CONVERGENCE OF NUMERICAL ADJOINT SCHEMES ARISING FROM OPTIMAL BOUNDARY CONTROL PROBLEMS OF HYPERBOLIC CONSERVATION LAWS*

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Abstract. We study the convergence of discretization schemes for the adjoint equation arising in the adjoint-based derivative computation for optimal boundary control problems governed by entropy solutions of conservation laws with source term. As boundary control we consider piecewise continuously differentiable controls with possible discontinuities at switching times, where the smooth parts as well as the switching times serve as controls. The derivative of tracking-type objective functionals with respect to the smooth controls and the switching times can then be represented by an adjoint-based formula. The main difficulties arise from the fact that the correct adjoint state is the reversible solution of a transport equation with discontinuous coefficient and that the boundary conditions lead in general to discontinuous adjoints. We prove that discrete adjoint schemes of monotone difference methods in conservation form such as the Engquist-Osher scheme in combination with the Godunov flux at the boundary converge to the reversible solution of the adjoint equation. We also allow that the state is computed by another numerical scheme satisfying certain convergence properties.

Key words. scalar conservation laws, optimal control, finite difference scheme, adjoint equation, convergence analysis

AMS subject classifications. 49K20, 49M25, 65M06, 65K10, 35B37

1. Introduction.

We consider optimal initial-boundary control problems for entropy solutions of scalar conservation laws

$$\begin{aligned} &y_t + f(y)_x = g \quad \text{on } \Omega_T \coloneqq]0,T[\times]0,\infty[, \\ &y(0,\cdot) = u_0 \quad \text{on } \Omega \coloneqq]0,\infty[, \quad y(\cdot,0+) = u_B \quad \text{on }]0,T[\text{ in the sense of [3]}, \end{aligned} \tag{1.1}$$

where $f \in C^2_{loc}(\mathbb{R})$ with $f'' \geq m_{f''} > 0$ and sonic point σ, i.e. $f'(\sigma) = 0$, and g is a source term satisfying for an $\varepsilon_g > 0$

$$g \in C^2_b(\Omega_T) \cap L^1(0,T;(L^1 \cap BV)(\Omega)), \quad g(t,x) = 0 \text{ for } t < \varepsilon_g \text{ or } x < \varepsilon_g. \tag{1.2}$$

For $k \in \mathbb{N}_0$, $\Omega \subset \mathbb{R}^n$, we denote by $C^k(\Omega)$ is the set of k-times continuously differentiable functions and by $C^k_b(\Omega)$ the subspace of C^k with bounded C^k-norm. $C^k_b(\Omega^{cl})$ is the subspace of $C^k_b(\Omega)$ where all derivatives admit a continuous extension to Ω^{cl}. $C^k_c(\Omega)$ is the space of C^k-functions with compact support. $C^{k,\alpha}(\Omega)$, $C^{k,\alpha}_b(\Omega)$ are the corresponding Hölder spaces. $B(\Omega)$ is the space of bounded functions with the sup-norm. $BV(\Omega)$ is the space of functions of bounded variation.

We consider controls $u = (u_0, u_B) \in (L^1 \cap BV)(\Omega) \times (L^1 \cap BV)(0,T)$. It is well known that in general weak solutions of (1.1) develop discontinuities after finite time and that uniqueness holds only in the class of entropy solutions (see [3]). By definition $y = y(u) \in L^\infty(\Omega_T)$ is an *entropy solution* of (1.1), if for all convex functions $\eta \in C^{0,1}(\mathbb{R})$ with corresponding fluxes $q(y) = \int_0^y \eta'(s)f'(s)\,\mathrm{d}s$ the entropy inequality

$$\eta(y)_t + q(y)_x \leq \eta'(y)g \quad \text{in } \mathcal{D}'(\Omega_T)$$

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is satisfied and the initial data are attained in L^1, i.e. for arbitrary $R > 0$ it holds $\lim_{\tau \searrow 0} \frac{1}{\tau} \int_0^\tau \|y(t,\cdot) - u_0\|_{1,]0,R[}\, \mathrm{d}t = 0$. To obtain well-posedness, it is standard to impose the boundary condition in (1.1) in the sense, see [3],

$$\operatorname{sgn}(u_B - y(\cdot,0+))(f(y(\cdot,0+) - f(k)) \geq 0,\ \forall k \in I(y(\cdot,0+),u_B) \text{ a.e. on }]0,T[\tag{1.3}$$

with $I(\alpha,\beta) \coloneqq [\min\{\alpha,\beta\}, \max\{\alpha,\beta\}]$. This ensures that the boundary data are only attained at inflow regions. We consider objective functionals of the form

$$J(y) \coloneqq \int_\Omega \gamma(x)\psi(y(T,x), y_d(x))\, \mathrm{d}x \tag{1.4}$$

with $\psi \in C^{1,1}(\mathbb{R}^2)$, desired state $y_d \in BV_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and a weight $\gamma(x) \in C_c^1(\Omega)$.

As initial and boundary data, we consider piecewise continuously differentiable functions $u_0 \in PC^1(\Omega; x_1, \ldots, x_{n_x})$ and $u_B \in PC^1([0,T]; t_1, \ldots, t_{n_t})$ with possible discontinuities at $0 < x_1 < \cdots < x_{n_x}$ and at $0 < t_1 < \cdots < t_{n_t} < T$, respectively. To this end, we set as control

$$\begin{aligned} w &= ((u_j^0)_{1\leq j\leq n_x+1}, (u_j^B)_{1\leq j\leq n_t+1}, (x_j)_{1\leq j\leq n_x}, (t_j)_{1\leq j\leq n_t}) \\ &\in C_b^1([0,\infty[)^{n_x+1} \times C_b^1([0,T])^{n_t+1} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_t} =: W \end{aligned}$$

and define with $x_0 = 0$, $x_{n_x+1} = \infty$, $t_0 = 0$, $t_{n_t+1} = T$

$$u_0(x;w) = \sum\nolimits_{j=1}^{n_x+1} u_j^0(x)\mathbf{1}_{]x_{j-1},x_j]}, \quad u_B(t;w) = \sum\nolimits_{j=1}^{n_t+1} u_j^B(x)\mathbf{1}_{]t_{j-1},t_j]}. \tag{1.5}$$

Moreover, we define the jumps

$$[u_0(x_j)] = u_j^0(x_j) - u_{j+1}^0(x_j), \quad [u_B(t_j)] = u_j^B(t_j) - u_{j+1}^B(t_j).$$

In [24, 26] it was shown that at a control w, for which $u_0(\cdot;w)$ and $u_B(\cdot;w)$ satisfy a generic non-degeneracy assumption (see (ND) below), the mapping

$$w \in W \mapsto J(y(u_0(\cdot;w), u_B(\cdot;w)))$$

is Fréchet-differentiable. Now let

$$\begin{aligned} W_{ad} = \{&((u_j^0)_{1\leq j\leq n_x+1}, (u_j^B)_{1\leq j\leq n_t+1}, (x_j)_{1\leq j\leq n_x}, (t_j)_{1\leq j\leq n_t}) \in W : \\ &0 < x_1 < \cdots < x_{n_x},\ 0 < t_1 < \cdots < t_{n_t} < T,\ f'(u_j^B) > 0,\ 1 \leq j \leq n_t+1\}. \end{aligned}$$

The boundary condition remains unchanged, if we assume $f'(u_B) > 0$, see [23].

As shown in [24, 26], the gradient of the objective functional (1.4) admits an adjoint representation. Let $\delta w = (\delta u_0, \delta u_B, \delta x, \delta t) \in W$ be an arbitrary variation of w In this paper, as in [24] the smooth control parts u_j^0 and u_j^B as well as shock generating switching locations x_j or switching times t_j can be controlled, i.e.,

$$\delta x_j = 0 \ \text{ if } [u_0(x_j)] \leq 0 \quad \text{and} \quad \delta t_j = 0 \ \text{ if } [u_B(t_j)] \geq 0. \tag{1.6}$$

In this case, the adjoint representation reads

$$d_u J(y(u)) \cdot \delta u = \int_\Omega p(0,x)\delta u_0(\mathrm{d}x) + \int_0^T p(t+,0+)\hat{f}'(y(t,0+))\delta u_B(\mathrm{d}t) \tag{1.7}$$

with $\hat{f}'(y(t,0+)) = \int_0^1 f'((1-\lambda)y(t+,0+) + \lambda y(t-,0+))\,\mathrm{d}\lambda$. Here, p is the reversible solution of the adjoint equation (see Definition 2.9)

$$
\begin{aligned}
(1.8)\qquad & p_t + f'(y)p_x = 0 \quad \text{on } \Omega_T,\\
& p(T,\cdot) = p^T \quad \text{on } \Omega, \quad p(\cdot,0+) = p^B \text{ on } \{t \in]0,T[: f'(y(t,0+)) < 0\}.
\end{aligned}
$$

with constant boundary data $p^B \in \mathbb{R}$. The adjoint formula (1.7) requires the data

$$
(1.9)\qquad p^B = 0, \quad p^T(x) = \begin{cases} \gamma(x)\psi_y(y(T,x), y_d(x)) & \text{if } y(T,\cdot) \text{ continuous at } x,\\ \gamma(x)\frac{[\psi(y(T,x),y_d(x))]}{[y(T,x)]} & \text{if } y(T,\cdot) \text{ discontinuous at } x. \end{cases}
$$

We note that the adjoint formula (1.7) can also be extended to the case, that rarefaction centers can be controlled (then (1.6) does not apply). In fact, then (1.7) holds with a particular definition of $p(0,x_j)$ and $p(t_j,0+)$ at rarefaction centers obtained by a weighted average of the adjoint state over the rarefaction wave, see [26]. Due to space limitations we will assume (1.6) for the convergence analysis of discrete adjoint schemes and will not treat the shift of rarefaction centers.

Hence, objective functions (1.4) lead to discontinuous end data (1.9), if $y(T,\cdot)$ has a shock in the support of γ. The computation of adjoints with discontinuous end data requires care, since the correct value has to be propagated in the shock funnel. This requires an increasing numerical smoothing at the shock location or an appropriate modification of the end data of the adjoint scheme, since otherwise the discrete adjoint may converge to wrong values in the shock funnels [4, 13, 14, 25].

In this paper, we will restrict the analysis to Lipschitz end data $p^T \in C^{0,1}(\Omega)$. The developed convergence results can then be applied to objective functionals using smoothed states, e.g., by convolution, or can be utilized when using the method from [25], where a suitable approximation of the discontinuous end data p^T is proposed.

As discussed in [8] the solution of (1.8) is not unique if the state contains a shock. Therefore, it requires an appropriate characterization of the "correct" adjoint state, which is for the Cauchy problem , i.e., $\Omega = \mathbb{R}$, the reversible solution introduced in [6]. Moreover, [6] develops a characterization of reversible solution by a monotonicity criterion, which is also well suitable for numerical approximations. For the case of Cauchy problems this approach was already used in [15, 29]. In this paper, we will first develop a similar monotonicity criterion for reversibility in the setting of transport equations with boundary conditions of the type (1.8). This criterion will then be used to prove convergence of the adjoint scheme.

There are only few publications discussing the discretization of the related optimal control problems. For the case of initial control problems, [15, 30, 31] considered first order approximations of transport equation having a coefficient satisfying at least a weak one-sided Lipschitz condition and sufficiently smooth end data. Moreover, the authors in [17] propose a second order Roe-type scheme whereas using total variation diminishing Runge-Kutta (TVD-RK) methods for the time discretization, a second order adjoint approximation is developed in [16]. For discontinuous end data p^T, [13, 14] propose a convergent modified Lax-Friedrich-scheme with sufficient viscosity.

The case of boundary control poses several additional challenges. Boundary conditions (1.3) [3, 21] lead to a free boundary problem, since the determination of the outflow region is part of the problem. While the state attains the boundary data on the inflow boundary, the adjoint attains its boundary data at the outflow boundary. After discretization, this requires a careful convergence study of the discrete state and the discrete adjoint on the first grid cell at the boundary to the correct values (boundary data or values of the state / adjoint propagated from the interior). Moreover, also the characterization of reversible solutions of the adjoint equation in a way suitable for the convergence analysis of adjoint schemes is more involved. We will show a convergence result, if the Godunov flux is used on the boundary cell.

The paper is organized as follows: In Section 2 we recall some facts on the state equation and introduce a convenient characterization for the reversible solution of the adjoint equation. In Section 3 we consider monotone difference schemes and the corresponding adjoint scheme. For a modified Engquist-Osher scheme that uses the Godunov flux at the boundary we show the convergence of the discrete adjoint to the correct reversible solution.

2. Continuous problem. We collect some important properties of entropy solutions and analyze the behavior at the boundary.

PROPOSITION 2.1. *Let $f \in C^2(\mathbb{R})$ with $f'' \geq m_{f''} > 0$, $g \in C_b^2(\Omega_T)$ and $M_u > 0$. Let $U_{ad} := \{(u_0, u_B) \in (L^\infty \cap BV)(\Omega) \times (L^\infty \cap BV)(0,T) : \|u_0\|_\infty + \|u_B\|_\infty \leq M_u\}$. Then for all $u \in U_{ad}$ there exists a unique entropy solution $y = y(u) \in L^\infty(\Omega_T)$. After modification on a set of measure zero, one has $y \in C([0,T]; L^1_{loc}(\Omega))$. There are $M_y, L_y > 0$ such that for all $u, \hat{u} \in U_{ad}$ and all $t \in [0,T]$ the following estimates hold*

$$\|y(t,\cdot;u)\|_\infty \leq M_y,$$
$$\|y(t,\cdot;u) - y(t,\cdot;\hat{u})\|_{1,[a,b]} \leq L_y(\|u_0 - \hat{u}_0\|_{1,[a-tM_{f'},b+tM_{f'}]} + \|u_B - \hat{u}_B\|_{1,[0,t]}),$$

where $a < b$ and $M_{f'} = \max_{|s|\leq \max(\|u\|_\infty, \|\hat{u}\|_\infty)} |f'(s)|$. Moreover, $y \in BV(\Omega_T)$.

Finally, for any interval $[\tau_0, \tau_1] \subset [0,T]$ such that $u_B'|_{[\tau_0,\tau_1]} \geq -C_B$ with some constant $C_B \geq 0$ there exist constants $c, C > 0$ such that

$$y_x(t,x) \leq \frac{1}{c(t-\tau_0) + 1/\sup((y_x(\tau_0,\cdot))_+)} + C, \quad (t,x) \in]\tau_0,\tau_1] \times \Omega. \tag{2.1}$$

Moreover, for any $\varepsilon > 0$ there is $C = C(\varepsilon) > 0$ such that (2.1) holds on $]0,T] \times [\varepsilon, \infty[$.

Proof. See, e.g., [3, 21, 24]. BV-regularity and one-sided Lipschitz condition follow by combining results in [8, 19, 18, 27], see [23]. (2.1) follows also from Lemma 3.5. □

Let us first consider the case where the controls only generate shocks, i.e., $u_0(x-) \geq u_0(x+)$ for all $x \in \Omega$ and $u_B(t-) \leq u_B(t+)$ for all $t \in]0,T[$, this assures that the one-sided Lipschitz condition (OSLC) is satisfied on Ω_T^{cl}, see [24], meaning that

$$(f'(y(t,\cdot)))_x \leq \alpha(t) \quad \text{for } \alpha \in L^1(0,T). \tag{2.2}$$

If the initial control has upward-jumps or the boundary control has downward-jumps generating rarefaction waves then only a weak OSLC is satisfied and a OSLC (2.2) holds by (2.1) outside of any neighborhood of the rarefaction centers. In any case, we have $y(t,x+) < y(t,x-)$ for almost all t and all $x \in \Omega$. This allows us to apply Dafermos' theory of generalized characteristics [10].

DEFINITION 2.2 (Generalized characteristics). *A Lipschitz curve $[\alpha,\beta] \subset [0,T] \to \Omega_T$, $t \mapsto (t,\xi(t))$ is called* (generalized) characteristic *on $[\alpha,\beta]$, if*

$$\dot{\xi}(t) \in [f'(y(t,\xi(t)+)), f'(y(t,\xi(t)-))] \quad \text{a.e. on } [\alpha,\beta].$$

A characteristic is called genuine *on $[\alpha,\beta]$, if $y(t,\xi(t)+) = y(t,\xi(t)-)$ for a.a $t \in [\alpha,\beta]$. A characteristic is called* minimal/maximal*, if for almost every $t \in [\alpha,\beta]$ it holds $\dot{\xi}(t) = f'(y(t,\xi(t)-))$ respectively $\dot{\xi}(t) = f'(y(t,\xi(t)+))$. We write $\xi = \xi_+$ if ξ is maximal and $\xi = \xi_-$ if ξ is minimal, respectively. A generalized characteristic is called* extreme*, if it is either minimal or maximal.*

It is known that generalized characteristics travel either with classical speed or with shock speed. Moreover, extreme backward characteristics are genuine. Moreover, if ξ is genuine, i.e., $y(t,\xi(t)-) = y(t,\xi(t)+)$ for almost all $t \in]\alpha,\beta[$, then

$$\xi(t) = \zeta(t), \quad y(t,\xi(t)) = v(t),\ t \in]\alpha,\beta[,$$

where (ζ, v) is a solution of the characteristic equation

$$\dot{\zeta}(t) = f'(v(t)) \quad \text{and} \quad \dot{v}(t) = g(t,\zeta(t)). \tag{2.3}$$

We refer to [10], [22], and [24, Prop. 2.4, 2.5].

Definition 2.3 (Transition points). *We call a point $\theta_i \in]0,T[$* transition point, *if for all sufficiently small $\delta > 0$ the sets $\{t \in]\theta_i - \delta, \theta_i[: f'(y(t,0+)) < 0\}$ and $\{t \in]\theta_i, \theta_i+\delta[: f'(y(t,0+)) \geq 0\}$ have positive measure. If the extreme (i.e., maximal) backward characteristic ξ_i through $(\theta_i, 0)$ ends at some point $(\vartheta_i, 0)$ with $\vartheta_i > 0$, we call ϑ_i* return point. *The set of transition points is denoted by $\mathbb{T}$.*

Note that the maximal backward characteristic through a transition point exists due to [22], see also [24, Proposition 2.5].

Lemma 2.4 (Structure at the boundary). *Let $f \in C^2(\mathbb{R})$ be strictly convex and let $y = y(u)$ be the entropy solution of (1.1) for $u = (u_0(w), u_B(w))$ given by (1.5), $w \in W_{ad}$. Assume that $\operatorname{ess\,inf}_{\{t\colon u_B(t) \neq y(t,0+)\}} |f(u_B(t)) - f(y(t,0+))| > 0$ holds, then the set $\mathbb{T}$ is finite with $|\mathbb{T}| = \mathfrak{n}_T$. Moreover, it holds that $0 < \theta_1 < \theta_2 < \cdots < \theta_{\mathfrak{n}_T-1} < \theta_{\mathfrak{n}_T} < T$. In addition, each $\theta \in \mathbb{T}$ is a shock generating point with $u_B(\theta+) > u_B(\theta-)$.*

Proof. A proof is given in [23, Lemma 3.1.17] and [24, Lemma 2.7]. ∎

Definition 2.5. *In the setting of Lemma 2.4 we define:*

(i) We call a transition point $\theta_i \in \mathbb{T}$ secondary, *if there exists $j \in \{1,\dots,\mathfrak{n}_T\}$ with $j \neq i$ such that $\theta_i \in]\vartheta_j, \theta_j[$. Otherwise, the transition point is called* primary. *We write $\mathbb{T}^p \subset \mathbb{T}$ for the set of primary transition points.*

(ii) If outflow occurs at T, i.e., $\lim_{x\searrow 0} f'(y(T,x)) < 0$, we denote by ξ_T the extreme backward characteristic through $(T,0)$, which either ends at some point (T_-, z) with $T_- = 0$, $z \in \Omega$ or at $(T_-, 0)$ with $0 \leq T_- < T$. If inflow occurs T, i.e., $\lim_{x\searrow 0} f'(y(T,x)) > 0$, we set $T_- = T$. With the maximal backward characteristic $\xi_i = \xi(\cdot;\theta_i, 0)$, the outflow domain *is defined by*

$$D^- := \bigcup_{\substack{i=1,\dots,\mathfrak{n}_T+1,\\ \theta_i \in \mathbb{T}^p}} \{(t,x) \in \Omega_T \colon t \in]\vartheta_i, \theta_i[,\ x \in]0, \xi_i(t)[\},$$

where we set $\theta_{\mathfrak{n}_T+1} := T$, $\vartheta_{\mathfrak{n}_T+1} := T_-$ and $\xi_{\mathfrak{n}_T+1} := \xi_T$.

