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Optimal Control of Elliptic Equations with Pointwise Constraints on the Gradient of the State in Nonsmooth Polygonal Domains

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OPTIMAL CONTROL OF ELLIPTIC EQUATIONS WITH POINTWISE CONSTRAINTS ON THE GRADIENT OF THE STATE IN NONSMOOTH POLYGONAL DOMAINS

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Abstract. This article is concerned with optimal control problems subject to a second order elliptic PDE and additional pointwise constraints on the gradient of the state. In particular, existence of solutions on nonsmooth polygonal or polyhedral domains is analyzed. In this situation the solution operator for the partial differential equation does not provide enough regularity to state the pointwise constraint for any right hand side due to the appearance of singularities associated to the corners, edges and vertices of the domain.

Further, necessary optimality conditions for the solution of the optimization problem in two and three space dimensions are derived. However, in the three dimensional case certain critical angles along the edges of the domain have to be circumvented in the derivation of the optimality conditions.

Finally, the derived optimality conditions are utilized to deduce additional regularity for the control variable.

Key words. optimization with PDEs, first order state constraints, nonsmooth domains, regularity, optimality conditions

AMS subject classifications. 49J20, 49K20, 49N60

1. Introduction. We are concerned with the existence of solutions to optimal control problems of second order elliptic equations subject to constraints on the gradient of the state. Such problems have some natural application for instance in cooling processes or structural optimization when high stresses have to be avoided.

Despite these interesting applications first order state constraints have hardly been recognized in mathematics. In the works [5, 6] the case of optimal control of semilinear elliptic equations with pointwise first order state constraints was studied under the assumption that the domain $\Omega \subset \mathbb{R}^n$ possesses a $C^{1,1}$ boundary. In particular, they studied the adjoint equation and derived first order necessary optimality conditions. It is immediately clear that their results carry over to the case of a polygonally bounded domain, as long as the linearized state equation (with homogeneous Dirichlet boundary values) defines an isomorphism between $W^{2,t}(\Omega) \cap H^1_0(\Omega)$ and $L^t(\Omega)$ for some $t > n$. However, even for $n = 2$ this requires a convex domain which is usually too restrictive for applications. In [21] a Moreau-Yosida based framework for PDE-constrained optimization with constraints on the derivative of the state is developed and used to obtain a semismooth Newton algorithm. In [25] an investigation of barrier methods for this problem class is conducted.

When concerned with the discretization of the infinite dimensional problem using finite elements, recent results where obtained in [11, 19, 24]. However in all cases the domain was either smooth or polygonally bounded with sufficiently small interior angles. Concerning adaptive discretization methods we refer to [29] and the recent contribution [20].

This article is based upon parts of the dissertation [28] of the author. We will consider a bounded, polygonal or polyhedral domain $\Omega \subset \mathbb{R}^n$ with $n = 2, 3$. Then using standard notation for the Lebesgue and Sobolev spaces we consider the following

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distributed optimal control problem

$$\min_{(q,u)\in L^r(\Omega)\times H_0^1(\Omega)} J(q,u) = \frac{1}{2}\|u-u^d\|^2 + \frac{\alpha}{r}\|q\|_{L^r}^r$$
$$\text{subject to } \begin{cases} (\nabla u, \nabla\varphi) = (q,\varphi) & \forall\,\varphi \in H_0^1(\Omega),\\ |\nabla u|^2 \le \psi & \text{a.e. in } \overline{\Omega},\\ a \le q \le b & \text{a.e. in } \Omega, \end{cases} \tag{1.1}$$

where $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$ and $r, \alpha, \psi \in \mathbb{R}$ with $r \ge 2$, $\alpha, \psi > 0$. To assert well posedness we assume that at least $r > n$ or $-\infty < a < b < \infty$. Furthermore, for any given right hand side $f \in L^2(\Omega)$ the state equation

