bar{M}_0 \leq \psi(\bar{M}_0) \leq \sqrt{L_g/c}$ and $C_1 \leq C_0 \leq \bar{M}_0$. Hence, $\bar{M}_1 = \psi(\bar{M}_0)$ and we obtain inductively $\bar{M}_{n+1} = \psi(\bar{M}_n)$, $\bar{M}_{n+1} \leq \sqrt{L_g/c}$ for all $n \geq 0$

Case 2: $\bar{M}_0 > \sqrt{L_g/c}$. Then we obtain similarly as in Case 1 that $(\bar{M}_n)$ is a decreasing sequence $> \sqrt{L_g/c}$. An elementary investigation of the quadratic function

$$\psi(C_n) - C_{n+1} = C_n - \Delta t c C_n^2 + \Delta t L_g - mC_n - C_B =: q(C_n)$$

yields that $q(C)$ has a minimum at $\bar{C} = (1-m)/(2c\Delta t)$ with value $q(\bar{C}) = (1-m)^2/(4c\Delta t) - C_B + \Delta t L_g$. Now (4.9) implies $\bar{M}_n \leq \bar{C}$, $n \geq 0$, as well as $q(\bar{C}) \geq 0$. Therefore, q has a unique zero at some some $C_\Delta \in [(C_B - \Delta t L_g)/(1-m), 2(C_B - \Delta t L_g)/(1-m)]$ and

$$\psi(C_n) - C_{n+1} \begin{cases} \geq 0 & \text{for } C_n \in [C_\Delta, (1-m)/(2\tilde{c}\Delta t)], \\ \leq 0 & \text{for } C_n \leq C_\Delta, \end{cases}$$

Hence, we obtain $\bar{M}_{n+1} = \psi(\bar{M}_n)$ until the first $n = n_1$ with $\bar{M}_{n_1} < C_\Delta, \psi(\bar{M}_{n_1}) \leq C_{n_1+1}$ and from that point on we obtain $\bar{M}_n = C_n$ for $n \geq n_1 + 1$.

In other words, if we define the sequence $\tilde{M}_{n+1} = \psi(\tilde{M}_n)$, $\tilde{M}_0 = M_0$ we obtain $\bar{M}_n = \max\{\tilde{M}_n, C_n\}$. To estimate $\tilde{M}_n$ we note that $\frac{1}{\Delta t}(\tilde{M}_{n+1} - \tilde{M}_n) = -c\tilde{M}_n^2$. The solution of the initial value problem $\dot{\alpha}(t) = -c\alpha(t)^2$, $\alpha(0) = \tilde{M}_0$ satisfies $\alpha(n\Delta t) \geq \tilde{M}_n$ and is given by $\alpha(t) = (ct + 1/\tilde{M}_0)^{-1}$. Hence, we can conclude $\bar{M}_n = \max\{C_n, \tilde{M}_n\} \leq \max\{C_n, \alpha(t_n)\}$. Finally, (4.9) implies easily that $\psi(\tilde{M}_n) \geq m\tilde{M}_n$ and thus $\tilde{M}_n \geq m^n C_0$ for $n \geq 0$ leading to

$$\bar{M}_n \leq \frac{C_B}{1-m} + \frac{1}{ct_n + \frac{1}{M_0}}.$$

It remains to prove that the modified Engquist-Osher scheme satisfies (4.8) under the assumed CFL condition. In [30, Lemma 6.5.2] it was shown that (4.8) holds for the Engquist-Osher scheme with $\nu = \min\{m_{f''/4, 1/(4\lambda M_y)}\}$. The mEO-scheme differs only for $j = 1$ in the transonic case $y_1^n < \sigma < y_0^n$, where

$$F^{EO}(y_0^n, y_1^n) = f(y_1^n) + f(y_0^n) - f(\sigma) \geq \max\{f(y_1^n), f(y_0^n)\} = F^G(y_0^n, y_1^n).$$

Hence, we have $(\ell_1^{n+1})^{EO} \geq (\ell_1^{n+1})^{mEO}$ and thus [30, Lemma 6.5.2] holds with $j = 1$ also for the mEO scheme. ☐