(iii) Denote by $\zeta(\cdot;\theta,s,w), v(\cdot;\theta,s,w)$ the solution of (2.3) for the initial condition

$$\zeta(\theta;\theta,s,w) = s, \quad v(\theta;\theta,s,w) = w.$$

We say the transition point $\theta_i \in \mathbb{T}$ is non-degenerated, *if the extreme backward characteristic ξ_i arrives in $(0,z)$ or $(\vartheta_i, 0)$ from the interior of a rarefaction wave or ends in a point $(0,z)$ or $(\vartheta_i,0)$ where u_0 or u_B, respectively, are continuously differentiable. In the latter case, we require that for some $\beta > 0$ the solution ζ of (2.3) fulfills*

$$\begin{aligned} &\tfrac{\mathrm{d}}{\mathrm{d}s}\zeta(t;0,s,u_0(s))|_{s=z} \geq \beta && \forall\ t \in [0,\theta_i], \\ \textit{resp.}\quad &\tfrac{\mathrm{d}}{\mathrm{d}\theta}\zeta(t;\theta,0,u_B(\theta))|_{\theta=\vartheta_i} \leq -\beta && \forall\ t \in [\vartheta_i,\theta_i]. \end{aligned}$$

Conversely, if ζ ends in the interior of a rarefaction wave, it has to hold

$$\begin{aligned} &\tfrac{\mathrm{d}}{\mathrm{d}w}\zeta(t;0,\bar{s},w)|_{w=\bar{w}} \geq \beta, && \forall t \in [0,\theta_i],\ \bar{w} \in]u_0(\bar{s}-),u_0(\bar{s}+)[, \\ \textit{resp.}\quad &\tfrac{\mathrm{d}}{\mathrm{d}w}\zeta(t;\theta,0,w)|_{w=\bar{w}} \geq \beta(t-\vartheta_i), && \forall t \in [\vartheta_i,\theta_i] \end{aligned}$$

with $\bar{w} \in]u_B(\vartheta_i+),u_B(\vartheta_i-)[$.

Remark 2.6. The non-degeneracy condition of transition points according to Definition 2.5(iii) coincides with the one in [24, Definition 4.12] which is:
If $\theta_i \in \mathbb{T}$ is non-degenerated, we can as in [29, Lemma 3.5.1, Lemma 3.5.6] or [24, Lemma 4.2, Lemma 4.4] construct a stripe $S_i \subset \Omega_T$ around ξ_i of the form

$$S_i = \{(t,x) : t \in]\vartheta_i, \theta_i + \varepsilon] : x \in [\xi(t;\theta_i - \varepsilon, 0), \xi(t;\theta_i+\varepsilon,0)]\} \tag{2.4}$$

with genuine backward characteristics $\xi(\cdot;\theta_i \mp \varepsilon, 0)$ emanating from $(\theta_i \mp \varepsilon, 0)$ and $\varepsilon > 0$ small enough (we set $\xi(t;\theta_i - \varepsilon, 0) = 0$ outside of its domain of definition) such that there exists a local solution $Y \in C_b^{0,1}(S_i)$ that coincides with the solution y of (1.1) on $S_i \cap ([\vartheta_i, \theta_i[\times \Omega)$. If η_i denotes the shock emanating from $(\theta_i, 0)$ then it is easy to see that also $y = Y$ on $\hat{S}_i := \{(t,x) \in S_i : t \geq \vartheta_i,\ x \geq \eta_i(t)\}$. With our regularity of the source term even $Y \in C_b^1(S_i)$ holds.

From a geometrical point of view, the description of non-degeneracy in Remark 2.6 is more convenient as the one in Definition 2.5, so we will mainly work with it.

Next, we define the non-degeneracy of shocks, see [24, 26, 30, 31].

Definition 2.7. *A discontinuity $\bar{x}$ of $y(T,\cdot)$ is called* non-degenerated*, if it is neither a shock interaction nor a shock generation point and the corresponding minimal/maximal backward characteristic through $(T,\bar{x})$ ends in some point $(0,z)$ or $(\tilde{t},0)$, where u_0 or u_B is continuously differentiable, respectively, or which lies in the interior of a rarefaction wave that is created either by a discontinuity of u_0 or u_B.*

Consequently one can construct a stripe $S^{\pm}$ along the minimal/maximal characteristic $\xi^{\pm}$, such that y is Lipschitz continuous on $\{(t,x) \in S^- : x < \eta(t;T,\bar{x})\}$ and $\{(t,x) \in S^+ : x > \eta(t;T,\bar{x})\}$, where $\eta(\cdot;T,\bar{x})$ denotes the shock through $(T,\bar{x})$, see [30].

We will work under the following non-degeneracy condition from [26].

- (ND) A control $u = (u_0(w), u_B(w))$ given by (1.5), $w \in W_{ad}$, is called non-degenerated if the following holds: There is no point $x \in \Omega$ or $t \in [0,T]$ such that u_0 or u_B are continuous, but not differentiable. The associated entropy solution $y(T,\cdot;u)$ has no shock generation points and a finite number of discontinuities $0 < \bar{x}_1 < \cdots < \bar{x}_K$, which are all non-degenerated in the sense of Definition 2.7. Moreover, $\operatorname{ess\,inf}_{\{t\colon u_B(t) \neq y(t,0+)\}} |f(y(t,0+)) - f(u_B(t))| > 0$ is fulfilled and all transition points $\theta_i \in \mathbb{T}$ are non-degenerated according to Definition 2.5(iii).

It was shown for the initial value problem in [30, Thm. 3.8], that the requirements of (ND) for $y(T,\cdot;u)$ hold for almost all times T under the slight stronger regularity assumption $f \in C^3$, $u_0 \in PC^2$. Hence (ND) is a generic situation. Moreover, the following result can be shown.

Theorem 2.8. *Let $f \in C^3(\mathbb{R})$, $f'^{-1} \in C^{2,\beta}(\mathbb{R})$ and $f'' \geq m_{f''} > 0$ for constants $\beta \in]0,1]$, $m_{f''} > 0$. Let $\bar{w} \in W$ be such that $(u_0(\cdot;\bar{w}), u_B(\cdot;\bar{w}))$ given by (1.5) satisfy (ND). Then there is a neighborhood $B_{\rho,W}(\bar{w}) \subset W$ of $\bar{w}$ such that the mapping*

$$w \in B_{\rho,W}(\bar{w}) \subset W \mapsto y(u_0(\cdot;w), u_B(\cdot;w)))(T,\cdot) \in PC^1(\Omega; \bar{x}_1(w), \ldots, \bar{x}_K(w))$$

is well defined, the shock positions $w \in B_{\rho,W}(\bar{w}) \subset W \mapsto \bar{x}_k(w)$, $k = 1, \dots, K$, *are continuously differentiable and between the shocks*

$$w \in B_{\rho,W}(\bar{w}) \subset W \mapsto y(u_0(\cdot;w), u_B(\cdot;w)))(T,\cdot)|_{]\bar{x}_k(w),\bar{x}_{k+1}(w)[}$$

can be extended to a continuously differentiable mapping $w \in B_{\rho,W}(\bar{w}) \subset W \mapsto Y_i(w) \in C([\bar{x}_k(\bar{w}) - \varepsilon, \bar{x}_{k+1}(\bar{w}) + \varepsilon])$ *for* $\varepsilon > 0$ *sufficiently small.*

Proof. See [26, Thm. 17]. ∎

2.1. The adjoint equation. We consider the backward problem for transport equations of the form (1.8) with end data $p^T \in C_b^{0,1}(\Omega)$ and constant boundary value $p^B \in \mathbb{R}$. As already mentioned, the solution is not unique if y contains a shock [8]. A suitable stable solution to (1.8) is the reversible solution defined as in [24, 26]:

Definition 2.9 (Reversible solution). *Let* p^T *be a bounded function that is the pointwise everywhere limit of a sequence* $(p_n^T) \subset C_b^{0,1}(\Omega)$, *with* (p_n^T) *bounded in* $C(\Omega) \cap W_{loc}^{1,1}(\Omega)$ *and let* $p^B \in \mathbb{R}$. *Then the* reversible solution *associated to* (1.8) *is defined by the following requirements:*

(i) For all $\bar{x} \in \Omega$ *and all generalized backward characteristics* ξ *trough* $(T, \bar{x})$ *the solution of* (1.8) *is given by the characteristic equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\, p(t,\xi(t)) = 0 \text{ for } t \in]0,T[,\ \ \xi(t) > 0 \quad \text{and} \quad p(T,\xi(T)) = p^T(\bar{x}). \tag{2.5}$$

(ii) $p(t,x) = p^B$ *for all* $(t,x) \in D^-$.

Remark 2.10. In [6], the authors verify that the space of reversible solutions on $\mathbb{R}_T :=]0,T[\times\mathbb{R}$ for $p^T \in C^{0,1}(\mathbb{R})$ is a vector space. It is easy to see that the space of reversible solutions to (1.8) in the sense of Definition 2.9 is a vector space on $\Omega_T \setminus D^-$ and since they are constant on D^-, then also on Ω_T.

Proposition 2.11. *Let* $p^T \in C_b^{0,1}(\Omega)$ *and let* (2.2) *hold. Then for any* $t \in]0,T[$ *the reversible solution* p *of* (1.8) *according to Definition 2.9 satisfies*

$$\|p\|_{\infty,\Omega_T \setminus D^-} \le \|p^T\|_{\infty,\Omega}, \quad \|p_x\|_{\infty,([t,T]\times\Omega)\setminus D^-} \le e^{\int_t^T \alpha} \|p_x^T\|_{\infty,\Omega}. \tag{2.6}$$

Proof. Set $\tilde{p}^T(x) = p^T(\max\{x,0\}+)$ and extend the coefficient by $\tilde{a}(t,x) = f'(y(t,x))$ for $x \ge 0$ and $\tilde{a}(t,x) = M_{f'}$ for $x < 0$ with $M_{f'} := \sup_{|y|\le M_y} |f'(y)|$. Then $\tilde{a}$ satisfies the OSLC $\tilde{a}_x(t,\cdot) \le \alpha(t)$. Let $\tilde{p}$ be the reversible solution of $\tilde{p}_t + \tilde{a}\tilde{p}_x = 0$ on $\mathbb{R}_T$ with end data $\tilde{p}^T$ along generalized backward characteristics according to [6, Prop. 4.1.16] analogously to Definition 2.9. Then by [6, Thm. 4.1.5] $\tilde{p}$ satisfies (2.6) and by [24, Lem. 2.7(ii)] $\tilde{p}$ coincides with p on $\Omega_T \setminus D^-$, see also the proof of Proposition 2.12 below. ∎

This characterization is not well suited for the convergence analysis of numerical schemes. Therefore, we develop a more convenient monotonicity criterion for reversible solutions of (1.8), which extends the one of [6] for transport equations without boundary conditions.

Proposition 2.12 (Characterization of reversible solution by monotonicity). *Let* $f \in C^2(\mathbb{R})$ *be strictly convex, let* $y = y(u)$ *be the entropy solution of* (1.1) *for* $u = (u_0(w), u_B(w))$ *given by* (1.5), $w \in W_{ad}$, *satisfying* (ND) *and let* y *fulfill the OSLC* (2.2). *Choose end data* $p^T \in C^{0,1}(\Omega)$ *with* $p_x^T \ge 0$ *and boundary data* $p^B \in \mathbb{R}$. *If* $p = p^B$ *on* D^-, p *is a Lipschitz-solution on* $\Omega_T \setminus (D^-)^{cl}$ *and* $p_x \ge 0$ *on* $\Omega_T \setminus (D^-)^{cl}$, *then* p *is the reversible solution of* (1.8) *in the sense of Definition 2.9.*

Proof. We extend p to a Lipschitz solution on $\mathbb{R}_T$ and apply [6, Lemma 4.1.8]. First assume that there is only one transition point $\theta_1 \in]0,T[$ in the sense of Definition 2.3 and denote by $\vartheta_1 \in [0,\theta_1[$ the associated return point. Then the outflow domain is given by $D^- = \{(t,x) \in]\vartheta_1, \theta_1[\times \Omega \colon x < \xi(t;\theta_1,0)\}$, where $\xi(\cdot;\theta_1,0)$ is the maximal backward characteristic through $(\theta_1, 0)$. By the non-degeneracy assumption, there exists a stripe S around $\xi(\cdot;\theta_1,0)$ such that the entropy solution y is a classical solution of (1.1) on $\hat{S} = S \cap (]\vartheta_1 - \kappa, \theta_1[\times \Omega)$ for small $\kappa \geq 0$. Set $\tilde{p}^T(x) = p^T(\max\{x,0\}+)$ and extend the coefficient by $\tilde{a}(t,x) = f'(y(t,x))$ for $x \geq 0$ and $\tilde{a}(t,x) = M_{f'}$ for $x < 0$ with $M_{f'} \coloneqq \sup_{|y| \leq M_y} |f'(y)|$. Define the sets

$$\begin{aligned}
A_1^+ &\coloneqq \{(t,x) \in \mathbb{R}_T \colon M_{f'} \min\{t - \vartheta_1, 0\} \leq x \leq 0\}, \\
A_1^- &\coloneqq \{(t,x) \in \mathbb{R}_T \colon M_{f'}(t-T) < x \leq M_{f'} \min\{t - \theta_1, 0\}\}, \\
A_1^0 &\coloneqq \{(t,x) \in \mathbb{R}_T \colon M_{f'} \min\{t - \theta_1, 0\} < x < M_{f'} \min\{t - \vartheta_1, 0\}\}.
\end{aligned}$$

Let $\tilde{p}$ be a function on $\mathbb{R}_T$ with $\tilde{p}\mathbf{1}_{\Omega_T \setminus D^-} = p$ and

$$\tilde{p}(t,x) = \begin{cases} p\left(-\frac{x}{M_{f'}} + t, 0+\right) & \text{for } \ (t,x) \in A_1^- \cup A_1^+, \\ p^T(0+) & \text{for } \ x \leq M_{f'}(t-T), \\ p(\theta_1+, 0+) & \text{for } \ (t,x) \in A_1^0 \cup D^-. \end{cases}$$

We verify that $\tilde{p}$ is a Lipschitz solution of the extended equation $p_t + \tilde{a} p_x = 0$ on $\mathbb{R}_T$ for end data $\tilde{p}^T$ satisfying $\tilde{p}_x \geq 0$. By construction of $\tilde{a}$, $\tilde{p}$ is a Lipschitz solution of the extended adjoint equation on $A_1^\pm$ and on $\{(t,x) \in \mathbb{R}_T \colon x \leq M_{f'}(t-T)\}$, see [6]. By the non-degeneracy assumption, y is a classical solution of (1.1) on $\hat{S}$, hence p is a classical solution of (1.8) on $\hat{S}$. Thus, there is small $\varepsilon > 0$ such that the solution is constant along the backward characteristics $\xi(\cdot;\theta_1,x)$ for $x \in]0,\varepsilon[$ with values $p(\theta_1+, x)$. The non-degeneracy assumption and the continuity of p on the stripe $\hat{S} \setminus D^-$ yield $\lim_{x \searrow 0} p(t, \xi(t;\theta_1,x)) = p(\theta_1+, 0+)$ for all $t \in]\vartheta_1, \theta_1[$. By using the construction of $\tilde{a}$, we deduce that $\tilde{p}(t,\cdot)$ is continuous on $\mathbb{R}$ for all $t \in]0,T[$. Moreover, we have $\tilde{p} = p(\theta_1+, 0+)$ on $A_1^0 \cup D^-$. Together with the previous observations, we deduce that $\tilde{p}$ is a Lipschitz solution of the extended homogeneous adjoint equation.

Next, we analyze the monotonicity of $\tilde{p}$. To this end, we show the monotonicity of $p(\cdot, 0+)$ on $]0, \vartheta_i[\cup]\theta_i, T[$. Denote by $\Sigma \subset [0,T]$ the set of discontinuities of $f'(y(\cdot, 0+))$. As $f'(y)$ satisfies the OSLC, $f'(y(t,\cdot)) \in BV_{loc}(\Omega)$ holds and we have

$$f'(y(t,x)) \to f'(y(t,0+)) \quad \text{for } \ x \searrow 0.$$

Let $\bar{t} \notin \Sigma$ and $f'(y(\bar{t}, 0+)) > \rho$ for some $\rho > 0$, thus there exists $\delta > 0$ such that $f'(y(\cdot,0+)) > \rho/2$ on $[\bar{t} - \delta, \bar{t} + \delta]$. Let $0 < h \leq \delta$. Choose $\varepsilon > 0$ and multiplying (1.8) with $\frac{1}{\varepsilon}\mathbf{1}_{\{x \leq \varepsilon\}}(x)$ and integration over $[\bar{t}, \bar{t}+h] \times \Omega$ leads by using $p_x \geq 0$ to

$$\begin{aligned}
\frac{1}{\varepsilon}\int_0^\varepsilon p(\bar{t}+h, x) - p(\bar{t}, x)\,\mathrm{d}x &= -\frac{1}{\varepsilon}\int_0^\varepsilon \int_{\bar{t}}^{\bar{t}+h} f'(y(t,x)) p_x(t,x)\,\mathrm{d}t\,\mathrm{d}x \\
&\leq -\int_{\bar{t}}^{\bar{t}+h} \frac{1}{\varepsilon}\int_0^\varepsilon \min\{f'(y(t,x)), 0\} p_x(t,x)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Now $\frac{1}{\varepsilon}\int_0^\varepsilon \min\{f'(y(t,x)),0\} p_x(t,x)\,\mathrm{d}x$ is bounded by $\|f'(y)p_x\|_{\infty,]\bar{t}, \bar{t}+h[\times]0,\varepsilon[}$ and converges to zero a.e. on $]\bar{t}, \bar{t}+h[$ for $\varepsilon \searrow 0$, since $\lim_{x \searrow 0} f'(y(t,x)) = f'(y(t,0+)) >$

$\rho/2 > 0$ on $]\bar t - h, \bar t + h[$. Hence, the Lebesgue dominated convergence theorem yields

$$p(\bar t + h, 0+) - p(\bar t, 0+) = \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_0^{\varepsilon} p(\bar t + h, x) - p(\bar t, x)\,\mathrm{d}x \le 0, \quad \forall\, h \in]0, \delta[$$

and we conclude that $p_t(\bar t, 0+) \le 0$ in all points of differentiability $\bar t \in \Sigma$ and thus by Rademacher's theorem in a.a. $\bar t \in \Sigma$. Since Σ is countable, we have shown that

$$p_t(t, 0+) \le 0 \quad \text{for a.a. } t \in]0, \vartheta_i[\cup]\theta_i, T[\tag{2.7}$$

and thus the Lipschitz function $p(\cdot, 0+)$ is monotone decreasing on $]0, \vartheta_i[\cup]\theta_i, T[$.

Since $\tilde p = p$ on $\Omega_T \setminus D^-$, we have $\tilde p_x = p_x \ge 0$ a.e. on $\Omega_T \setminus D^-$. Moreover, $\tilde p$ is constant on $A_1^0 \cup D^-$ and on $\{(t,x) \in \Omega_T \colon x \le M_{f'}(t-T)\}$. Let $(t, x_1), (t, x_2) \in A_1^+$ with $x_1 \le x_2$, then (2.7) yields

$$p(t, x_1) = p\left(-\tfrac{x_1}{M_{f'}} + t, 0+\right) \le p\left(-\tfrac{x_2}{M_{f'}} + t, 0+\right) = p(t, x_2).$$

As this holds for all $(t, x_1), (t, x_2) \in A_1^+$, it follows that $\tilde p_x \ge 0$ on A_1^+. The same arguments are applicable on A_1^-.

Together with the previous observations we obtain $\tilde p_x \ge 0$ on $\mathbb{R}_T$. Now [6, Lemma 4.1.8] shows that $\tilde p$ is the reversible solution of $p_t + \tilde a p_x = 0$ on $\mathbb{R}_T$ and therefore p is the reversible solution on $\Omega_T \setminus D^-$, since on this set the definition of reversible solutions along generalized backward characteristics according to Definition 2.9 and for reversible solutions on $\mathbb{R}_T$, see [6] and [30, Def. 7.5], coincide. As $p = p^B$ on D^- holds, p is the reversible solution of (1.8) on Ω_T in the sense of Definition 2.9.