$$(\nabla u, \nabla\varphi) = (f,\varphi) \ \ \forall\,\varphi \in H_0^1(\Omega) \tag{1.2}$$

defines a unique element $u \in H_0^1(\Omega)$ which we will frequently denote by u_f. Moreover, the solution of (1.2) exhibits more regularity, i.e., $u_f \in H^s(\Omega)$ for some $s \ge 3/2$. Because we wish to consider the constraint $|\nabla u|^2 \le \psi$ we need $u \in W^{1,\infty}(\Omega)$ and are thus interested in additional regularity $u \in W = W^{2,t}(\Omega) \cap H_0^1(\Omega)$ for some $t \ge 2$. In particular, if $t > n$ then $W \subset W^{1,\infty}(\Omega)$ and we can pose the pointwise gradient constraint. For given t we denote by t' the dual number, i.e., $t' = \frac{t}{t-1}$.

With this we can define the reduced cost functional

$$\hat{J}(q) := J(q, u_q).$$

The admissible set for the control variable will be denoted by

$$Q^{\text{ad}} := \{q \in L^r(\Omega) \,|\, a \le q \le b \text{ a.e. in } \Omega\}$$

whereas we denote the set of controls feasible for the reduced problem by

$$Q^{\text{feas}} := Q^{\text{ad}} \cap \{q \in L^r(\Omega) \,|\, |\nabla u_q|^2 \le \psi \text{ a.e. in } \overline{\Omega}\}.$$

Remark 1.1. *We note that the consideration of other linear and semilinear elliptic operators with constant coefficients in the main part of the operator as well as different boundary conditions would be possible using the same techniques used in this article because our results depend only on the asymptotics of the singularities near corners, edges, and vertices and not on the precise form of the singular functions. However, as the exponents in the radial asymptotic of the singular functions may differ the angle dependencies –present in the three dimensional case for* (1.2)*– may need adjustment. Because our exposition is already quite technical we will refrain from such generalizations.*

The case of non polygonal domains or, likewise, non constant coefficients in the main part, quasilinear or nonlinear equations will require more additional work due to complications from ,,crossing" of singular exponents. See, e.g., [7], for a 3d edge with variable angle.

The rest of this article is structured as follows. In Section 2, we show that problem (1.1) is well posed and derive a regularity result for the state equation. Namely we will show additional regularity of the state variable due to the first order state constraint. This will be based upon well known singular expansion of the solution to (1.2). Then, in Section 3 we will obtain first order necessary conditions for arbitrary polygonal domains in $n = 2$ and certain polyhedral domains in $n = 3$. Finally, we will use the derived optimality conditions to show additional regularity of the control variable in Section 4.

2. Existence. The main theorem of this section will be the following

THEOREM 2.1. *Assume* (1.1) *has at least one feasible control, then* (1.1) *has a unique solution* $(\overline{q},\overline{u}) \in L^r(\Omega) \times W^{2,t}(\Omega) \cap H^1_0(\Omega)$. *With some* $t > 2$ *depending only on the angles in the corners, vertices and edges of the domain.*

Before we come to the proof of Theorem 2.1 we will require some additional preparation. In particular, we will show that the constraint $|\nabla u| \le \psi$ gives us additional regularity of the state variable. In order to make the idea more clear we split the proof in two parts. One for the case $n = 2$ and one for the case $n = 3$. The proofs in both cases follow the same line of arguments, but the case $n = 3$ is by far more technical.

LEMMA 2.2. *Let* $\Omega \subset \mathbb{R}^2$ *be a polygonal domain. Further, let* $f \in L^p(\Omega)$ *for some* $p \ge 2$. *If the solution* $u = u_f \in H^1_0(\Omega)$ *of* (1.2) *satisfies* $u \in W^{1,\infty}(\Omega)$, *then* $u \in W^{2,t}(\Omega)$ *for some* $t \in [2, p]$ *whose value can be determined by knowledge of the angles in the corners of the domain. Moreover, if* $p > 2$ *then* $t \in (2, p]$ *is possible.*

Proof. The proof is based on the well known singular behavior of the solution near the corners of the domain, cf., [14] for the 2d case. The idea of the proof is as follows, the solution to the state equation can be split into a regular part that exhibits the regularity introduced by the right-hand side f and a singular part corresponding to the non-smooth boundary. By the bound on the gradient of the solution one obtains, that the singular part may not exist.