If there is more than one transition point, we have to consider the sets

$$\begin{aligned}
A_i^+ &\coloneqq \{(t,x) \in \mathbb{R}_T \colon M_{f'} \min\{t - \vartheta_i, 0\} \le x \le 0\},\\
A_i^- &\coloneqq \{(t,x) \in \mathbb{R}_T \colon M_{f'} \min\{t - \vartheta_{i+1}, 0\} < x \le M_{f'} \min\{t - \theta_i, 0\}\},\\
A_i^0 &\coloneqq \{(t,x) \in \mathbb{R}_T \colon M_{f'} \min\{t - \theta_i, 0\} < x < M_{f'} \min\{t - \vartheta_i, 0\}\}
\end{aligned}$$

for $i = 1, \dots, \mathfrak{n}_T$ and $\vartheta_{\mathfrak{n}_T + 1} = T$. Choose

$$\tilde p(t,x) = \begin{cases} p\left(-\tfrac{x}{M_{f'}} + t, 0+\right) & \text{for } \ (t,x) \in \bigcup_{i=1}^{\mathfrak{n}_T} A_i^{\pm},\\ p^T(0) & \text{for } \ x \le M_{f'}(t-T),\\ p(\theta_i +, 0+) & \text{for } \ (t,x) \in A_i^0 \cup D_i^-, \ i = 1, \dots, \mathfrak{n}_T \end{cases}$$

whereby $D_i^- \coloneqq \{(t,x) \in]\vartheta_i, \theta_i[\times \Omega \colon x < \xi(t; \theta_i, 0)\}$. By the same arguments as before, $\tilde p$ is a Lipschitz solution of $p_t + \tilde a p_x = 0$ on $\mathbb{R}_T$ with $\tilde p_x \ge 0$ and applying [6, Lemma 4.1.8] yields the desired result. ∎

Theorem 2.13. *Let the assumptions of Proposition 2.12 hold,* $p^T \in C^{0,1}(\Omega)$, $p = p^B \in \mathbb{R}$ *on* D^- *and* p *be a Lipschitz-solution on* $\Omega_T \setminus (D^-)^{cl}$. *Then* p *is reversible if and only if there exist Lipschitz-solutions* p_i *on* $\Omega_T \setminus (D^-)^{cl}$ *and values* $p_i^B \in \mathbb{R}$ *such that* $\partial_x p_i \ge 0$ *on* $\Omega_T \setminus (D^-)^{cl}$ *for* $i = 1, 2$ *and* $p = p_1 - p_2$.

Proof. We proceed partially as in [6]. We write $p^T = p_1^T - p_2^T$ with $p_i^T \in C^{0,1}(\Omega)$ satisfying $\partial_x p_i^T \ge 0$ and choose $p_1^B, p_2^B \in \mathbb{R}$ such that $p^B = p_1^B - p_2^B$. Let p be reversible and let p_i be the unique reversible solutions for end data p_i^T and boundary data p_i^B for $i = 1, 2$. Then $p = p_1 - p_2$, since the unique reversible solutions form a vector space. As the end data are propagated along the generalized backward characteristics, we deduce $\partial_x p_i \ge 0$ on $\Omega_T \setminus (D^-)^{cl}$ for $i = 1, 2$.

The converse follows directly from Proposition 2.12 and the fact that the unique reversible solutions of (1.8) form a vector space on Ω_T, see Remark 2.10. ∎

Lemma 2.14. *Let $a \in C^1(\mathbb{R})$ and $p^T \in C^{0,1}(\mathbb{R})$. Then a function $p \in BV_{loc}(\mathbb{R}_T)$ which solves*

$$p_t + ap_x = 0 \quad on \quad \mathbb{R}_T, \quad p(T,\cdot) = p^T \quad on \quad \mathbb{R}$$

in the sense of distributions is unique.

Proof. It is sufficient to consider the case $p^T \equiv 0$. Let $p \in BV_{loc}(\mathbb{R}_T)$ be a function, which solves (1.8) for $p^T = 0$ in the sense of distributions.

Let $\tau \in [0,T[$ be arbitrary and for $0 < \delta < (T-\tau)/2$ let $\psi_\delta \in C_c^\infty(]\tau,T[)$ with $0 \le \psi_\delta \le 1$ and $\psi_\delta \equiv 1$ on $]\tau+\delta, T-\delta[$.

Let $x_l < x_r$ be arbitrary and denote by $\xi_{l/r}(t)$ the forward characteristics starting in $(0,x_l)$ and $(0,x_r)$. Now consider any $\tau \in [0,T[$, let $\phi^\tau \in C_c^\infty(]\xi_l(\tau),\xi_r(\tau)[)$ be arbitrary and denote by ϕ the classical solution of $\phi_t + a\phi_x = 0$ on $]\tau,T[\times\mathbb{R}$ for initial data $\phi(\tau,\cdot) = \phi^\tau$. Then the support of ϕ is compact, since it is confined by the forward characteristics $\xi_{l/r}$. The product rule for BV-function gives by using $\phi_t + a\phi_x = 0$

$$(\psi_\delta(t)\phi p)_t + a(\psi_\delta(t)\phi p)_x = \psi_\delta'(t)(\phi p) + \psi_\delta(t)\phi(p_t + ap_x).$$

After integrating over $]\tau,T[\times\mathbb{R}$ the last term vanishes, since $\psi_\delta(t)\phi(t,x)$ is C^1 with compact support and $p_t + ap_x = 0$ in the sense of distributions, leading to

$$-\int_\tau^T \int_{\mathbb{R}} \psi_\delta(t)(\phi p a_x)(t,x)\,\mathrm{d}x\,\mathrm{d}t = \int_\tau^T \int_{\mathbb{R}} (\psi_\delta'(t)(\phi p)(t,x)\,\mathrm{d}x\,\mathrm{d}t.$$

Since p admits traces at $t = \tau$ and $t = T$ and $p(T-,\cdot) = p^T = 0$, $\delta \searrow 0$ yields

$$-\int_\tau^T \int_{\mathbb{R}} \phi p a_x \,\mathrm{d}x\,\mathrm{d}t = \int_{\mathbb{R}} (\phi^\tau(x)p(\tau+,x) - (\phi p)(T-,x))\,\mathrm{d}x = \int_{\mathbb{R}} \phi^\tau(x)p(\tau+,x)\,\mathrm{d}x.$$

By taking the supremum over all $\phi^\tau \in C_c^\infty(]\xi_l(\tau),\xi_r(\tau)[)$ with $|\phi^\tau| \le 1$ we obtain

$$\|p(\tau+,\cdot)\|_{\infty,]\xi_l(\tau),\xi_r(\tau)[} \le \int_\tau^T \|a_x(t,\cdot)\|_{1,]\xi_l(t),\xi_r(t)[} \|p(t,\cdot)\|_{\infty,]\xi_l(t),\xi_r(t)[}\,\mathrm{d}t.$$

and the Gronwall lemma implies $\|p(t,\cdot)\|_{\infty,]\xi_l(t),\xi_r(t)[} = 0$ for a.a. $t \in]0,T[$. Since $x_l < x_r$ were arbitrary, the proof is complete. $\square$

3. Discrete approximation. For the discretization of the state equation (1.1) we consider conservative finite difference schemes. Let $\lambda > 0$ be fixed and set for a grid size $\Delta > 0$ with $\Delta x := \Delta$

$$\Delta t := \lambda \Delta x, \quad t_n := n\Delta t, \quad x_j := j\Delta x, \quad R_j := [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}[, \quad Q_j^n := [t_n, t_{n+1}[\times R_j.$$

For grid values y_j^n, v_j, w^n, $n,j \in \mathbb{N}_0$ we associate piecewise constant functions by

$$y_\Delta = \sum_{j,n\ge 0} y_j^n \mathbf{1}_{Q_j^n}, \quad y_\Delta^n(x) = y_\Delta(t_n,x), \quad v_\Delta = \sum_{j\ge 0} v_j \mathbf{1}_{R_j}, \quad w_\Delta = \sum_{n\ge 0} w^n \mathbf{1}_{[t_n,t_{n+1}[}$$

and use the convention $(y_\Delta)_j^n \equiv y_j^n$, $(y_\Delta^n)_j \equiv y_j^n$, $(v_\Delta)_j \equiv v_j$, $(w_\Delta)^n \equiv w^n$. Given a function $v \in L^1_{loc}(\mathbb{R})$ we obtain a grid function by the averaging operators

$$T_\Delta v(x) := \frac{1}{\Delta x}\int_{R_j} v(\xi)\,\mathrm{d}\xi, \; x \in R_j \quad \text{and} \quad T_\Delta^B v(t) := \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} v(s)\,\mathrm{d}s, \; t \in [t_n, t_{n+1}[.$$

Finally, we define the difference operators for arbitrary functions $\psi \in L^1(\Omega_T)$ by

$$\Delta^+\psi(t,x) := \psi(t,x+\Delta x) - \psi(t,x), \quad \Delta^-\psi(t,x) := \psi(t,x) - \psi(t,x-\Delta x).$$

Let N_T such that $T \in [t_{N_T}, t_{N_T+1}[$ (analogously we define N_τ for any $\tau \in]0,T]$).

3.1. Analysis of the discretized state equation. To discretize (1.1) we consider conservative finite difference schemes of the form

$$\begin{aligned}
&y_j^0 = u_{0,j}, \quad j \geq 1, \qquad y_0^n = u_B^n, \quad n = 1, \dots, N_T+1, \\
&y_j^{n+1} = \underbrace{y_j^n - \lambda\left(F(y_j^n, y_{j+1}^n) - F(y_{j-1}^n, y_j^n)\right)}_{=:H(y_{j-1}^n, y_j^n, y_{j+1}^n)} + \Delta t G_j^n, \quad j \geq 1,\ n = 1, \dots, N_T,
\end{aligned} \tag{3.1}$$

with consistent numerical flux function $F\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying

$$F \in C^{1,1}(\mathbb{R}^2) \quad \text{and} \quad F(y,y) = f(y) \quad \text{for all} \quad y \in \mathbb{R}. \tag{3.2}$$

For abbreviation we define the finite difference operator

$$\Delta^+ F(y_{j-1}^n, y_j^n) := F(y_j^n, y_{j+1}^n) - F(y_{j-1}^n, y_j^n).$$

For $(u_0, u_B) \in (L^\infty \cap L^1)(\Omega) \times (L^\infty \cap L^1)(0,T)$ we approximate the controls and source term by cell averages

$$u_{0,j} = (T_\Delta u_0)_j,\ u_B^n = (T_\Delta^B u_B)^n,\ \ G_j^n = g_\Delta(t_n, x_j),\ \ g_\Delta := T_\Delta^B T_\Delta g,\ j \geq 1,\ \ G_0^n = 0$$

and as shown in [9] it holds (note that g vanishes for $x \leq \varepsilon_g$ by (1.2))

$$u_{0,\Delta} \to u_0 \quad \text{in} \quad L^1_{loc}(\Omega), \quad u_{B,\Delta} \to u_B \quad \text{in} \quad L^1([0,T]), \quad g_\Delta \to g \quad \text{in} \quad B(\Omega_T). \tag{3.3}$$

The discrete control-to-state mapping is given by

$$(u_{0,\Delta}, u_{B,\Delta}) \mapsto y_\Delta$$

where y_Δ denotes the grid function corresponding to y_j^n determined by (3.1). As discrete approximation of the objective functional (1.4) we choose for example

$$J_\Delta(y_\Delta) = \int_\Omega \gamma_\Delta(x)\psi(y_\Delta(T,x), y_{d,\Delta}(x))\,\mathrm{d}x = \sum\nolimits_{j\geq 1} \Delta x \gamma_j \psi(y_j^{N_T}, y_{d,j})$$

with $\gamma_j = (T_\Delta \gamma)_j$ and $y_{d,j} = (T_\Delta y_d)_j$ and associated grid functions γ_Δ and $y_{d,\Delta}$.

Definition 3.1. *The difference approximation* (3.1) *is called* monotone *on* $[l,r]$, *if* $H : [l,r]^3 \to \mathbb{R}$ *is nondecreasing in each argument. If the numerical Flux* $(a,b) \mapsto F(a,b)$ *is differentiable on a neighborhood of* $[l,r]^2$ *and* F_a, F_b *denote the partial derivatives with respect to first and second argument, respectively, this is equivalent to*

$$F_a(c,d) \geq 0, \quad F_b(c,d) \leq 0, \quad 1 - \lambda(F_a(d,e) - F_b(c,d)) \geq 0 \quad \forall c,d,e \in [l,r]. \tag{3.4}$$

By $\mathcal{H}(y_\Delta^n)$ we denote the corresponding difference operator for the grid function

$$\mathcal{H}(y_\Delta^n)_j = H(y_{j-1}^n, y_j^n, y_{j+1}^n) \tag{3.5}$$

which we will also use for y_Δ^n defined on $\mathbb{R}$. The difference operator $\mathcal{H}_B(y_\Delta^n)$ for the scheme (3.1) with boundary condition is then given by

$$y_\Delta^{n+1} = \mathcal{H}_B(y_\Delta^n) + \Delta t g_\Delta^n = \begin{cases} \mathcal{H}(y_\Delta^n) + \Delta t g_\Delta^n & \text{on} \quad [\Delta x/2, \infty[, \\ u_B^{n+1} & \text{on} \quad [-\Delta x/2, \Delta x/2[. \end{cases} \tag{3.6}$$

In the following analysis, it will sometimes be convenient to extend the difference scheme (3.1) on $\mathbb{R}$ by setting $y_j^n = y_0^n$ for $j < 0$.

Lemma 3.2. *Let the difference operator $\mathcal{H}_B$ be monotone on $[l, r]$. Then for grid functions v_Δ, w_Δ with initial and boundary data $v_0, w_0 \in (L^1 \cap L^\infty)(\Omega)$, $v_B, w_B \in (L^1 \cap L^\infty)(0, T)$ with $l \leq v_0, w_0, v_B, w_B \leq r$ it holds for $n = 0, \dots, N_T$.*

(i) $\min_{k\in\{-1,0,1\}} v^n_{j+k} \leq \mathcal{H}_B(v^n_\Delta)_j \leq \max_{k\in\{-1,0,1\}} v^n_{j+k}$ *for all* $j \geq 1$,

(ii) $\|\mathcal{H}_B(v^n_\Delta) - \mathcal{H}_B(w^n_\Delta)\|_1 \leq \|v^n_\Delta - w^n_\Delta\|_1 + \lambda\Delta t|v^{n+1}_B - w^{n+1}_B|$,

(iii) $\|\mathcal{H}_B(v^n_\Delta)\|_{TV} \leq \|v^n_\Delta\|_{TV} + |v^{n+1}_B - v^n_B|$.

Proof. (i) holds by [7, Lemma 3.1] and (3.6). (ii) and (iii) can be shown as in [9], see the supplementary material in the appendix. ∎

Now let $u_0 \in (L^1 \cap L^\infty)(\Omega)$ and $u_B \in (L^1 \cap L^\infty)(0, T)$ with $\|u_0\|_\infty, \|u_B\|_\infty \leq M_y$ and let the scheme $\mathcal{H}_B$ be monotone on $[-M_y, M_y]$. Then there exists $M_1 > 0$ such that

$$\|y_\Delta\|_\infty \leq M_y \quad \text{and} \quad \|y_\Delta\|_1 \leq M_1. \tag{3.7}$$

We recall the convergence of monotone difference schemes to the entropy solution of (1.1). Similar results and proofs for higher dimension can be found in [5, 7, 32].

Theorem 3.3. *Consider data $u_0 \in (L^1 \cap L^\infty)(\Omega)$ and $u_B \in (L^1 \cap L^\infty)(0, T)$. Let $l, r \in \mathbb{R}$ such that $l \leq u_0, u_B \leq r$, let F be a consistent numerical flux and let the scheme (3.1) be monotone on $[l - T\|g\|_\infty, r + T\|g\|_\infty]$. Then the approximate solutions of (3.1) converge to the entropy solution of (1.1)*

$$y_\Delta \to y \quad in \quad L^\infty(0, T; L^1_{loc}(\Omega)) \ as \ \Delta \searrow 0 \tag{3.8}$$

and for the trace of the flux function F the following weak convergence holds

$$\int_{[0,T[} F(u_{B,\Delta}(t), y_\Delta(t, \Delta x))\phi(t)\,\mathrm{d}t \to \int_{[0,T[} f(y(t, 0+))\phi(t)\,\mathrm{d}t \quad as \ \Delta x \searrow 0. \tag{3.9}$$

Proof. The proof extends standard arguments and is given in the supplementary material in the appendix. See also [7, 32]. ∎

In the continuous case, the entropy solution satisfies the OSLC (2.2) which is needed to guarantee the existence of a reversible solution to (1.8). To obtain a discrete analogue of this condition, we consider OSLC consistency of the numerical approximation y_Δ as proposed in [29]. To this end, we define for an interval $I \subset \Omega$

$$Lip^+_\Delta(y_\Delta(t, \cdot); I) := \Delta x^{-1} \sup_{x\in I} \max\{y_\Delta(t, x + \Delta x) - y_\Delta(t, x), 0\}.$$

Definition 3.4. *A family of grid functions $(y_\Delta)_{0<\Delta\leq\Delta_0}$ is called* OSLC consistent on $[\tau_0, \tau_1] \subset [0, T]$ *if there exist constants $c, C > 0$ such that with N_{τ_i} such that $\tau_i \in [t_{N_{\tau_i}-1}, t_{N_{\tau_i}}[$, $i = 0, 1$, it holds for $N_{\tau_0} \leq n \leq N_{\tau_1}$*

$$Lip^+_\Delta(y_\Delta(t_n, \cdot); \Omega) \leq \left(c(t_n - t_{N_{\tau_0}}) + (Lip^+_\Delta(y_\Delta(\tau_{N_{\tau_0}}, \cdot)))^{-1}\right)^{-1} + C. \tag{3.10}$$

3.2. Suitable numerical flux functions. The Engquist-Osher scheme (EO-scheme) has the numerical flux function

$$F^{EO}(y_0, y_1) = f(\bar{y}) + \int_{\bar{y}}^{y_0} f'(y)_+\,\mathrm{d}y + \int_{\bar{y}}^{y_1} f'(y)_-\,\mathrm{d}y$$

with $f'(y)_+ := \max\{f'(y), 0\}$, $f'(y)_- := \min\{f'(y), 0\}$, and $\bar{y} \in \mathbb{R}$ is fixed, see [11]. F^{EO} is indepent of $\bar{y}$ and for convenince we choose the sonic point $\bar{y} = \sigma$, i.e. $f'(\sigma) = 0$, which exists in our setting. The Godunov scheme (G-scheme) has numerical flux

$$F^G(y_0, y_1) = f(w(\Delta t, 0)),$$

where w solves the Riemann-Problem

$$w_t + f(w)_x = 0, \quad w(0,x) = y_1 \mathbf{1}_{\{x>0\}}(x) + y_0 \mathbf{1}_{\{x\le 0\}}(x).$$

Both fluxes are monotone, cf. Definition 3.1, if the CFL condition

$$\lambda \sup_{|y|\le M_y} |f'(y)| \le 1-\rho, \quad \rho \in [0,1[.$$

is satisfied. In the considered case, where f is strictly convex, the G-flux coincide with the EO-flux except for the transonic case $y_1 < \sigma < y_0$ where

$$F^{EO}(y_0,y_1) = f(y_0) + f(y_1) - f(\sigma) \quad \text{and} \quad F^G(y_0,y_1) = \max\{f(y_0), f(y_1)\}.$$

Note that $F^G \in C^{0,1}(\mathbb{R}^2)$ is not everywhere differentiable. Later, we will use a *modified Engquist-Osher scheme* (mEO-scheme), which applies the Engquist-Osher flux in the interior and the Godunov flux at the boundary. This yields the scheme

$$\begin{aligned} y_1^{n+1} &= y_1^n - \lambda(F^{EO}(y_1^n,y_2^n) - F^G(y_0^n,y_1^n)), \\ y_j^{n+1} &= y_j^n - \lambda\Delta^+ F^{EO}(y_{j-1}^n,y_j^n), \quad j \ge 2, \end{aligned} \tag{3.11}$$

for all $n = 0,\dots,N_T-1$. We call this scheme *modified Engquist-Osher scheme* (mEO-scheme). The choice of the Godunov flux on the boundary will be critical to show the critical convergence results (3.13), (3.14) and Lemma 3.7 on the boundary cells and to obtain the correct coefficients in the adjoint scheme at the boundary.