Let $\mathcal{C}$ be the (finite) set of corners of the domain. For a corner $c \in \mathcal{C}$ we denote the interior angle by ω_c. We introduce polar coordinates with respect to the corner c denoted by (ρ_c, θ_c). Now, because Ω is bounded we have $f \in L^t(\Omega)$ for all $t \in [2, p]$. Assume further that t is such that for $t' = \frac{t}{t-1}$ it holds $\frac{2\omega_c}{\pi t'} \notin \mathbb{N}$ for all $c \in \mathcal{C}$. Then there exist numbers $C_{c,j}$ such that the solution u to (1.2) satisfies

$$u - \sum_{c\in\mathcal{C}} \sum_{\substack{j=1\\ \frac{j\pi}{\omega_c}\neq 1}}^{j<\frac{2\omega_c}{\pi t'}} C_{c,j} s_{c,j}(\rho_c, \theta_c) \in W^{2,t}(\Omega)$$

where the singular functions $s_{c,j}$ are given by

$$s_{c,j}(\rho_c, \theta_c) = \eta_c(\rho_c)\rho_c^{\frac{j\pi}{\omega_c}} \sin\left(\frac{j\pi}{\omega_c}\theta_c\right)$$

with suitable cutoff functions η_c, cf., [14, Theorem 4.4.3.7].

To proceed further, note that if $t > 2 = n$ we have $t' < 2$ and $W^{2,t}(\Omega) \subset C^1(\overline{\Omega})$. Further, we have that $\lim_{t\downarrow 2} t' = 2$.

Now we intend to show that we can choose $t > 2$ (sufficiently small) to assert that the inner sum in the singular expansion above contains at most the index $j = 1$. To do so, first, we assume that $\omega_c \ne 2\pi$, then by considering $t > 2$ small enough $\frac{2\omega_c}{\pi t'} < 2$ and the second sum in the singular expansion contains at most the value $j = 1$. If $\omega_c = 2\pi$, then $t > 2$ small enough implies $\frac{2\omega_c}{\pi t'} < 3$, and because the case $j = 2$ is prohibited by the condition $\frac{j\pi}{\omega_c} \ne 1$ we obtain again that the second sum in the singular expansion contains at most the value $j = 1$.

Now, we will discuss the behavior of the derivative of the singular solutions. We obtain for all $c \in \mathcal{C}$ where $j = 1$ appears in the singular expansion that $1 = j < \frac{2\omega_c}{\pi t'}$ and thus $\frac{j\pi}{\omega_c} = \frac{\pi}{\omega_c} < \frac{2}{t'}$. Now, we have to distinguish two cases:

First, assume that $\frac{\pi}{\omega_c} < 1$ then the first derivative of $s_{c,j}$ is unbounded because $\partial_{\rho_c} s_{c,j} \approx \rho_c^{\frac{\pi}{\omega_c}-1} \to \infty$ as $\rho_c \to 0$. Thus the assumption $u \in W^{1,\infty}(\Omega)$ implies $C_{c,1} = 0$.

Second, if $\frac{\pi}{\omega_c} > 1$, then it is necessary for the singular function to appear in the representation above that we have $\frac{\pi}{\omega_c} < \frac{2}{t'}$. Because $\frac{2}{t'} \to 1$ as $t \downarrow 2$ we can reduce $t > 2$ even further to obtain that $\frac{2}{t'} < \frac{\pi}{\omega_c}$, and hence this case can be excluded by choosing t sufficiently close to 2.

It is clear that the same argument remains true in the limit if $t = t' = 2$.