LEMMA 3.5. *Let $f \in C^2(\mathbb{R})$ be strictly convex with $f'' \ge m_{f''} > 0$, let $u_0 \in L^\infty(\Omega)$ and let $u_B \in L^\infty(0,T)$ with $f'(u_B) \ge \beta > 0$ on $[0,T]$. Moreover, let L_g be a Lipschitz constant of g with respect to x. Let the CFL-condition*

$$\lambda \sup_{|y|\le M_y} |f'(y)| \le \frac{1-\rho}{2}, \quad \rho \in [0,1[\tag{3.12}$$

hold and the solution of (3.11) fulfill $\|y_\Delta\|_\infty \le M_y$ for all $\Delta \le \Delta_0$ and some $M_y > 0$. Finally, let $[\tau_0,\tau_1] \subset [0,T]$ be an interval with $u_B'|_{[\tau_0,\tau_1]} \ge -C_B$ for a constant $C_B \ge 0$. Then for all $0 < \Delta \le \Delta_0$ the solution y_Δ of (3.11) satisfies with some $c > 0$ and $C = \max\left\{\frac{C_B}{\lambda\beta}, \sqrt{L_g/c}\right\}$ the OSLC consistency (3.10).

Proof. The proof refines arguments for initial value problems and is given in the supplementary material in the appendix. ∎

LEMMA 3.6. *Let the assumptions of Theorem 3.3 hold and assume that the boundary control u_B satisfies $f'(u_B) \ge \beta$ for small $\beta > 0$. Let the CFL-condition (3.12) hold and $y_\Delta(\cdot,\Delta x)$ be determined by the mEO-scheme from (3.11). Then it holds*

$$F^G(u_{B,\Delta}, y_\Delta(\cdot,\Delta x)) \to f(y(\cdot,0+)) \quad \textit{in} \ \ L^1(0,T) \ \ \textit{as} \ \ \Delta \to 0 \tag{3.13}$$

and one has on the outflow boundary $\mathcal{T}_B := \{t \in]0,T[\,:\, f'(y(t,0+)) < 0\}$ for $\Delta \to 0$

$$F_a^G(u_{B,\Delta}, y_\Delta(\cdot,\Delta x)) \to 0, \ \ F_b^G(u_{B,\Delta}, y_\Delta(\cdot,\Delta x)) \to f'(y(\cdot,0+)) \ \ \textit{in} \ L^1(\mathcal{T}_B). \tag{3.14}$$

Proof. Let $\Delta_0 > 0$ be sufficiently small and denote by σ the sonic point of f, i.e., $f'(\sigma) = 0$. Under the given assumptions the Godunov flux reads

$$F^G(u_B^n, y_1^n) = \begin{cases} f(y_1^n) & \text{for} \ \ y_1^n < \sigma, \ f(y_1^n) > f(u_B^n), \\ f(u_B^n) & \text{else.} \end{cases}$$

Let $M \subset [0,T]$ be a set where $Lip_\Delta^+(y_\Delta(t,\cdot);\Omega) \le L$ with some $L>0$ for all $t \in M$ uniformly in $\Delta \le \Delta_0$. This is the case, if M has distance $> \kappa$ from rarefaction centers with arbitrary $\kappa > 0$.

Since y is a BV function, we have $\lim_{x\searrow 0} \|y(\cdot,0+) - y(\cdot,x)\|_{1,M} = 0$. Let $\delta > 0$ be arbitrary and define the set $M_\Delta^-(\delta) = \{t \in M : y_\Delta(t,\Delta x) \le y(t,0+) - \delta\}$. Then

$$y_\Delta(t,x) \le y(t,0+) - \delta/2, \quad \forall\,(t,x) \in M_\Delta^-(\delta) \times]0, \delta/(2L)].$$

We show that

$$\|(y(\cdot,0+) - y_\Delta(\cdot,\Delta x))_+\|_{1,M} \to 0 \tag{3.15}$$

where $(a)_+ := \max\{a,0\}$, $a \in \mathbb{R}$. If not, there exists $(\Delta_k) \to 0$ and $\varepsilon > 0$ with

$$\|(y(\cdot,0+) - y_{\Delta_k}(\cdot,\Delta_k x))_+\|_{1,M} \ge \varepsilon, \quad \forall\, k. \tag{3.16}$$

Since $\|y_\Delta - y\|_{1,M\times]0,\delta/(2L)]} \to 0$ by Theorem 3.3, we find a subsequence again denoted by (Δ_k) and $(x_k) \to 0$ with $\|y_{\Delta_k}(\cdot,x_k) - y(\cdot,x_k)\|_{1,M\times]0,\delta/(2L)]} \to 0$ as $\Delta_k \to 0$. Hence,

$$\begin{aligned} &\|y(\cdot,0+) - y(\cdot,x_k)\|_{1,M^-_{\Delta_k}(\delta)} + \|y(\cdot,x_k) - y_{\Delta_k}(\cdot,x_k)\|_{1,M^-_{\Delta_k}(\delta)} \\ &\qquad \ge \|y(\cdot,0+) - y_{\Delta_k}(\cdot,x_k)\|_{1,M^-_{\Delta_k}(\delta)} \ge \Lambda(M^-_{\Delta_k}(\delta))\delta/2 \end{aligned}$$

and the left hand side tends to 0 for $\Delta_k \to 0$. Here $\Lambda(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}$. We conclude that $\Lambda(M^-_{\Delta_k}(\delta)) \to 0$ for $\Delta_k \to 0$ and thus

$$\|(y(\cdot,0+) - y_{\Delta_k}(\cdot,\Delta_k x))_+\|_{1,M} \le \delta\Lambda(M) + \Lambda(M^-_{\Delta_k}(\delta))2M_y \to \delta\Lambda(M) \quad \text{for } \Delta_k \to 0.$$

Since $\delta > 0$ was arbitrary, we conclude that $\|(y(\cdot,0+) - y_{\Delta_k}(\cdot,\Delta_k x))_+\|_{1,M} \to 0$ as $\Delta_k \to 0$, a contradiction to (3.16). Hence, (3.15) is shown.

Let $A = \{t \in M \colon y(t,0+) = u_B(t)\}$. By our assumptions on u_B we find $\delta > 0$ with $u_B \ge \sigma + \delta$ and thus also $u_{B,\Delta} \ge \sigma + \delta$. Hence, we have

$$F^G(u_{B,\Delta}(t), y_\Delta(t,\Delta x)) = \begin{cases} f(u_{B,\Delta}(t)) & \text{if } t \in A \setminus M^-_\Delta(\delta), \\ \in \{f(u_{B,\Delta}(t)), f(y_\Delta(t,\Delta x))\} & \text{if } t \in A \cap M^-_\Delta(\delta). \end{cases}$$

Now (3.15) implies $\Lambda(M^-_\Delta(\delta)) \to 0$ as $\Delta \to 0$ and we conclude that

$$F^G(u_{B,\Delta}, y_\Delta(\cdot,\Delta x)) \to f(u_B) \ \text{ in } \ L^1(A) \ \text{ as } \ \Delta \to 0. \tag{3.17}$$

It remains to study the convergence on $M \setminus A$. If the boundary control is not attained in the BLN sense then $y(t,0+) < \sigma$ and $f(u_B(t)) < f(y(t,0+))$. The non-degeneracy assumption yields $\varepsilon > 0$ such that $f(u_B(t)) + \varepsilon < f(y(t,0+))$ is satisfied. Consider $M^+_\Delta(\delta) = \{t \in M : y_\Delta(t,\Delta x) > y(t,0+) + \delta\}$ for arbitrary $\delta > 0$. We will show that

$$\Lambda(M^+_\Delta(\delta) \setminus A) \to 0 \ \text{ as } \ \Delta \to 0. \tag{3.18}$$

Since $\delta > 0$ is arbitrary, this implies with (3.15) that

$$\|y(\cdot,0+) - y_\Delta(\cdot,\Delta x)\|_{1,M\setminus A} \to 0 \ \text{ as } \ \Delta \to 0 \tag{3.19}$$

and since $F^G(u_B, y(\cdot,0+)) = f(y(\cdot,0+))$ and F^G is Lipschitz on $[-M_y, M_y]^2$,

$$F^G(u_{B,\Delta}, y_\Delta(\cdot,\Delta x)) \to f(y(\cdot,0+)) \ \text{ in } \ L^1(M \setminus A) \ \text{ as } \ \Delta \to 0. \tag{3.20}$$

Now (3.14) follows immediately, since $f(u_B)+\varepsilon < f(y(\cdot,0+))$ on $M\setminus A$.

We still have to show (3.18). For $\Delta_0>0$ small enough we have $f(u_{B,\Delta})+\varepsilon/2 < f(y(\cdot,0+))$ on $M\setminus A$ and for all $\Delta\le\Delta_0$ it holds

$$F^G(u_{B,\Delta},y_\Delta(\cdot,\Delta x)) = \begin{cases} f(y_\Delta(\cdot,\Delta x)) & \text{on } M_{1,\Delta} := M_\Delta^-(0)\setminus A,\\ f(y_\Delta(\cdot,\Delta x)) & \text{on } M_{2,\Delta} := (M_\Delta^+(0)\setminus A)\\ & \quad \cap\{y_\Delta(\cdot,\Delta x)<\sigma,\ f(y_\Delta(\cdot,\Delta x))>f(u_{B,\Delta})\},\\ f(u_{B,\Delta}) & \text{on the remaining part } M_{3,\Delta} \text{ of } M\setminus A. \end{cases}$$

Now we obtain by (3.15)

$$\|F^G(u_{B,\Delta},y_\Delta(\cdot,\Delta x)) - f(y(\cdot,0+))\|_{L^1(M_{1,\Delta})}\to 0, \tag{3.21}$$

$$F^G(u_{B,\Delta},y_\Delta(\cdot,\Delta x)) = \begin{cases} f(y_\Delta(\cdot,\Delta x))\le f(y(\cdot,0+)) & \text{on } M_{2,\Delta},\\ f(y_\Delta(\cdot,\Delta x))\le f(y(\cdot,0+)) - \frac{m_{f''}}{2}\delta^2 & \text{on } M_{2,\Delta}\cap M_\Delta^+(\delta),\\ f(u_{B,\Delta})\le f(y(\cdot,0+))-\varepsilon/2 & \text{on } M_{3,\Delta}. \end{cases}$$

Now consider (3.9) for arbitrary $\phi\in C_c^1(M)$, $\phi\ge 0$. Then

$$\begin{aligned} &\int_M (f(y(t,0+)) - F^G(u_{B,\Delta}(t),y_\Delta(t,\Delta x)))\phi(t)\,\mathrm{d}t\\ &= \int_{A\cup M_{1,\Delta}} (f(y(t,0+)) - F^G(u_{B,\Delta}(t),y_\Delta(t,\Delta x)))\phi(t)\,\mathrm{d}t\\ &\quad + \int_{(M_{2,\Delta}\setminus M_\Delta^+(\delta))\cup(M_{2,\Delta}\cap M_\Delta^+(\delta))} (f(y(t,0+)) - f(y_\Delta(t,\Delta x)))\phi(t)\,\mathrm{d}t\\ &\quad + \int_{M_{3,\Delta}} (f(y(t,0+)) - f(u_{B,\Delta}(t)))\phi(t)\,\mathrm{d}t\\ &\ge \int_{A\cup M_{1,\Delta}} (f(y(t,0+)) - F^G(u_{B,\Delta}(t),y_\Delta(t,\Delta x)))\phi(t)\,\mathrm{d}t\\ &\quad + \int_{M_\Delta^+(\delta)\setminus A} \min\{\delta^2 m_{f''}/2,\varepsilon/2\}\phi(t)\,\mathrm{d}t. \end{aligned}$$

For $\Delta\to 0$ the left hand side tends to 0 by (3.9) and the first term on the right tends to 0 by (3.17) and (3.21). Since $\phi\in C_c^1(M)$, $\phi\ge 0$, is arbitrary, (3.18) follows.

Hence, the claim is shown on M and $M\cap\mathcal{T}_B$. Since $]0,T[$ can be exhausted by sets M and the involved functions are uniformly bounded, the proof is complete. ∎

Lemma 3.7. *Under the assumptions of Lemma 3.6 let y_Δ be computed by (3.11). Let $I_B=[\tau_1,\tau_2]\subset\mathcal{T}_B$ be a subinterval of the outflow boundary that has distance $\ge\kappa$ from rarefaction centers with arbitrary $\kappa>0$. Then for every $\delta>0$ there is $\Delta'<\Delta_0$ such that it holds*

$$y_\Delta(t,\Delta x)<\sigma \quad \textit{and} \quad f(y_\Delta(t,\Delta)) > f(u_{B,\Delta}(t)) \quad \textit{for all} \quad \Delta\le\Delta',\quad t\in[\tau_1+\delta,\tau_2].$$

Proof. Let $I_B=[\tau_1,\tau_2]$ be as above and $\delta>0$ be arbitrary. By Lemma 3.5 for all $0<\Delta\le\Delta_0$ the discrete OSLC (3.10) holds with $\tau_0=\tau_1-\kappa$ for $y_\Delta(t,\cdot)$ for all $t\in I_B$. Thus, we find $C>0$ such that $Lip_\Delta^+(y_\Delta(t,\cdot);\Omega)\le C$ for all $t\in I_B$.

We show that possibly after reducing Δ_0 for all $0<\Delta\le\Delta_0$ the implication holds

$$t_n\in I_B,\quad y_1^n<\sigma,\quad f(y_1^n)>f(y_0^n) \quad\Longrightarrow\quad y_1^{n+1}-y_1^n\le L_f C\Delta t. \tag{3.22}$$

In fact, by assumption we have $f'(u_B) \geq \beta > 0$ and thus also $f'(y_0^n) \geq \beta > 0$. Now let the left hand side hold. Then $f'(y_0^n) \geq \beta > f'(\sigma) = 0$ implies the existence of $\varepsilon > 0$ with $y_1^n < \sigma - \varepsilon$. Hence, for $\Delta_0 \leq \varepsilon/C$ the discrete OSLC ensures $y_2^n < \sigma$. Now (3.22) follows, since the scheme (3.11) yields

$$y_1^{n+1} - y_1^n = -\lambda(f(y_2^n) - f(y_1^n)) \leq \lambda L_f (y_2^n - y_1^n)_+ \leq L_f C \Delta t.$$

Now the non-degeneracy assumption ensures $y(\cdot, 0+) < \sigma - \varepsilon$ and $f(y(\cdot, 0+)) - f(u_B) > \varepsilon$ on I_B for small $\varepsilon > 0$. From the proof in Lemma 3.6, see (3.19), we know that $y_\Delta(\cdot, \Delta x) \to y(\cdot, 0+)$ in $L^1(I_B)$. Therefore the measure of the set

$$E_\Delta \coloneqq \{t \in I_B : f(y_\Delta(t, \Delta x)) - f(u_{B,\Delta}(t)) \leq \varepsilon/2 \ \text{ or } \ y_\Delta(t, \Delta x) \geq \sigma - \varepsilon/2\}$$

tends to zero for $\Delta \to 0$. Hence, for any $0 < \delta' < \delta$ we find $0 < \Delta_{\delta'} \leq \Delta_0$ such that $\Lambda(E_\Delta) < \delta'$ for all $\Delta < \Delta_{\delta'}$ and thus for any $t \in [\tau_1 + \delta, \tau_2]$ there exists $s \in [t - \delta', t] \subset I_B$ with $s \in I_B \setminus E_\Delta$ for all $\Delta < \Delta_{\delta'}$.

Now choose $0 < \delta' \leq \min\{\delta, \varepsilon/(4 \max\{1, L_f\} L_f C)\}$. Assume that the assertion of the lemma is wrong. Then we find sequences $\Delta_k \searrow 0$ and $(s_k) \subset [\tau_1 + \delta, \tau_2]$ with $\Delta_k \leq \Delta_{\delta'/2}$ and

$$y_{\Delta_k}(s_k, \Delta_k x) \geq \sigma \quad \text{or} \quad f(y_{\Delta_k}(s_k, \Delta_k x)) \leq f(u_{B,\Delta_k}(s_k)) \quad \text{for all } k. \tag{3.23}$$

Then $s_k \in E_{\Delta_k}$ for all k. As shown above, we find $s_k' \in [s_k - \delta'/2, s_k]$ with $s_k' \in I_B \setminus E_{\Delta_k}$. Starting with $t_n = s_k'$ we can apply (3.22) iteratively at least up to any $t_m > t_n$ with $L_f C(t_m - t_n) \leq \varepsilon/(2 \max\{1, L_f\})$ and thus up to some $t_m \geq t_n + \delta'$. Hence, $y_{\Delta_k}(\cdot, \Delta_k)$ has by (3.22) a discrete one-sided Lipschitz constant $L_f C$ on the interval $[s_k', s_k' + \delta']$ that contains s_k. But since $s_k' \in I_B \setminus E_{\Delta_k}$, the discrete one-sided Lipschitz constant $L_f C$ implies that $y_{\Delta_k}(s_k, \Delta_k) \leq y_{\Delta_k}(s_k', \Delta x) + L_f C \delta' \leq \bar{y} - \varepsilon/4$ and $f(y_{\Delta_k}(s_k, \Delta_k)) \geq f(y_{\Delta_k}(s_k', \Delta_k)) - L_f L_f C \delta' \geq f(u_{B,\Delta_k}(s_k)) + \varepsilon/4$. This is a contradiction to (3.23) and the proof is complete. ∎

3.3. Analysis of the adjoint scheme. The adjoint scheme reads

$$\begin{aligned} p_j^n &= \sum\nolimits_{k=-1}^{1} B_{j,k}^{n+1} p_{j-k}^{n+1}, \quad j \geq 1, \quad n = 0, \dots, N_T - 1 \\ p_j^{N_T} &= (T_\Delta p^T)_j, \quad j \geq 1, \quad p_0^n = p^B, \quad n = 0, \dots, N_T \end{aligned} \tag{3.24}$$

$$B_{j,-1}^n = \lambda F_a(y_j^n, y_{j+1}^n), \quad B_{j,1}^n = -\lambda F_b(y_{j-1}^n, y_j^n), \quad B_{j,0}^n = 1 - B_{j,-1}^n - B_{j,1}^n. \tag{3.25}$$

with constant boundary data $p^B \in \mathbb{R}$. A derivation of the adjoint scheme can be found in [15, 25, 31] using the discrete Lagrangian. Let $\delta u_\Delta = (\delta u_{0,\Delta}, \delta u_{B,\Delta})$ with the cell averages $\delta u_{0,\Delta} = T_\Delta \delta u_0$ and $\delta u_{B,\Delta} = T_\Delta^B \delta u_B$. Together with standard adjoint calculus the discrete adjoint formulation of the derivative of the objective $J(y(u))$ is given by

$$\begin{aligned} d_{u_\Delta} J_\Delta(y_\Delta(u_\Delta)) \cdot \delta u_\Delta = &\int_\Omega p_\Delta(0, x) \delta u_{0,\Delta}(x) \, \mathrm{d}x \\ &+ \int_0^{T - \Delta t} p_\Delta(t + \Delta t, \Delta x) F_a(u_{B,\Delta}(t), y_\Delta(t, \Delta x)) \delta u_{B,\Delta}(t) \, \mathrm{d}t \end{aligned} \tag{3.26}$$

where p_Δ denotes the grid function to p_j^n determined by (3.24) with $p^B = 0$.

As in the continuous case, we require the function F to fulfill a discrete OSLC.

LEMMA 3.8. *Let the flux F satisfy (3.2) and let F_a, F_b be monotone increasing with respect to both arguments (this is the case for F^{EO} and F^{G}). If the solution of (3.1) fulfills (3.10) on $[\tau_0, \tau_1] \subset [0,T]$ then for all $\nu > 0$ there is $\alpha \in L^1(\tau_0+\nu, \tau_1)$ such that the weak discrete OSLC*

$$\Delta^+ F_a(y_j^n, y_{j+1}^n) + \Delta^+ F_b(y_{j-1}^n, y_j^n) \leq \frac{\Delta x}{\Delta t} \int_{t_n}^{t_{n+1}} \alpha(s)\, \mathrm{d}s$$

is satisfied for all $j \geq 1$ and $n = N_{\tau_0+\nu}, \ldots, N_{\tau_1} - 1$.

Proof. By (3.10) we find for all $\nu > 0$ a $\gamma \in L^1(\tau_0+\nu, \tau_1)$ with $\Delta^+ y_j^n \leq \frac{\Delta x}{\Delta t} \int_{t_n}^{t_{n+1}} \gamma(s)\, \mathrm{d}s$. Since F_a, F_b are Lipschitz and monotone, the assertion follows. ∎

Taking the difference of the discrete adjoint state for $j+1$ and j leads to

$$\Delta^+ p_j^n = \sum\nolimits_{k=-1}^{1} C_{j,k}^n \Delta^+ p_{j-k}^{n+1}, \tag{3.27}$$

$$C_{j,-1}^n = \lambda F_a(y_{j+1}^n, y_{j+2}^n), C_{j,1}^n = -\lambda F_b(y_{j-1}^n, y_j^n), C_{j,0}^n = 1 - C_{j-1,-1}^n - C_{j+1,1}^n. \tag{3.28}$$

3.3.1. A priori estimates for the adjoint scheme. Consider end data $p^T \in C_b^{0,1}(\Omega)$ and constant boundary data $p^B \in \mathbb{R}$. Since the discrete end data are given by cell averages, there are $M_T, L_T, \Delta_0 > 0$ such that for $\Delta \leq \Delta_0$ it holds, e.g. [9],

$$\|p_\Delta^T\|_\infty \leq M_T, \quad |p_\Delta^T|_{Lip_\Delta(\Omega)} \leq L_T, \quad p_\Delta^T \to p^T \ \text{ in } \ B_{loc}(\Omega) \ \text{ as } \ \Delta \to 0. \tag{3.29}$$

LEMMA 3.9. *Let $y_j^n \in [l,r]$ and let the monotonicity condition (3.4) hold. Then p_j^n is a convex combination of $p_{j-1}^{n+1}, p_j^{n+1}, p_{j+1}^{n+1}$. Hence, the linear operator*

$$A : ((p_j^{N_T})_{j\in\mathbb{N}}, (p_0^n)_{1\leq n\leq N_T}) \in \ell^\infty \mapsto (p_j^n)_{j\in\mathbb{N}, 0\leq n\leq N_T} \in \ell^\infty$$

has operator norm ≤ 1 and values in the convex hull of the boundary and end data.

Proof. This results from (3.24), the non-negativity of $B_{j,k}^n$ and $\sum_{k=-1}^{1} B_{j,k}^n = 1$. ∎

By this we obtain the following.

LEMMA 3.10. *Let $y_j^n \in [l,r]$ for all $j \in \mathbb{N}_0, n \in \{0, \ldots, N_T - 1\}$ and let the monotonicity condition (3.4) hold. Then the solution of the adjoint scheme satisfies*

$$|p_j^n| = \|p_\Delta\|_{B(Q_j^n)} \leq \max\{\|p_\Delta^T\|_{B(J_j^n)}, |p_B|\}$$

with $J_j^n :=] \max\{0, x_j - (N_T - n)\Delta x\}, x_j + (N_T - n)\Delta x[$.

Proof. This follows directly by applying Lemma 3.9.

LEMMA 3.11. *Let $y_j^n \in [l,r]$ for all $j \in \mathbb{N}_0$ and let the monotonicity and CFL condition (3.4) hold. Then the solution of the adjoint scheme satisfies the BV-estimate*

$$\sum\nolimits_{j\in\mathbb{N}_0} |p_{j+1}^n - p_j^n| \leq \sum\nolimits_{j\in\mathbb{N}_0} |p_{j+1}^{n+1} - p_j^{n+1}|.$$

Proof. If we set $p_j^{n+1} = p^B$ and $y_j^n = y_0^n$ for $j < 0$ and denote the corresponding values obtained by the scheme by $\tilde{p}_j^n$, we obtain by (3.27), (3.28)

$$\sum\nolimits_{j\in\mathbb{Z}} |\tilde{p}_{j+1}^n - \tilde{p}_j^n| \leq \sum\nolimits_{j\in\mathbb{Z}} |p_{j+1}^{n+1} - p_j^{n+1}|.$$

Now $p_j^n = \tilde{p}_j^n$ for $j \geq 1$ and $\tilde{p}_j^n = p^B$ for $j \leq -1$. Hence, $|p_1^n - p_0^n| = |\tilde{p}_1^n - \tilde{p}_{-1}^n| \leq |\tilde{p}_1^n - \tilde{p}_0^n| + |\tilde{p}_0^n - \tilde{p}_{-1}^n|$. This shows that

$$\sum_{j\in\mathbb{N}_0} |p_{j+1}^n - p_j^n| \leq \sum_{j\in\mathbb{Z}} |\tilde{p}_{j+1}^n - \tilde{p}_j^n| \leq \sum_{j\in\mathbb{Z}} |p_{j+1}^{n+1} - p_j^{n+1}| = \sum_{j\in\mathbb{N}_0} |p_{j+1}^{n+1} - p_j^{n+1}|.$$

∎

3.3.2. Convergence of the adjoint scheme. We proceed partially as in [25, 29]. The main difficulty arises from the fact that the Lipschitz bound of the end data is not transferred to the discretized adjoint state, as it fails to hold near the boundary. This property was crucial to prove convergence in [15, 25, 29]. We overcome this problem by using a Lip^+-interpolation estimate of [28, Lemma 2.1].

THEOREM 3.12 (Convergence of the discrete adjoint to the reversible solution). *Let $p^T \in C^{0,1}(\Omega)$ and p^B be constant and let the non-degeneracy condition* (ND) *be satisfied. Assume that y_Δ is generated by the scheme* (3.11) *and let the CFL-condition* (3.12) *be satisfied, then the solution of* (3.24) *based on the mEO-scheme from* (3.11) *converges to the unique reversible solution p of* (1.8)*, more precisely,*

$$p_\Delta \to p \quad in \quad B([0,T]; L^1_{loc}(\Omega)) \quad as \quad \Delta \to 0. \tag{3.30}$$

Moreover, the convergence is uniform on all bounded sets that have a positive distance from rarefaction centers, the boundary of D^- and from shock points at $\{x = 0\}$.

Proof. Step 1: Convergence to some function p. Under the given CFL-condition the mEO-scheme is monotone and OSLC consistent by Lemma 3.5. Due to Lemma 3.11 the grid function $p_\Delta(t, \cdot)$ is bounded in BV, so by [2, Theorem 3.23] we can select a diagonal subsequence (Δ'_i) such that $p_{\Delta'_i}(\bar{t}, \cdot)$ converges for all $\bar{t} \in S$ in a dense subset $S \subset [0,T]$ in $L^1(0,R)$ for all $R > 0$. If we now show that $t \in [0,T] \mapsto p_{\Delta'_i}(t, \cdot) \in L^1(0,R)$ is equicontinuous, (3.30) follows by an Arzela-Ascoli argument. For arbitrary $0 \le t_1 < t_2 \le T$ it holds

$$p_{\Delta'_i}(t_2, x) - p_{\Delta'_i}(t_1, x) = \sum_{n=N_{t_1}}^{N_{t_2}-1} (p^{n+1}_{\Delta'_i}(x) - p^n_{\Delta'_i}(x)). \tag{3.31}$$

The adjoint scheme (3.24) yields

$$|p^{n+1}_j - p^n_j| \le \lambda \left| F_a(y^n_j, y^n_{j+1}) \Delta^+ p^{n+1}_j + F_b(y^n_{j-1}, y^n_j) \Delta^+ p^{n+1}_{j-1} \right|.$$

Set $I =]z_1, z_2[\subset \Omega$. Summing the above inequality for $\{j \colon \Lambda(R_j \cap I) > 0\} = \{j \colon x_{j+1/2} \in]z_1, z_2 + \Delta'_i x[\}$ with weights $\Lambda(I \cap R_j)$ yields

$$\|p^{n+1}_{\Delta'_i}(x) - p^n_{\Delta'_i}(x)\|_{1,I} \le \Delta'_i t \, 2 M_{f'} \|p^{n+1}_{\Delta'_i}\|_{TV(]z_1 - \Delta'_i x, z_2 + \Delta'_i x[)}$$

where we set again $p^{n+1}_j = p^B$ for $j < 0$ for all n. Altogether, there is a constant depending on p^T_Δ such that $\|p_{\Delta'_i}(t + \Delta'_i t, x) - p_{\Delta'_i}(t, x)\|_{1,I} \le \text{const} \cdot \Delta'_i t$ for all $t \in [0, T - \Delta t]$. With (3.31) we deduce that $p_{\Delta'_i}$ is equicontinuous in time and by an Arzela-Ascoli argument (3.30) is satisfied. Moreover, the equicontinuity in time gives $p \in B([0,T]; BV_{loc}(\Omega)) \cap BV_{loc}(\Omega_T)$ by the lower semicontinuity of the BV-norm under L^1-convergence. To shorten the notation, we write in the sequel Δ instead of Δ'_i.

Step 2: Limit p attains end data. In the next step, we prove that the limit function p is a solution to the adjoint equation (1.8). By the above inequality $p_\Delta(t, \cdot)$ is equicontinuous, thus for all $t \in [T - \Delta t, T]$ it holds

$$\|p_\Delta(t, \cdot) - p_\Delta(T, \cdot)\|_{1,[0,R]} \le \text{const} \cdot (|T - t| + \Delta t). \tag{3.32}$$

Moreover, (3.29) ensures $\|p_\Delta(T, \cdot) - p^T_\Delta\|_{B([0,R])} \to 0$, thus including (3.30) yields as required $\lim_{t \nearrow T} \|p(t, \cdot) - p^T\|_{1,[0,R]} = 0$.

Step 3: Limit p attains boundary data. The previous statements are satisfied for arbitrary monotone numerical flux functions. To show that the limit function p attains the boundary data in the correct sense, we use the modified Engquist-Osher scheme from (3.11). Let $\{\theta_1, \dots, \theta_{\mathfrak{n}_T}\}$ be the transition points of $f'(y(\cdot, 0+))$, see Definition 2.3, and denote as above the outflow boundary by $\mathcal{T}_B \coloneqq \{t \in]0, T[\,:\, f'(y(t, 0+)) < 0\}$. By Lemma 2.4 $\mathcal{T}_B$ consists of $\mathfrak{n}_T$ intervals with endpoints θ_i, $i = 1, \dots, \mathfrak{n}_T$. Choose an arbitrary interval $I_B = [\tau_1, \tau_2] \subset \mathcal{T}_B$ having an arbitrary distance $\kappa > 0$ from rarefaction centers.

By Lemma 3.5 for all $0 < \Delta \leq \Delta_0$ the discrete OSLC (3.10) holds with $\tau_0 = \tau_1 - \kappa$ for $y_\Delta(t, \cdot)$ for all $t \in I_B$. Thus, we find $C > 0$ such that $Lip^+_\Delta(y_\Delta(t, \cdot); \Omega) \leq C$ for all $t \in I_B$. Then by Lemma 3.7 for any $\delta > 0$ there is $\Delta' > 0$ such that for all $0 < \Delta < \Delta'$ it holds $f(y_1^n) > f(y_0^n)$ and $y_1^n < \sigma$ for $n \in \{N_{\tau_1+\delta}, \dots, N_{\tau_2} - 1\}$. By assumption we have $f'(u_B) \geq \beta > 0$ and thus also $f'(y_0^n) \geq \beta > f'(\sigma) = 0$. Hence, by $f(y_1^n) > f(y_0^n)$ and $y_1^n < \sigma$ there exists $\varepsilon > 0$ with $y_1^n < \sigma - \varepsilon$. Since the grid function y_Δ is OSLC-consistent on $[\tau_1, \tau_2] \times \Omega$, there is a sufficiently small $\rho > 0$ such that for $\Delta \leq \Delta'$ it holds $y_j^n < \sigma$ and thus $f'(y_j^n) < 0$ for all $n, j \in \mathbb{N}$ with $(t_n, x_j) \in [\tau_1 + \delta, \tau_2] \times [0, \rho[$.

Hence, $F^G(y_0^n, y_1^n) = f(y_1^n)$ and $F^{EO}(y_j^n, y_{j+1}^n) = f(y_{j+1}^n)$ and therefore (3.24) yields $p_j^n = (1 + \lambda f'(y_j^n))p_j^{n+1} - \lambda f'(y_j^n)p_{j-1}^{n+1}$ for $j \geq 1$ with $x_j < \rho$ and $n \in \{N_{\tau_1+\delta}, \dots, N_{\tau_2} - 1\}$. Since $f'(y_j^n) < 0$ holds, we obtain with $\alpha_j^n = \lambda |f'(y_j^n)|$ that $p_j^n = (1 - \alpha_j^n)p_j^{n+1} + \alpha_j^n p_{j-1}^{n+1}$. For brevity, we set $N_1 := N_{\tau_1+\delta}$, $N_2 = N_{\tau_2}$. Then, since $p_0^n = p_B$ for all n, we obtain inductively

$$p_j^{N_1} = \sum\nolimits_{l=0}^{j} w_l^{N_2} p_l^{N_2}, \quad w_l^{N_2} \geq 0, \quad \sum\nolimits_{l=0}^{j} w_l^{N_2} = 1, \tag{3.33}$$

$$w_l^{N_2} = \sum_{\{n_1 < \ldots < n_{j-l}\} \subset \{N_1, \dots, N_2 - 1\}} \prod_{k=1}^{j-l} \left(\alpha_{j-k+1}^{n_k} \prod_{n=n_k+1}^{n_{k+1}} (1 - \alpha_{j-k+1}^n) \right), \quad l = 1, \dots, j,$$

with $n_{j-l+1} = N_2 - 1$. If now $\alpha \in]0, \alpha_l^n]$ and $\beta \in [\alpha_l^n, 1[$ holds for all $1 \leq l \leq j$ and $N_1 \leq n \leq N_2$, the weights can be estimated by

$$w_l^{N_2} \leq \tbinom{N_2 - N_1}{j-l} \beta^{j-l} (1 - \alpha)^{N_2 - N_1 - j + l}, \quad l = 1, \dots, j.$$

Let $N = N_2 - N_1$, fix $m > 0$ such that $\frac{1}{m} < \alpha$ and consider j with $jm \in]N - m, N]$ for fixed m such that $\frac{1}{m} < \alpha$. Observe that m is independent of the grid size Δ. We want to derive an upper bound for $\sum_{l=1}^{j} w_l^{N_2}$. We have $w_l^{N_2} \leq (\beta/\alpha)^{j-l} \binom{N}{j-l} \alpha^{j-l} (1 - \alpha)^{N-j+l}$, where the right hand side is an integral over the tail of a binomial distribution X with expected value $E(X) = \alpha N$ and variance $V(X) = \alpha(1 - \alpha)N$. As a consequence of Stirlings formula, one obtains the estimate

$$\tbinom{N}{l} \alpha^l (1 - \alpha)^{N-l} \leq \frac{C}{\sqrt{2\pi V(X)}} e^{-\frac{(l - E(X))^2}{2V(X)}}$$

for $0 < \alpha < 1/2$ and $l < \alpha N$, see [12, Thm. VII.3.1, Prob. VII.7.13, Prob. VII.7.15] and a straightforward extension of the arguments there to the tail $\frac{\alpha N - l}{N} = O(1)$.

Hence, the substitution $s = \frac{t - E(X)}{\sqrt{V(X)}}$ leads to

$$\begin{aligned}
\sum\nolimits_{l=1}^{j} w_l^{N_2} &\leq (\beta/\alpha)^j \sum\nolimits_{l=0}^{j} \tbinom{N}{l} \alpha^l (1 - \alpha)^{N-l} \leq (\beta/\alpha)^j \frac{C}{\sqrt{2\pi V(X)}} \int_{-\infty}^{j} e^{-\frac{(t - E(X))^2}{2V(X)}} \,\mathrm{d}t \\
&= (\beta/\alpha)^j \frac{C}{\sqrt{2\pi}} \int_{-\infty}^{\frac{j - E(X)}{\sqrt{V(X)}}} e^{-\frac{s^2}{2}} \,\mathrm{d}s \leq e^{\ln(\frac{\beta}{\alpha}) \frac{N}{m}} \frac{C}{\sqrt{2\pi}} \int_{-\infty}^{\frac{(\frac{1}{m} - \alpha)N}{\sqrt{N\alpha(1-\alpha)}}} e^{-\frac{s^2}{2}} \,\mathrm{d}s \\
&\leq e^{\ln(\frac{\beta}{\alpha}) \frac{N}{m}} \frac{C}{\sqrt{2\pi}} \int_{-\infty}^{2(\frac{1}{m} - \alpha)\sqrt{N}} e^{-\frac{s^2}{2}} \,\mathrm{d}s = \frac{C}{\sqrt{2\pi}} \int_{-\infty}^{2(\frac{1}{m} - \alpha)\sqrt{N}} e^{-\frac{s^2}{2} + \ln(\frac{\beta}{\alpha}) \frac{N}{m}} \,\mathrm{d}s
\end{aligned}$$

for a constant $C > 0$. Now choose m such that $\alpha - \frac{1}{m} > \frac{\alpha}{2}$ and $\frac{1}{m}\ln(\frac{\beta}{\alpha}) < \frac{\alpha^2}{4}$. Then for $s \leq 2(\frac{1}{m} - \alpha)\sqrt{N} \leq -\alpha\sqrt{N}$ it holds $-s^2/4 \leq -N\alpha^2/4 \leq -\ln(\beta/\alpha)\, N/m$ and therefore $-\frac{s^2}{2} + \ln\left(\frac{\beta}{\alpha}\right)\frac{N}{m} \leq -\frac{s^2}{4}$. Altogether, we obtain

$$\sum_{l=1}^{j} w_l^{N_2} \leq (\beta/\alpha)^j \sum_{l=0}^{j} \binom{N}{l}\alpha^l(1-\alpha)^{N-l} \leq \frac{C}{\sqrt{2\pi}}\int_{-\infty}^{-\alpha\sqrt{N}} e^{-\frac{s^2}{4}}\,\mathrm{d}s \ \to 0$$

as $N \to \infty$ and together with (3.33), for all $x_j < \rho$ we deduce

$$p_\Delta^{N_1}(x_j) - p^B = \sum_{l=1}^{j} w_l^{N_2}(p_\Delta^{N_2}(x_l) - p^B) + (w_0^{N_2} - 1)p^B \to 0 \quad \text{as} \quad \Delta \to 0. \tag{3.34}$$

Hence, varying δ and m we have shown that the limit function p satisfies $p = p_B$ on $\{(t,x) \in]\tau_1,\tau_2]\times]0,\rho[: x < \min\{\alpha/2, \alpha^2/(4\ln(\beta/\alpha))\}(\tau_2 - t)\}$. Moreover, the convergence is for each $\delta > 0$ uniform on

$$\{(t,x) \in]\tau_1+\delta,\tau_2]\times]0,\rho[: x < \min\{\alpha/2, \alpha^2/(4\ln(\beta/\alpha))\}(\tau_2 - t)\}. \tag{3.35}$$

By varying τ_1 and τ_2 we can exhaust the outflow domain $\mathcal{T}_B$ up to a set of measure zero and conclude that the trace of p satisfies $p(\cdot,0+)|_{\mathcal{T}_B} = p_B$ a.e. and thus the limit function p attains the boundary data in the correct sense.

Step 4: Limit p is the reversible solution. We now prove that the limit function of the adjoint scheme is actually the reversible solution of the adjoint equation (1.8). The adjoint scheme yields for $(t,x) \in [\Delta t, T[\times[\Delta x/2,\infty[$

$$\begin{aligned} \tfrac{p_\Delta(t,x)-p_\Delta(t-\Delta t,x)}{\Delta t} &+ \big(F_a(y_\Delta(t,x), y_\Delta(t,x+\Delta x))\tfrac{\Delta^+ p_\Delta(t,x)}{\Delta x} \\ &+ F_b(y_\Delta(t,x-\Delta x), y_\Delta(t,x))\tfrac{\Delta^+ p_\Delta(t,x-\Delta x)}{\Delta x}\big) = 0 \end{aligned} \tag{3.36}$$

Since Lemma 3.11 implies that $\frac{\Delta^+ p_\Delta}{\Delta x}$ is bounded in $L^1_{loc}(\Omega_T)$ uniformly in Δ, we conclude that also $\frac{p_\Delta(\cdot,\cdot)-p_\Delta(\cdot-\Delta t,\cdot)}{\Delta t}$ is bounded in $L^1_{loc}(\Omega_T)$ uniformly in Δ. Hence, the convergence (3.30) yields $p \in BV_{loc}(\Omega_T)$ and

$$\tfrac{p_\Delta - p_\Delta(\cdot-\Delta t,\cdot)}{\Delta t} \to p_t,\ \ \tfrac{\Delta^+ p_\Delta}{\Delta x} \to p_x \text{ in } \mathcal{D}'(\Omega_T) \text{ and } L^1_{loc}(\Omega_T)\text{-weak}^* \text{ as } \Delta \to 0. \tag{3.37}$$

We split p^T in monotone parts $p^T = p_1^T - p_2^T$ with $p_l^T \in C^{0,1}(\Omega)$, $(p_l^T)_x \geq 0$, $l = 1,2$. Let $p_{B,l} \leq p_l^T(0)$, $l = 1,2$, with $p_B = p_{B,1} - p_{B,2}$. Denote by p, p_l the reversible solutions of the adjoint equation for data p^T, p_B and $p_l^T, p_{B,l}$, $l = 1,2$, respectively. Now let $p_\Delta^T = T_\Delta p^T$, $p_{l,\Delta}^T = T_\Delta p_l^T$, $l = 1,2$, and p_Δ the solution of the adjoint scheme for data p_Δ^T, p_B and $p_{l,\Delta}$ the solution of the adjoint scheme for data $p_{l,\Delta}^T, p_{B,l}$, $l = 1,2$, respectively. Then by linearity $p_\Delta = p_{1,\Delta} - p_{2,\Delta}$, we show now that $p_{l,\Delta} \to p_l$ in the sense of Theorem 3.12. Then $p_\Delta \to p$ converges in the same sense.

Therefore, we can without restriction consider monotone data $p^T \in C^{0,1}(\Omega)$, $p_x^T \geq 0$ and $p_B \leq p^T(0)$. By the previous results there exists a subsequence $\Delta_i \to 0$ such that $p_{\Delta_i} \to p$ in $B([0,T]; L^1_{loc}(\Omega))$. Moreover, p attains the end data and boundary data in the right sense. Under the CFL-condition we have $B_{j,k}^n, C_{j,k}^n \geq 0$ for the coefficients in (3.25), (3.28). Therefore, the monotonicity of p_Δ and $p_B \leq p_\Delta^T(0)$ yield with (3.24) $p_\Delta(t,0) = p_B \leq p_\Delta(t,x)$ for $(t,x) \in \Omega_T$ and thus by (3.27) $\frac{\Delta^+ p(t,x)}{\Delta x} \geq 0$ for all $(t,x) \in \Omega_T$. Hence, the limit function satisfies $p_x \geq 0$.

To prove that the limit function p is a solution of the adjoint equation, we split the domain Ω_T into three parts; 4.1) the domain outside of the shock funnels, where

y is a classical solution; 4.2) the shock funnels without the closure $(D^-)^{cl}$ of the outflow domain; 4.3) the outflow domain D^-. By our previous results there exists a subsequence $\Delta_i \to 0$ such that $p_{l,\Delta_i} \to p_l$ in $B([0,T];L^1_{loc}(\Omega))$. With the CFL-condition the adjoint scheme is monotone leading to $\frac{\Delta^+ p(t,x)}{\Delta x} \geq 0$ for all $(t,x) \in \Omega_T$,

By the nondegeneracy assumption (ND) the entropy solution $y(T,\cdot)$ has finitely many nondegenerated shocks at $0 < \bar{x}_1 < \cdots < \bar{x}_K$ and is piecewise C^1, see also Theorem 2.8. Let $\bar{x}_k$ be a shock location of $y(T,\cdot)$ and $\xi_k^\pm$ the minimal/maximal characteristic through $(T,\bar{x}_k)$.

Denote by $D_k \subset \Omega_T$ the shock funnel confined by ξ_k^-, ξ_k^+ and by the lines $\{x=0\}$ and $\{t=0\}$. Similarly, denote for arbitrary $0<\delta \ll 1$ by $D_{k,\delta}$ the domain confined by $\xi_k^- - \delta$, $\xi_k^+ + \delta$ and by the lines $\{x=0\}$ and $\{t=0\}$. Let $S := \Omega_T \setminus (\bigcup_{k=1}^K D_k)$ be the domain between the shock funnels and set $S_\delta := (]\delta,T[\times]\delta,\frac{1}{\delta}[) \setminus (\bigcup_{k=1}^K D_k)$.

4.1) We consider the domain S. By the theory of generalized characteristics [10, 22], see also [24], S is covered by genuine backward characteristics that end at $\{x=0\}$ or $\{t=0\}$ and y coincides on S with the solution of the characteristic equations (2.3) and is thus C^1 on S outside of any neighborhood of rarefaction centers. Let $0<\delta\ll 1$ be arbitrary. Then y is in $C^1_b(S^{cl}_\delta)$ and y_Δ satisfies by Lemma 3.5 on $[\delta,T]\times[\delta,\frac{1}{\delta}] \supset S_\delta$ the discrete OSLC (3.10) with $\tau_0=0$. Moreover, Theorem 3.3 yields $\|y_\Delta - y\|_{B([0,T];L^1(]\delta,\frac{1}{\delta}[)} \to 0$ as $\Delta \searrow 0$. Applying now for all $t \in]\delta,T[$ the interpolation inequality [28, Lemma 2.1] for the difference of a one-sided Lipschitz continuous function and a C^1-function, which can be applied uniformly in t, we obtain with a constant $C>0$ and $I_\delta := [\delta,\frac{1}{\delta}]$ for $\Delta\searrow 0$ with $e_\Delta := y_\Delta - y$

$$\|e_\Delta\|_{B(S_{2\delta})} \leq C\|e_\Delta\|^{\frac{1}{2}}_{B([0,T];L^1(I_\delta))}\left(\sup_{t\in[\delta,T]} Lip^+_\Delta(y_\Delta(t,\cdot);I_\delta) + \|y\|_{C^1(S^{cl}_\delta)}\right)^{\frac{1}{2}} \to 0. \tag{3.38}$$

Now we show that p satisfies the adjoint equation (1.8) on S in the distributional sense. Let $\psi \in C_c(S)^\infty$ be arbitrary. Then there is $\delta>0$ such that $\mathrm{supp}(\psi) \subset S_{3\delta}$. For the Engquist-Osher-flux F_a and F_b are continuous. Therefore, the uniform convergence on $S_{2\delta}$ and the continuity of y on S_δ yield $F_a(y_\Delta(\cdot,\cdot), y_\Delta(\cdot,\cdot+\Delta x)) \to F_a(y,y)$ and $F_b(y_\Delta(\cdot,\cdot-\Delta x), y_\Delta(\cdot,\cdot)) \to F_b(y,y)$ in $B(S_{2\delta})$. This together with (3.37) allows limit transition for (3.36) in $L^1(S_{2\delta})$-weak* and thus in $\mathcal{D}'(S)$ and since $F_a(y,y)+F_b(y,y) = f'(y)$, this shows that the limit function satisfies $p_t + f'(y)p_x = 0$ in $\mathcal{D}'(S)$. By Lemma 2.14 and $p(T,\cdot) = p^T$, p is thus the unique reversible solution on S.

The convergence $p_\Delta \to p$ is uniform on $S_{2\delta}$. In fact, p is Lipschitz coninuous on S_δ and $p_\Delta \to p$ in $B([0,T];L^1_{loc}(\Omega))$. Moreover, since p_Δ is bounded and $p_\Delta(t,\cdot)$ is monotone increasing and therefore satisfies a discrete OSLC, we can apply the interpolation result (3.38) with p, p_Δ instead of y, y_Δ (it is easy to check that the C^1 requirement can be weakened to $C^{0,1}$). Hence, $p_\Delta \to p$ in $B(S_{2\delta})$ for all $\delta>0$.

4.2) We show now that p is constant on $D_k \setminus (D^-)^{cl}$. Consider first D_k such that $\xi_k^\pm$ both intersect the line $\{t=0\}$. Then $D_k = D_k\setminus(D^-)^{cl}$. For $\varepsilon_0>0$ small enough all backward characteristics $\xi(\cdot;T,\bar{x}_k\mp\varepsilon)$, $0<\varepsilon\leq\varepsilon_0$, are genuine and travel in S. Since p is the reversible solution on S, we have

$$p(t,\zeta(t;T,\bar{x}_k\mp\varepsilon)) = p^T(\bar{x}_k\mp\varepsilon).$$

We find $\delta(\varepsilon) \in]0,\varepsilon[$ with $(t,\xi(t;T,\bar{x}_k\mp\varepsilon)) \in S_\delta(\varepsilon)$ for all $t\in[\delta(\varepsilon),T]$. By 4.1) $\|(p_\Delta - p)(\cdot,\xi(\cdot;T,\bar{x}_k\mp\varepsilon))\|_{B([\delta(\varepsilon),T])} \to 0$. The monotonicity of $p_\Delta(t,\cdot)$ yields

$$\begin{aligned} p^T(\bar{x}_k-\varepsilon) - \|(p_\Delta - p)(\cdot,\xi(\cdot;T,\bar{x}_k-\varepsilon))\|_{B([\delta(\varepsilon),T])} &\leq p_\Delta(t,\xi(t;T,\bar{x}_k-\varepsilon)) \leq p_\Delta(t,x) \\ \leq p_\Delta(t,\xi(t;T,\bar{x}_k+\varepsilon)) &\leq p^T(\bar{x}_k+\varepsilon) + \|(p_\Delta-p)(\cdot,\xi(\cdot;T,\bar{x}_k+\varepsilon))\|_{B([\delta(\varepsilon),T])} \end{aligned}$$

for all $(t,x) \in D_k$ with $t \geq \delta(\varepsilon)$. Now $\Delta \to 0$ and then $\varepsilon \to 0$ shows that $p_\Delta(t,x) \to p^T(\bar{x}_k)$ for all $(t,x) \in D_k$ with $t > 0$. By the equicontinuity of p_Δ in time we conclude that this extends to $t = 0$ except possibly to the boundary points $\xi_k^\pm(0)$, if these are rarefaction centers. By using (3.38) and the monotonicity of p_Δ we conclude that $p_\Delta(0,\cdot) \to p^T(\bar{x}_k)$ in $B([\xi_k^- + \delta, \xi_k^+ - \delta])$ for all $\delta > 0$.

Now consider D_k, where ξ_k^- intersects the boundary in a point $(s_k^-, 0)$ with $s_k^- > 0$. Moreover, let $0 \leq s_k^+ < s_k^-$ be the time at which ξ_k^+ intersects the boundary.

Set $s_1 := s_k^-$. Then we can argue exactly as before with $t \in]s_1, T[$ instead of $t \in]0, T[$ to show that $p_\Delta(t,x) \to p^T(\bar{x}_k)$ for all $(t,x) \in D_k$ with $t > s_1$ and also for $t = s_1$ except possibly the point $(s_k^-, 0)$ if it is a rarefaction center. Moreover, $p_\Delta(s_1,\cdot) \to p^T(\bar{x}_k)$ in $B([\delta, \xi_k^+(s_1) - \delta])$ for all $\delta > 0$.

Since the minimal characteristic ξ_k^- is genuine, $(s_1, 0)$ is a continuity point of u_B or the center of a rarefaction wave generated by u_B. Now by assumption u_B is C^1 on an interval $]s_2, s_1[$, where $s_2 = s_k^+$ or $s_2 > s_k^+$ and $(s_2, 0)$ is a rarefaction center or a shock point. Since u_B' is uniformly bounded in the smooth regions, we find a fixed $\rho = \rho(u_B) > 0$ only depending on u_B such that for $s_{1,1} := \max\{s_2, s_1 - \rho\}$ all forward characteristics $\xi(\cdot; s, 0)$, $s \in]s_{1,1}, s_k^-[$ are classical up to a time $t > s_1 = s_k^-$ and cover the region $G := \{(t,x) : t \in]s_{1,1}, s_1], 0 \leq x \leq \zeta(t; s_{1,1}, 0)\}$. Thus, y is a classical C^1-solution on G and we can argue as for S in 4.1) that $y_\Delta \to y$ uniformly on any subset $G_\delta := \{(t,x) : t \in]s_{1,1}, s_1], \delta \leq x \leq \xi(t; s_{1,1}, 0) - \delta\}$, $\delta > 0$. Moreover, $p_\Delta(s_1,\cdot) \to p^T(\bar{x}_k)$ in $B([\delta, \xi_k^+ - \delta])$ for all $\delta > 0$. Hence, again as for S in 4.1) we obtain that the limit p of p_Δ is the unique classical solution of the adjoint equation on any G_δ with $p(s_1,\cdot) = p^T(\bar{x}_k)$ on $\{x : (s_1, x) \in G_\delta\}$, which is given by $p|_{G_\delta} \equiv p^T(\bar{x}_k)$. Moreover, the convergence is uniform on $G_{2\delta}$.

Using now the uniform convergence to $p^T(\bar{x}_k)$ on $G_{2\delta}$ and on $S_{2\delta}$ we obtain with the monotonicity of p_Δ also the uniform convergence $p_\Delta(t,x) \to p(t,x) = p^T(\bar{x}_k)$ for all $(t,x) \in D_k$ with $t > s_{1,1} + \delta$ and $x > \delta$. Thus, $\delta \to 0$ yields $p(t,x) = p^T(\bar{x}_k)$ for all $(t,x) \in D_k$ with $t \geq s_{1,1}$ (where we use the equicontinuity in time) and $x > 0$. Moreover, $p_\Delta(s_{1,1},\cdot) \to p^T(\bar{x}_k)$ in $B([\delta, \xi_k^+(s_{1,1}) - \delta])$ for all $\delta > 0$.

Since ρ depends only on u_B, we can repeat the argument with $s_{1,1}$ instead of s_1 and $s_{1,2} := \max\{s_2, s_{1,1} - \rho\}$ finitely many times until we reach $s_{1,l} = s_2$ and have thus shown that $p_\Delta(t,x) \to p(t,x) = p^T(\bar{x}_k)$ uniformly for all $(t,x) \in D_k$ with $t \geq s_2$ and $x > \delta$. If $s_2 = s_k^+$, we are done. Otherwise, $(s_2, 0)$ is a rarefaction center or a shock point.

If $f'(y(s_2-, 0+)) > 0$, then the flow remains incoming and by assumption u_B is C^1 on an interval $]s_3, s_2[$, where $s_3 = s_k^+$ or $s_3 > s_k^+$ and $(s_3, 0)$ is a rarefaction center or a shock point. We can now proceed exactly as before on $]s_3, s_2[$ instead of $]s_2, s_1[$ to show that $p_\Delta(t,x) \to p(t,x) = p^T(\bar{x}_k)$ uniformly for all $(t,x) \in D_k$ with $t \geq s_3$ and $x > \delta$.

Otherwise, s_2 is a transition point θ_i. Let ξ_i be the maximal backward characteristic through starting at $(s_2, 0)$. Then ξ_i ends in a return point $(\vartheta_i, 0)$ with $0 < \vartheta_i < s_k^+$ or reaches $t = 0$, i.e., $\vartheta_i = s_k^+ = 0$. Set $s_3 := \vartheta_i$. Let η_i denote the shock emanating from $(\theta_i, 0)$. By Remark 2.6 the non-degeneracy condition of transition points according to Definition 2.5(iii) we can construct a stripe $S_i \subset \Omega_T$ around ξ_i of the form (2.4) such that there exists a function $Y \in C_b^1(S_i)$ that coincides with y on $\hat{S}_i := \{(t,x) \in S_i : t > s_3,\ t < s_2 \text{ or } t \geq s_2 \text{ and } x \geq \eta_i(t)\}$. $\hat{S}_i$ is covered by genuine characteristics.

For all $0 < \delta < \varepsilon/2$ we have $G_\delta := \{(t,x) : t \in]\vartheta_i + \delta, \theta_i + \varepsilon - \delta[, x \in [\xi(t; \theta_i + \delta, 0), \xi(t; \theta_i + \varepsilon - \delta, 0)]\} \subset \hat{S}_i$. Since y is C^1 on $\hat{S}_i$ and we know already that $p(s_2,\cdot) =$

$p^T(\bar{x}_k)$ on $]0, \xi(s_2; \theta_i + \varepsilon, 0)[$, we can argue as above that $p_\Delta \to p^T(\bar{x}_k)$ uniformly on $G_{2\delta}$. Using also the uniform convergence on $S_{2\delta}$ we obtain with the monotonicity of p_Δ as above also the uniform convergence $p_\Delta(t,x) \to p(t,x) = p^T(\bar{x}_k)$ for all $(t,x) \in D_k$ with $t > s_3 + \delta$ and $x > \delta$. Thus, $\delta \to 0$ yields $p(t,x) = p^T(\bar{x}_k)$ for all $(t,x) \in D_k$ with $t \geq s_3$ (where we use the equicontinuity in time) and $x > 0$. Moreover, $p_\Delta(s_3, \cdot) \to p^T(\bar{x}_k)$ in $B([\delta, \xi_k^+(s_3) - \delta])$ for all $\delta > 0$.

Now with s_3 instead of s_2 we can continue with the above cases finitely many times until we reach the time s_k^+. We have thus shown that $p_\Delta(t,x) \to p(t,x) = p^T(\bar{x}_k)$ for all $(t,x) \in D_k$ with $t > s_k^+ + \delta$ and $x > \delta$. In addition, $p(t,x) = p^T(\bar{x}_k)$ for all $(t,x) \in D_k$ with $t \geq s_k^+$ (where we use the equicontinuity in time) and $x > 0$. Moreover, $p_\Delta(s_k^+, \cdot) \to p^T(\bar{x}_k)$ in $B([\delta, \xi_k^+(s_k^+) - \delta])$ for all $\delta > 0$

4.3) Finally, we show that $p \equiv p_B$ on D^-. Let as above θ_i be a transition point, ξ_i be the maximal backward characteristic starting at $(\theta_i, 0)$. Then ξ_i ends in a return point $(\vartheta_i, 0)$ with $\vartheta_i > 0$ or reaches $t = 0$, i.e., $\vartheta_i = 0$. Let η_i denote the shock emanating from $(\theta_i, 0)$. By Remark 2.6 the non-degeneracy condition yields as above a stripe $S_i \subset \Omega_T$ around ξ_i of the form (2.4) such that there exists a function $Y \in C_b^1(S_i)$ that coincides with y on $\hat{S}_i := \{(t,x) \in S_i : t \geq \vartheta_i,\ t < \theta_i \text{ or } t \geq \theta_1 \text{ and } x \geq \eta_i(t)\}$.

For all $0 < \delta < \varepsilon/2$ we have $G_\delta := \{(t,x) : t \in]\vartheta_i + \delta, \theta_i - \delta[, x \in [\xi(t; \theta_i - \varepsilon + \delta, 0), \xi(t; \theta_i - \delta, 0)]\} \subset \hat{S}_i$. Since y is C^1 on $\hat{S}_i$, we obtain as above uniform convergence of $y_\Delta \to y$ on $G_{2\delta}$. Moreover, we know by Step 3 that $p_\Delta \to p_B$ uniformly on $[\theta_i - \varepsilon + \delta, \theta_i - \delta] \times]0, \rho[$ for $\rho > 0$ small enough, since this is contained in a set of the form (3.35). Hence, we can argue as above that $p_\Delta \to p_B$ uniformly on $G_{2\delta}$. Since also $p_\Delta(t, 0) = p_B$, the monotonicity of p_Δ yields as before also the uniform convergence $p_\Delta(t,x) \to p_B$ on $\{(t,x) : t \in]\vartheta_i + 2\delta, \theta_i - 2\delta[, x \in [0, \xi(t; \theta_i - 2\delta, 0)]\}$. Since for $\delta \to 0$ the set D^- can be exhausted by these sets, we obtain that $p = p_B$ on D^-.

Hence, we have shown that p is the reversible solution on S, is constant on all $D_k \setminus (D^-)^{cl}$, continuous on $\Omega_T \setminus (D^-)^{cl}$, $p \equiv p_B$ on D^-, and $p(T, \cdot) = p^T$. Moreover, $p_x \geq 0$. Hence, p is a locally Lipschitz continuous solution on $\Omega_T \setminus (D^-)^{cl}$, $p_x \geq 0$ and $p \equiv p_B$ on D^-. Therefore, p is the unique reversible solution of (1.8) by Proposition 2.12. The convergence of the whole sequence $p_\Delta \to p$ follows by a subsequence-subsequence argument.

The uniform convergence on all bounded sets that have a positive distance from rarefaction centers, the boundary of D^- and of the boundary $\{x = 0\}$ follows from the fact that the reversible solution p is locally Lipschitz continuous there, $p_\Delta \to p$ in $B([0,T]; L^1_{loc}(\Omega))$, the monotonicity of p_Δ and the interpolation inequality (3.38) applied to p, p_Δ.

Step 5: Uniform convergence up to interior of inflow boundaries. It remains to show the uniform convergence up to $\{x = 0\}$ on sets with a positive distance from rarefaction centers, D^- and shock points of $\{x = 0\}$.

Let $[\tau_1, \tau_2] \subset]0, T[$ with $f'(y(\cdot, 0+)) > 0$ on $[\tau_1, \tau_2]$ be arbitrary such that $[\tau_1, \tau_2] \times \{0\}$ has distance $\kappa > 0$ from D^-, from rarefaction centers and from shocks. Then (1.3) implies $y(\cdot, 0+) = u_B$ on $[\tau_1, \tau_2]$. By continuity we find $0 < \varepsilon < \kappa$ such that the same holds for $[\tau_1 - \varepsilon, \tau_2 + \varepsilon]$. Since u_B is C^1 on $[\tau_1 - \varepsilon, \tau_2 + \varepsilon]$, we find $\rho > 0$ such that the characteristics emanating from $[\tau_1 - \varepsilon, \tau_2 + \varepsilon] \times \{0\}$ cover the region $G := [\tau_1, \tau_2 + \varepsilon] \times [0, \rho]$ and are classical there. Hence y is C^1 on G.

Now let $\tilde{y}$ be the extension of y to $[\tau_1, \tau_2 + \varepsilon] \times \mathbb{R}$ by setting $\tilde{y}(t,x) = u_B(t) = y(t, 0+)$ for $x < 0$ and analogously $\tilde{y}_\Delta$ be the extension of y_Δ by $\tilde{y}_\Delta(t,x) = u_{B,\Delta}(t) = y_\Delta(t, 0)$ for $x < 0$. Then $\tilde{y}$ is in $C_b^{0,1}([\tau_1, \tau_2 + \varepsilon] \times] - \infty, \rho])$ and C^1 outside of the line $x = 0$. Moreover, $\tilde{y}_\Delta$ inherits the discrete OSLC on $[\tau_1, \tau_2 + \varepsilon] \times] - \infty, \rho]$ from the

OSLC of y_Δ guaranteed by Lemma 3.5 with $\tau_0 = \tau_1 - \kappa$. Moreover, the convergence $y_\Delta \to y$ in $B([0,T]; L^1_{loc}(\Omega))$ implies also that $\tilde{y}_\Delta \to \tilde{y}$ in $B([0,T]; L^1_{loc}(\mathbb{R}))$. Now the interpolation inequality [28, Lemma 2.1], see (3.38), yields $\tilde{y}_\Delta \to \tilde{y}$ in $B([\tau_1, \tau_2 + \varepsilon] \times [-\rho/2, \rho/2])$. Since, $f'(y) \geq \beta/2 > 0$ on $[\tau_1, \tau_2 + \varepsilon] \times [0, \rho/2]$ after a possible reduction of $\rho > 0$, for $\Delta_0 > 0$ small enough, we obtain $f'(y_\Delta) \geq \beta/4 > 0$ on $[\tau_1, \tau_2 + \varepsilon] \times [0, \rho/2]$ for all $0 < \Delta \leq \Delta_0$. Hence, the adjoint scheme (3.36) of the modified Engquist-Osher scheme has for $t_n \in [\tau_1, \tau_2 + \varepsilon - \Delta t]$ and $j = 1$ the form

$$p_1^n = p_1^{n+1} + \lambda f'(y_1^{n+1})(p_2^{n+1} - p_1^{n+1}).$$

and consequently the computation of p_Δ on $[\tau_1, \tau_2 + \varepsilon - \Delta t] \times [\Delta x, \infty[$ depends only on $p_\Delta(\tau_2 + \varepsilon, x)$, $x \geq \Delta x$. Hence, if we set $\tilde{p}_\Delta(\tau_2 + \varepsilon, x) = p_\Delta(\tau_2 + \varepsilon, x)$ for $x \geq -\Delta x/2$ and $\tilde{p}_\Delta(\tau_2 + \varepsilon, x) = p_B$ for $x < -\Delta x/2$ and determine $\tilde{p}_\Delta$ on $[\tau_1, \tau_2 + \varepsilon - \Delta t] \times \mathbb{R}$ by applying (3.36) with the state $\tilde{y}_\Delta$ then we have $\tilde{p}_\Delta = p_\Delta$ on $[\tau_1, \tau_2 + \varepsilon - \Delta t] \times [\Delta x, \infty[$.

Now $\tilde{p}_\Delta(\tau_2 + \varepsilon, \cdot) \to p(\tau_2 + \varepsilon, \cdot)\mathbf{1}_{\{x>0\}} + p_B \mathbf{1}_{\{x \leq 0\}} =: \tilde{p}^{\tau_2+\varepsilon}$ and the unique corresponding reversible solution $\bar{p}$ is for $t \in [\tau_1, \tau_2 + \varepsilon]$ given by

$$\begin{cases} \bar{p}(t,x) = p(t,x), & x > 0, \\ \bar{p}(t, f'(u_B(s))(t-s)) = p(s, 0+), & t \leq s < \tau_2 + \varepsilon, \\ \bar{p}(t,x) = p_B, & x \leq f'(u_B(\tau_2 + \varepsilon))(t - \tau_2 - \varepsilon), \end{cases}$$

and is obviously Lipschitz on $\{(t,x) : t \in [\tau_1, \tau_2 + \varepsilon],\ \ f'(u_B(\tau_2 + \varepsilon))(t - \tau_2 - \varepsilon) < x \leq \rho\} \supset [\tau_1, \tau_2 + \varepsilon/2] \times [-\tilde{\rho}, \rho]$ for $0 < \tilde{\rho} \leq \rho$ small enough.

Exactly as in Step 1, BV bound and equicontinuity in time yield for a subsequence $\Delta_k \to 0$ that $\tilde{p}_{\Delta_k} \to \tilde{p}$ in $B([\tau_1, \tau_2 + \varepsilon]; L^1_{loc}(\mathbb{R}))$ and clearly $\tilde{p} = p$ on $[\tau_1, \tau_2 + \varepsilon] \times \Omega$. As already observed, y is piecewise C^1 and Lipschitz on $[\tau_1, \tau_2 + \varepsilon] \times [-\rho, \rho]$ and $y_\Delta \to y$ uniformly on $[\tau_1, \tau_2 + \varepsilon] \times [-\rho/2, \rho/2]$ (on $x < 0$ this is trivial, since $u_{B,\Delta} \to u_B$ uniformly). Hence, as in 4.1) above we obtain that $\tilde{p}$ is a distributional solution of the adjoint equation on $[\tau_1, \tau_2 + \varepsilon] \times [-\rho/2, \rho/2]$ and coincides on $[\tau_1, \tau_2 + \varepsilon] \times [0, \rho/2]$ with the unique reversible solution p. By an obvious variant of Lemma 2.14 with end and boundary data we obtain that $\tilde{p} = \bar{p}$ on $[\tau_1, \tau_2 + \varepsilon/2] \times [-\tilde{\rho}/2, \rho/2]$. Thus, $\tilde{p} = \bar{p}$ is Lipschitz on this set. By a subsequence-subsequence argument we obtain $\tilde{p}_\Delta \to \tilde{p}$ in $B([\tau_1, \tau_2 + \varepsilon/2]; L^1(-\tilde{\rho}/2, \rho/2))$ for $\Delta \to 0$.

Since $\tilde{p} = \bar{p}$ is Lipschitz on $[\tau_1, \tau_2 + \varepsilon/2] \times [-\tilde{\rho}/2, \rho/2]$, we can apply (3.38) with $\tilde{p}, \tilde{p}_\Delta$ instead of y, y_Δ and obtain $\tilde{p}_\Delta \to \tilde{p}$ in $B([\tau_1, \tau_2 + \varepsilon/2] \times [-\tilde{\rho}/4, \rho/4])$ and thus $p_\Delta \to p$ in $B([\tau_1, \tau_2 + \varepsilon/2] \times [\Delta x, \rho/4])$. This completes the proof. ∎

Theorem 3.12 ensures the convergence of the discrete adjoint-based gradient for a smoothed version of the objective (1.4). For $\delta > 0$ and $\varphi_\delta \in C_c^1((-\delta, \delta))$ consider

$$J^\delta(y(u)) = \int_I \gamma(x)\psi((\varphi_\delta *_x y)(T,x), (\varphi_\delta *_x y_d)(x))\, \mathrm{d}x. \tag{3.39}$$

The adjoint formula for $d_u J^\delta(y(u)) \cdot \delta u$ and its discretization $d_{u_\Delta} J^\delta_\Delta(y_\Delta(u_\Delta)) \cdot \delta u_\Delta$ are given by (1.7) and (3.26) with $p^B = 0$ and end data $p^T = \gamma\psi_{y,\delta} \in C_c^\infty(\Omega)$, where

$$\psi_{y,\delta}(x) := \int_I \varphi_\delta(z - x)\psi_y((\varphi_\delta *_x y)(T,z), (\varphi_\delta *_x y_d)(z))\, \mathrm{d}z$$

and its discrete counterpart $p^T_\Delta = \gamma_\Delta \psi^\Delta_{y,\delta} := T_\Delta \gamma\psi_{y,\delta}$, respectively.

Lemma 3.13. *With the assumptions of Theorem 3.12 let $\delta > 0$ be arbitrary. Assume that no rarefaction center of the initial data u_0 and no rarefaction center or* *shock point of the boundary data u_B is shifted. Let the discrete adjoint state p_Δ be computed by (3.24) using the mEO-scheme (3.11) for end data $p_\Delta^T = \gamma_\Delta \psi_{y,\delta}^\Delta$ and zero boundary data $p^B = 0$. Then the discrete gradient adjoint representation of $d_{u_\Delta} J_\Delta^\delta(y_\Delta(u_\Delta)) \cdot \delta u_\Delta$ converges to the gradient $d_u J^\delta(y(u)) \cdot \delta u$ of the smoothed objective (3.39).*

Proof. We have to take the limit of

$$
\begin{aligned}
d_{u_\Delta} J_\Delta^\delta(y_\Delta(u_\Delta)) \cdot \delta u_\Delta &= \int_\Omega p_\Delta(0,x) \delta u_{0,\Delta}(x)\, \mathrm{d}x \\
&\quad + \int_0^{T-\Delta t} p_\Delta(t+\Delta t, \Delta x) F_a(u_{B,\Delta}(t), y_\Delta(t,\Delta x)) \delta u_{B,\Delta}(t)\, \mathrm{d}t
\end{aligned}
\tag{3.40}
$$

where p_Δ is determined by (3.24) based on the mEO-scheme for end data $p_\Delta^T = \gamma_\Delta \psi_{y,\delta}^\Delta$. By the non-degeneracy assumption u_0 has no shocks or rarefaction centers in a small neighborhood of $D^- \cap \{t=0\}$, thus by Theorem 3.12, the first term on the right hand side in (3.40) has the correct limit. As we use the mEO-scheme the last term on the right hand side in (3.40) is equal to

$$
\int_0^{T-\Delta t} p_\Delta(t+\Delta t, \Delta x) \mathbf{1}_{\{t:\ f(u_{B,\Delta}(t)) \geq f(\min\{y_\Delta(t,\Delta x), \sigma\})\}} f'(u_{B,\Delta})_+ \delta u_{B,\Delta}\, \mathrm{d}t.
$$

By Lemma 3.7 the last term on the right hand side in (3.40) becomes

$$
\int_0^{T-\Delta t} p_\Delta(t+\Delta t, \Delta x) \mathbf{1}_{\{t:\ f'(y_\Delta(t,\Delta x))>0\}} f'(u_{B,\Delta}) \delta u_{B,\Delta}\, \mathrm{d}t.
$$

As no shocks or rarefaction centers are shifted, the sensitivity δu_B does not include Dirac-measures and it holds $\delta u_{B,\Delta} \to \delta u_B$ in $L^1(0,T)$. Moreover, using the same arguments as in Lemma 3.6 gives $\mathbf{1}_{\{t:\ f'(y_\Delta(t,\Delta x))>0\}} \to \mathbf{1}_{\{t:\ f'(y(t,0+))>0\}}$ pointwise on $]0,T[$. Now Lebesgue's dominated convergence theorem yields

$$
\|\mathbf{1}_{\{t:\ f'(y_\Delta(t,\Delta x))>0\}} - \mathbf{1}_{\{t:\ f'(y(t,0+))>0\}}\|_{L^q(\mathbb{R})} \to 0 \quad \text{for all} \quad q \in [1,\infty[\quad \text{as} \quad \Delta \to 0.
$$

Together with Theorem 3.12 the last term on the right hand side in (3.40) has the correct limit which finishes the proof. □

Remark 3.14.

1. One can extend Lemma 3.13 such that shock positions can be shifted by smoothing u_B suitably in a κ-neighborhood of shock points and by using the resulting state $y_{\Delta,\kappa}$ in the adjoint scheme. Setting $\kappa = \kappa(\Delta) = \Delta^q$ for some $0 < q < 1/2$ one can take the limit $\Delta \to 0$ in (3.40), see [1].
2. Theorem 3.12 and Lemma 3.13 do not require that the state y_Δ is generated by the scheme to which the adjoint scheme belongs. y_Δ has only to ensure (3.7), (3.8), (3.10), (3.14) and the assertions of Lemma 3.7. Hence, also an optimize-then-discretize approach is covered.
3. Using the post-processing method for the discontinuous end data p^T from [25], Lemma 3.13 can be applied to the original gradient $d_u J(y(u)) \cdot \delta u$.

4. Numerical example. As an illustrating example consider on $(0,2) \times (0,1)$

$$
y_t + (y^2/2)_x = 0, \quad y(0,x) = -1, \quad u_{B,0}(t) = \begin{cases} \frac{1}{2} & t \in [0, \frac{1}{2}] \cup (\frac{7}{5}, 2] \\ 2 & (\frac{1}{2}, \frac{7}{5}] \end{cases}, \quad u_{B,1}(t) = -1,
$$

where $u_{B,0}$ and $u_{B,1}$ are boundary data at $x = 0$ and $x = 1$ in the sense of [3]. The boundary data generate a shock at $(\frac{1}{2}, 0)$ with speed $\frac{1}{2}$ and a rarefaction wave at $(\frac{7}{5}, 0)$, which start to interact at $(\frac{7}{5}, \frac{3}{5})$. The corresponding adjoint state, i.e., the reversible solution of (1.8), can be computed analytically. We choose $T = 2$ and

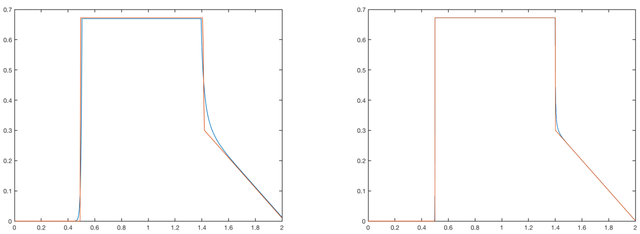

FIG. 1. *$p(\cdot, 0)$ (red) and $p_\Delta(\cdot, 0)$ (blue) for $\Delta x = 10^{-2}$ (left) and $\Delta x = 10^{-3}$ (right).*

$p^T(x) = x$ corresponding to $J(y) = \int_0^1 xy(2, x)\,\mathrm{d}x$, set $\lambda = \frac{\Delta t}{\Delta x} = \frac{1}{2}$ and apply the modified Engquist-Osher scheme (3.11) to compute y_Δ and its discrete adjoint scheme (3.24), (3.25) to compute the discrete adjoint state p_Δ. Figure 1 shows the exact adjoint state $p(\cdot, 0)$ at the left boundary in red (the gradient of $J(y(u))$ with respect to $u_{B,0}$), and the discrete adjoint $p_\Delta(\cdot, 0)$ in blue (discrete gradient) for $\Delta x = 10^{-2}$ and $\Delta x = 10^{-3}$, respectively. As shown in Theorem 3.12, the convergence is uniform ouside of any neighborhood of the two discontinuities located at the boundary of the ouflow domain D^- $(t = \frac{1}{2})$ and at the center of the rarefaction wave $(t = 7/5)$.

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Supplementary material for the paper

Convergence of numerical adjoint schemes arising from optimal boundary control problems of hyperbolic conservation laws

by P. Schäfer Aguilar and S. Ulbrich

Proof of Lemma 3.2, Theorem 3.3 and Lemma 3.5.

Proof of Lemma 3.2. The first assertion holds by [7, Lemma 3.1] and (3.6).

To show (ii), set $\bar v_j^n = v_j^n, \bar w_j^n = w_j^n$ for $j \geq 0$ and $\bar v_j^n = \bar w_j^n = 0$ for $j < 0$. Then we have by [9] with the scheme (3.5) $\|\mathcal{H}(\bar v_\Delta^n) - \mathcal{H}(\bar w_\Delta^n)\|_{1,\mathbb{R}} \leq \|\bar v_\Delta^n - \bar w_\Delta^n\|_{1,\mathbb{R}} = \|v_\Delta^n - w_\Delta^n\|_{1,[-\Delta x/2,\infty[}$. We obtain with (3.6) $\|\mathcal{H}_B(v_\Delta^n) - \mathcal{H}_B(w_\Delta^n)\|_{1,[\Delta x/2,\infty[} \leq \|v_\Delta^n - w_\Delta^n\|_{1,[-\Delta x/2,\infty[}$ and it follows

$$\|\mathcal{H}_B(v_\Delta^n) - \mathcal{H}_B(w_\Delta^n)\|_{1,[-\Delta x/2,\infty[} \leq \|v_\Delta^n - w_\Delta^n\|_{1,[-\Delta x/2,\infty[} + \Delta x|v_0^{n+1} - w_0^{n+1}|.$$

Now, set $v_j^n = v_B^n$ for $j < 0$ and denote by $\tilde v_j^{n+1}$ the corresponding grid values obtained by the scheme (3.5). Then $\tilde v_j^{n+1} = v_j^{n+1}$ for $j \geq 1$, $\tilde v_j^{n+1} = v_B^n$ for $j \leq -1$ and by (i) the resulting scheme is monotone. To verify (iii) set $w_\Delta^n = v_\Delta^n(\cdot + \Delta x)$, then by using again $\|\mathcal{H}(v_\Delta^n) - \mathcal{H}(w_\Delta^n)\|_{1,\mathbb{R}} \leq \|v_\Delta^n - w_\Delta^n\|_{1,\mathbb{R}}$, see [9], we deduce

$$\|\tilde v_\Delta^{n+1}\|_{TV,]\Delta x/2,\infty[} + |v_1^{n+1} - \tilde v_0^{n+1}| + |\tilde v_0^{n+1} - v_B^n| \leq \|v_\Delta^n\|_{TV}$$

and using the triangle inequality we obtain

$$\begin{aligned}
\|v_\Delta^{n+1}\|_{TV} &= \|\tilde v_\Delta^{n+1}\|_{TV,]\Delta x/2,\infty[} + |v_1^{n+1} - v_B^{n+1}| \\
&\leq \|v_\Delta^n\|_{TV} - |v_1^{n+1} - \tilde v_0^{n+1}| - |\tilde v_0^{n+1} - v_B^n| + |v_1^{n+1} - v_B^{n+1}| \\
&\leq \|v_\Delta^n\|_{TV} - |v_1^{n+1} - v_B^n| + |v_1^{n+1} - v_B^{n+1}| \leq \|v_\Delta^n\|_{TV} + |v_B^{n+1} - v_B^n|. \quad \square
\end{aligned}$$

Proof of Theorem 3.3. First assume that $u_0 \in BV_{loc}(\Omega)$ and $u_B \in BV(0,T)$. Lemma 3.2 yields $y_j^n \in [l - T\|g\|_\infty, r + T\|g\|_\infty]$ and $y_\Delta^n \in BV_{loc}(\Omega)$, since $g \in L^1(0,T;BV(\Omega))$. We extend the boundary data and source by $y_j^n = y_0^n$, $G_j^n = 0$ for $j < 0$ and apply the operator $\mathcal{H}(y_\Delta^n) + \Delta t g_\Delta^n$ with $\mathcal{H}$ in (3.5). For monotone fluxes with Lipschitz constant L_F it is known that $\|\mathcal{H}(y_\Delta^n) - y_\Delta^n\|_1 \leq 3\Delta t L_F \|y_\Delta^n\|_{BV}$, see e.g. [9, Prop. 3.5]. We have $y_j^{n+1} = (\mathcal{H}(y_\Delta^n) + \Delta t g_\Delta^n)_j$ for $j \geq 1$ and $\mathcal{H}(y_\Delta^n)_j - y_j^n = 0$ for $j \leq -1$. Hence, we obtain $\|y_\Delta^{n+1} - y_\Delta^n\|_{1,[-\Delta x/2,\infty[} \leq 3\Delta t L_F \|y_\Delta^n\|_{BV} + \Delta t|u_B^{n+1} - u_B^n| + \Delta t\|g_\Delta^n\|_1$. Since y_Δ^n are uniformly bounded in BV_{loc} and by using the equicontinuity in time, there is a subsequence converging in $L^\infty(0,T;L^1_{loc}(\Omega))$ to a function $y \in L^\infty(0,T;BV_{loc}(\Omega)) \cap C([0,T];L^1_{loc}(\Omega))$. We still have to show that y is an entropy solution of (1.1). For monotone fluxes the discrete entropy inequality holds

$$U_k(y_j^{n+1}) \leq U_k(y_j^n) - \frac{\Delta t}{\Delta x}\Delta^+ Q_k(y_{j-1}^n, y_j^n) + \Delta t U_k'(y_j^{n+1}) G_j^n \tag{4.1}$$

with $U_k(u) = |u - k|$, $Q_k(u,v) = F([u,k]_+, [v,k]_+) - F([u,k]_-, [v,k]_-)$, $U_k'(u) = \operatorname{sgn}(u-k)$, for any $k \in \mathbb{R}$ and $[\alpha,\beta]_+ := \max\{\alpha,\beta\}$ and $[\alpha,\beta]_- := \min\{\alpha,\beta\}$. In fact, since H is monotone increasing w.r.t. all arguments and $k = H(k,k,k)$, we thus have

$$\begin{aligned}
&[y_j^{n+1}, k]_+ \leq H([y_{j-1}^n, k]_+, [y_j^n, k]_+, [y_{j+1}^n, k]_+) + \Delta t \mathbf{1}_{\{y_j^{n+1} > k\}} G_j^n, \\
&[y_j^{n+1}, k]_- \geq H([y_{j-1}^n, k]_-, [y_j^n, k]_-, [y_{j+1}^n, k]_-) + \Delta t \mathbf{1}_{\{y_j^{n+1} < k\}} G_j^n.
\end{aligned}$$

Taking the difference yields the discrete entropy inequality (4.1). Let $\phi_j^n \geq 0$ be grid values of a test function $\phi \in C_c^1([0,T[\times[0,\infty[)$, $\phi \geq 0$. We obtain

$$\sum_{j\geq 1, n\geq 0} \phi_j^n \Delta x \left(U_k(y_j^{n+1}) - U_k(y_j^n) - \Delta t U_k'(y_j^{n+1}) G_j^n + \lambda \Delta^+ Q_k(y_{j-1}^n, y_j^n) \right) \leq 0$$

and summation by parts yields

$$\begin{aligned} &\sum_{j\geq 1, n\geq 0} \Delta x \left(U_k(y_j^{n+1})(\phi_j^n - \phi_j^{n+1}) - \lambda \Delta^+ \phi_j^n Q_k(y_j^n, y_{j+1}^n) - \Delta t \phi_j^n U_k'(y_j^{n+1}) G_j^n \right) \\ &- \sum_{j\geq 1} \Delta x U_k(y_j^0)\phi_j^0 - \sum_{n\geq 0} \Delta t Q_k(u_B^n, y_1^n)\phi_1^n \leq 0. \end{aligned} \tag{4.2}$$

Doing the same for the original scheme instead of the discrete entropy inequality yields

$$\begin{aligned} &\sum_{j\geq 1, n\geq 0} \Delta x \left(y_j^{n+1}(\phi_j^n - \phi_j^{n+1}) - \lambda \Delta^+ \phi_j^n F(y_j^n, y_{j+1}^n) - \Delta t \phi_j^n G_j^n \right) \\ &- \sum_{j\geq 1} \Delta x y_j^0 \phi_j^0 - \sum_{n\geq 0} \Delta t F(u_B^n, y_1^n)\phi_1^n = 0. \end{aligned} \tag{4.3}$$

Now $y_0^n = u_B^n$ and with $\bar{u} = y_0^n - \frac{\Delta t}{\Delta x}(F(y_0^n, y_1^n) - F(y_0^n, y_0^n))$ we obtain as above

$$U_k(\bar{u}) \leq U_k(y_0^n) - \frac{\Delta t}{\Delta x}(Q_k(y_0^n, y_1^n) - Q_k(y_0^n, y_0^n))$$

and by the convexity of U_k (note that $\operatorname{sgn}(u-k) \in \partial U_k(u)$)

$$U_k(\bar{u}) \geq U_k(y_0^n) - \frac{\Delta t}{\Delta x} \operatorname{sgn}(y_0^n - k)(F(y_0^n, y_1^n) - F(y_0^n, y_0^n)).$$

By combining the last two inequalities, we obtain

$$Q_k(y_0^n, y_1^n) - Q_k(y_0^n, y_0^n) - \operatorname{sgn}(y_0^n - k)(F(y_0^n, y_1^n) - F(y_0^n, y_0^n)) \leq 0.$$

Inserting this in (4.2) yields with $F(y_0^n, y_0^n) = f(y_0^n)$

$$\begin{aligned} &\sum_{j\geq 1, n\geq 0} \Delta x \left(U_k(y_j^{n+1})(\phi_j^n - \phi_j^{n+1}) - \lambda \Delta^+ \phi_j^n Q_k(y_j^n, y_{j+1}^n) - \Delta t \phi_j^n U_k'(y_j^{n+1}) G_j^n \right) \\ &- \sum_{j\geq 1} \Delta x U_k(y_j^0)\phi_j^0 - \sum_{n\geq 0} \Delta t \left(Q_k(y_0^n, y_0^n) + \operatorname{sgn}(y_0^n - k)(F(y_0^n, y_1^n) - f(y_0^n)) \right) \phi_1^n \leq 0. \end{aligned} \tag{4.4}$$

Taking test functions $\max\{0, 1 - x/\delta\}\phi(t)$ with $\phi \in C_c^1(]0,T[)$, $\phi \geq 0$ in (4.3) yields for $\Delta x = \Delta t/\lambda \to 0$ and $\delta \searrow 0$

$$\int_{[0,T[} f(y(t, 0+))\phi(t)\,\mathrm{d}t = \lim_{\Delta x \to 0} \int_{[0,T[} F(u_{B,\Delta}(t), y_\Delta(t, \Delta x))\phi(t)\,\mathrm{d}t.$$

Which verifies the first statement (3.9). By continuity this also holds for test functions $\phi \in L^1(0,T)$. Inserting this in (4.4) with the same test functions such that for all k

the set $\{u_B = k\}$ has measure 0 yields

$$\int_{[0,T[} (\operatorname{sgn}(y(t,0+)-k)(f(y(t,0+))-f(k)) - \operatorname{sgn}(u_B(t)-k)(f(u_B(t))-f(k)))$$
$$- \operatorname{sgn}(u_B(t)-k)(f(y(t,0+))-f(u_B(t)))\phi(t)\,\mathrm{d}t \le 0,$$

and thus

$$\int_{[0,T[} (\operatorname{sgn}(y(t,0+)-k)+\operatorname{sgn}(k-u_B(t)))(f(y(t,0+))-f(k))\phi(t)\,\mathrm{d}t \le 0. \tag{4.5}$$

This is equivalent to

$$\min_{k\in I(y(t,0+),u_B(t))} \operatorname{sgn}(k-y(t,0+))(f(y(t,0+))-f(k)) = 0,$$

thus the limit function satisfies the boundary condition in the BLN sense. Now, let $\phi \in C_c^1([0,T[\times[0,\infty[)$, $\phi \ge 0$ be arbitrary. Using test functions $\min\{1, x/\delta\}\phi(t,x)$ in (4.2) the limit $\Delta x = \Delta t/\lambda \to 0$ and $\delta \searrow 0$ results in

$$\int_{\Omega_T} (-|y-k|\phi_t - \operatorname{sgn}(y-k)(f(y)-f(k))\phi_x - \phi\operatorname{sgn}(y-k)g)\,\mathrm{d}x\,\mathrm{d}t$$
$$-\int_\Omega \operatorname{sgn}(y(t,0+)-k)(f(y(t,0+))-f(k))\phi(t,0)\,\mathrm{d}t - \int_\Omega |u_0(x)-k|\phi(0,x)\,\mathrm{d}x \le 0.$$

Including (4.5) yields the weak formulation of the entropy inequality

$$\int_{\Omega_T} (-|y-k|\phi_t - \operatorname{sgn}(y-k)(f(y)-f(k))\phi_x - \phi\operatorname{sgn}(y-k)g)\,\mathrm{d}x\,\mathrm{d}t$$
$$-\int_\Omega \operatorname{sgn}(u_B(t)-k)(f(y(t,0+))-f(k))\phi(t,0)\,\mathrm{d}t - \int_\Omega |u_0(x)-k|\phi(0,x)\,\mathrm{d}x \le 0$$

see also [3, 20]. The limit $\lim_{t\searrow 0}\|y(t,\cdot)-u_0\|_{L^1_{loc}(\Omega)} = 0$ follows from the equicontinuity in time and $u_{0,\Delta} \to u_0$ in $L^1_{loc}(\Omega)$. Consequently the limit function y is the entropy solution of (1.1). By a subsequence-subsequence argument the convergence holds for the whole sequence Δ. Using an approximation argument as in [9] the same holds for controls in $u \in (L^1\cap L^\infty)(\Omega)\times(L^1\cap L^\infty)(0,T)$ without the additional BV-bound. □

Proof of Lemma 3.5. In [29] the assertion was proved for Cauchy problems, see also [27]. We proceed similarly, but take the boundary data into account. Let without restriction $\tau_0 = 0$. Define $\ell_j^n := \frac{y_{j+1}^n - y_j^n}{\Delta x}$. We analyze first ℓ_0^n, $n = 1,\dots,N_T$, i.e., the behavior at the boundary. By assumption, $f'(u_B) \ge f'(\gamma) = \beta > 0$ on $[0,T]$ holds for some $\gamma > \sigma$, hence we have $y_0^n > \sigma$ for all $n = 1,\dots,N_T$. Moreover, $g = 0$ for $x \le \varepsilon_g$ and thus $G_1^n = 0$. We have to distinguish the following cases:

Case 1: $y_1^n \ge \sigma$, $y_2^n \ge \sigma$. The mEO-scheme reads $y_1^{n+1} = y_1^n - \lambda(f(y_1^n) - f(y_0^n))$, which yields $y_1^{n+1} - y_0^{n+1} = y_0^n - y_0^{n+1} + y_1^n - y_0^n - \lambda(f(y_1^n) - f(y_0^n))$. Now, since $f'' \ge m_{f''} > 0$ holds, we obtain

$$f(y_1^n) - f(y_0^n) \ge f'(y_0^n)(y_1^n - y_0^n) + \frac{m_{f''}}{2}(y_1^n - y_0^n)^2.$$

Hence,

$$y_1^{n+1} - y_0^{n+1} \le y_0^n - y_0^{n+1} + (y_1^n - y_0^n)\left(1 - \lambda f'(y_0^n) - \frac{\lambda m_{f''}}{2}(y_1^n - y_0^n)\right)$$
$$\le y_0^n - y_0^{n+1} + (y_1^n - y_0^n)_+(1 - \lambda f'(y_0^n)).$$

Thus, we have

$$y_1^{n+1} - y_0^{n+1} \le y_0^n - y_0^{n+1} + (y_1^n - y_0^n)_+(1 - \lambda f'(\gamma)).$$

Case 2: $y_1^n \ge \sigma,\ y_2^n < \sigma$. Then the mEO-scheme reads $y_1^{n+1} = y_1^n - \lambda(f(y_1^n) + f(y_2^n) - f(\sigma) - f(y_0^n))$. This yields

$$y_1^{n+1} - y_0^{n+1} \le y_0^n - y_0^{n+1} + y_1^n - y_0^n - \lambda(f(y_1^n) - f(y_0^n))$$

and we can proceed as in Case 1.

Case 3: $y_1^n < \sigma,\ y_2^n \ge \sigma$. The mEO-scheme is given by $y_1^{n+1} = y_1^n - \lambda(f(\sigma) - \max\{f(y_0^n), f(y_1^n)\})$ and we obtain

$$y_1^{n+1} - y_0^{n+1} = y_0^n - y_0^{n+1} + y_1^n - y_0^n - \lambda(f(\sigma) - \max\{f(y_0^n), f(y_1^n)\}).$$

The grid points y_j^n are bounded, so f has a Lipschitz constant L_f. Therefore

$$-\lambda(f(\sigma) - \max\{f(y_0^n), f(y_1^n)\} \le \lambda L_f \max\{y_0^n - \sigma, \sigma - y_1^n\} \le \lambda L_f (y_0^n - y_1^n).$$

Hence, under the CFL-condition $\lambda L_f \le 1$ it holds

$$y_1^{n+1} - y_0^{n+1} \le y_0^n - y_0^{n+1} + (y_0^n - y_1^n)(-1 + \lambda L_f) \le y_0^n - y_0^{n+1}.$$

Case 4: $y_1^n < \sigma,\ y_2^n < \sigma$. The mEO-scheme reads $y_1^{n+1} = y_1^n - \lambda(f(y_2^n) - \max\{f(y_0^n), f(y_1^n)\})$. This yields

$$y_1^{n+1} - y_0^{n+1} \le y_0^n - y_0^{n+1} + y_1^n - y_0^n - \lambda(f(\sigma) - \max\{f(y_0^n), f(y_1^n)\})$$

and we proceed as in Case 3. Altogether, we obtain

$$y_1^{n+1} - y_0^{n+1} \le y_0^n - y_0^{n+1} + (y_1^n - y_0^n)_+(1 - \lambda f'(\gamma)). \tag{4.6}$$

Let $\tilde{C}_0 = \max\{0, \frac{y_1^0 - y_0^0}{\Delta x}\}$. Since the boundary data generate by assumption no rarefaction centers, we find a constant $C_B \ge 0$ with $\lambda \frac{y_0^n - y_0^{n+1}}{\Delta t} \le C_B$ for $n = 0, \dots, N_T - 1$. We define a sequence a_n by $a_{n+1} \le C_B + m a_n$ with $a_0 \le \tilde{C}_0$ and $m = 1 - \lambda f'(\gamma) \in]0, 1[$. This yields $a_n \le m^n \tilde{C}_0 + C_B \sum_{i=0}^{n-1} m^i \le m^n \tilde{C}_0 + \frac{C_B}{1-m}$ and with (4.6) we deduce

$$\ell_0^n \le m^n \tilde{C}_0 + \frac{C_B}{1-m} =: C_n, \quad n = 0, \dots, N_T. \tag{4.7}$$

Let $L_g \ge 0$ be the Lipschitz constant of g with respect to x. By extending results of [29, Lem. 6.5.2] for the EO-scheme to the mEO-scheme, see below, we obtain with some $\nu > 0$ for all $0 < c \le \nu$ the estimate

$$\ell_j^{n+1} \le \ell_{j,1}^{n,+} - \Delta t c (\ell_{j,1}^{n,+})^2 + \Delta t L_g, \quad \ell_{j,1}^{n,+} := \max_{k=-1,0,1} \{(\ell_{j+k}^n)_+\},\ j \ge 1. \tag{4.8}$$

Set $\psi(\ell) := \ell - \Delta t c \ell^2 + \Delta t L_g$. We derive now an upper bound for ℓ_j^n. We observe that $\ell_j^n \leq 2M_y/\Delta x + C_B/(1-m)$ for $j \geq 0$. We clearly find a maximal $0 < c \leq \nu$ such that it holds for all $0 < \Delta \leq \Delta_0$

$$\psi'(\ell) = 1 - 2c\Delta t \ell \geq m \quad \forall\, \ell \in \left[0, \max\{2M_y/\Delta x + 2C_B/(1-m), \sqrt{L_g/c}\}\right]. \tag{4.9}$$

The latter interval contains all $\ell_{j,1}^{n,+}$, $j \geq 1$ and all C_n and ψ has the unique fixed point $\ell = \sqrt{L_g/c}$ on the interval. With (4.7), (4.8) we obtain $\ell_j^{n+1} \leq \max\{C_{n+1}, \psi(\ell_{j,1}^{n,+})\}$ for all $j \geq 0$.

Define $M_n := \sup_{j\geq 1} \max\{\ell_j^n, C_n\}$, then the monotonicity of ψ yields $\ell_j^{n+1} \leq \max\{C_{n+1}, \psi(M_n)\}$ and thus

$$M_{n+1} \leq \max\{C_{n+1}, \psi(M_n)\}. \tag{4.10}$$

Now define

$$\bar{M}_{n+1} := \max\{\tilde{C}_{n+1}, \psi_{\tilde{c}}(\bar{M}_n)\}, \quad \bar{M}_0 = M_0$$

then the monotonicity of $\psi_{\tilde{c}}$ yields $\bar{M}_n \geq M_n$ for all $n \geq 0$. We consider two cases.

Case 1: $\bar{M}_0 \leq \sqrt{L_g/c}$. Then by (4.8) and the fact that $\sqrt{L_g/c}$ is the unique fixed point on the interval in (4.8), we have $\bar{M}_0 \leq \psi(\bar{M}_0) \leq \sqrt{L_g/c}$ and $C_1 \leq C_0 \leq \bar{M}_0$. Hence, $\bar{M}_1 = \psi(\bar{M}_0)$ and we obtain inductively $\bar{M}_{n+1} = \psi(\bar{M}_n)$, $\bar{M}_{n+1} \leq \sqrt{L_g/c}$ for all $n \geq 0$

Case 2: $\bar{M}_0 > \sqrt{L_g/c}$. Then we obtain similarly as in Case 1 that $(\bar{M}_n)$ is a decreasing sequence $> \sqrt{L_g/c}$. An elementary investigation of the quadratic function

$$\psi(C_n) - C_{n+1} = C_n - \Delta t c C_n^2 + \Delta t L_g - mC_n - C_B =: q(C_n)$$

yields that $q(C)$ has a minimum at $\bar{C} = (1-m)/(2c\Delta t)$ with value $q(\bar{C}) = (1-m)^2/(4c\Delta t) - C_B + \Delta t L_g$. Now (4.9) implies $\bar{M}_n \leq \bar{C}$, $n \geq 0$, as well as $q(\bar{C}) \geq 0$. Therefore, q has a unique zero at some some $C_\Delta \in [(C_B - \Delta t L_g)/(1-m), 2(C_B - \Delta t L_g)/(1-m)]$ and

$$\psi(C_n) - C_{n+1} \begin{cases} \geq 0 & \text{for } C_n \in [C_\Delta, (1-m)/(2\tilde{c}\Delta t)], \\ \leq 0 & \text{for } C_n \leq C_\Delta, \end{cases}$$

Hence, we obtain $\bar{M}_{n+1} = \psi(\bar{M}_n)$ until the first $n = n_1$ with $\bar{M}_{n_1} < C_\Delta, \psi(\bar{M}_{n_1}) \leq C_{n_1+1}$ and from that point on we obtain $\bar{M}_n = C_n$ for $n \geq n_1 + 1$.

In other words, if we define the sequence $\tilde{M}_{n+1} = \psi(\tilde{M}_n)$, $\tilde{M}_0 = M_0$ we obtain $\bar{M}_n = \max\{\tilde{M}_n, C_n\}$. To estimate $\tilde{M}_n$ we note that $\frac{1}{\Delta t}(\tilde{M}_{n+1} - \tilde{M}_n) = -c\tilde{M}_n^2$. The solution of the initial value problem $\dot{\alpha}(t) = -c\alpha(t)^2$, $\alpha(0) = \tilde{M}_0$ satisfies $\alpha(n\Delta t) \geq \tilde{M}_n$ and is given by $\alpha(t) = (ct + 1/\tilde{M}_0)^{-1}$. Hence, we can conclude $\bar{M}_n = \max\{C_n, \tilde{M}_n\} \leq \max\{C_n, \alpha(t_n)\}$. Finally, (4.9) implies easily that $\psi(\tilde{M}_n) \geq m\tilde{M}_n$ and thus $\tilde{M}_n \geq m^n C_0$ for $n \geq 0$ leading to

$$\bar{M}_n \leq \frac{C_B}{1-m} + \frac{1}{ct_n + \frac{1}{M_0}}.$$

It remains to prove that the modified Engquist-Osher scheme satisfies (4.8) under the assumed CFL condition. In [29, Lemma 6.5.2] it was shown that (4.8) holds for the Engquist-Osher scheme with $\nu = \min\{m_{f''/4,1/(4\lambda M_y)}\}$. The mEO-scheme differs only for $j = 1$ in the transonic case $y_1^n < \sigma < y_0^n$, where

$$F^{EO}(y_0^n, y_1^n) = f(y_1^n) + f(y_0^n) - f(\sigma) \geq \max\{f(y_1^n), f(y_0^n)\} = F^G(y_0^n, y_1^n).$$

Hence, we have $(\ell_1^{n+1})^{EO} \geq (\ell_1^{n+1})^{mEO}$ and thus [29, Lemma 6.5.2] holds with $j = 1$ also for the mEO scheme. □