Summing up we obtain that for t sufficiently small,

$$\sum_{c\in\mathcal{C}} \sum_{j=1}^{j<\frac{2\omega_c}{\pi t'}} C_{c,j} s_{c,j}(\rho_c,\theta_c) \in W^{1,\infty}(\Omega)$$

if and only if all the singular coefficients fulfill $C_{c,j} = 0$ and hence that $u \in W^{2,t}(\Omega)$ for some $t \in [2, p]$ sufficiently small, and if $p > 2$ we can actually choose $t > 2$. □

LEMMA 2.3. *Let $\Omega \subset \mathbb{R}^3$ be a polyhedral domain. Further, let $f \in L^p(\Omega)$ for some $p \geq 3$. If the solution $u = u_f \in H_0^1(\Omega)$ of (1.2) satisfies $u \in W^{1,\infty}(\Omega)$ then $u \in W^{2,t}(\Omega)$ for some $t \in (2, p]$. The value of t can be determined by knowledge of the angles in the edges and vertices of the domain.*

Proof. As for Lemma 2.2 the proof is based on well known singular behavior of the solution near the edges and vertices. The 3d case was considered in [9, 17] in Hilbert spaces. Its extension to the non Hilbert space case can be found in [18].

As in the case $n = 2$, we have $f \in L^t(\Omega)$ for all $t \in [2, p]$ because Ω is bounded. In contrast to the case $n = 2$ we will have to consider contributions by vertices and edges. Therefore we denote the set of vertices on $\partial\Omega$ by $\mathcal{V}$ and the set of edges by $\mathcal{E}$.

We will begin by considering a vertex $v \in \mathcal{V}$. We introduce spherical coordinates $(\rho_v, \theta_v, \varphi_v)$ with respect to this vertex v. Let now B_v be a sufficiently small ball around v, let $G_v = \partial B_v \cap \Omega$, then let $w_{j,v}(\theta_v, \varphi_v)$ be the sequence of eigenfunctions of the Laplace-Beltrami operator with homogeneous Dirichlet boundary conditions on G_v ordered by the magnitude of the corresponding eigenvalues $\lambda_{j,v}$. Now, let t be such that $\lambda_{j,v} \neq \left(\frac{3}{t} - 2\right)\left(\frac{3}{t} - 3\right)$ for all j. Then the corresponding singular expansion for the vertex v reads

$$\sum_{\lambda_{j,v} < \left(\frac{3}{t}-2\right)\left(\frac{3}{t}-3\right)} C_{j,v} s_{v,j}(\rho_v, \theta_v, \varphi_v)$$

with the singular functions

$$s_{v,j}(\rho_v, \theta_v, \varphi_v) = \eta_v(\rho)\rho_v^{\beta_{j,v}-\frac{1}{2}} w_{j,v}(\theta_v, \varphi_v).$$

Here $\beta_{j,v}$ is given by $\beta_{j,v} = \sqrt{\left(\frac{3}{t} - 1\right)^2 + \lambda_{j,v}}$, see [18, Theorem 4.6].

As in the 2d case we wish to show that for t sufficiently small we can reduce this sum to at most one summand $s_{v,j}$ with singular derivatives. One immediately calculates that for $t \downarrow 3$ the upper bound $\left(\frac{3}{t} - 2\right)\left(\frac{3}{t} - 3\right)$ on $\lambda_{j,v}$ in the singular expansion converges to two. Hence for $t > 3$ small enough $\beta_{j,v} \leq \sqrt{2} + \varepsilon < 1.5$ and we obtain that the first derivative of these singular functions $s_{v,j}$ is not bounded because $\partial_{\rho_v} s_{v,j} \approx \rho_v^{\beta_{j,v}-\frac{3}{2}} \to \infty$ as $\rho_v \to 0$. Hence $C_{j,v} = 0$ if t is chosen sufficiently small.

We remark that we can choose $t \in (3, p]$ independent of the angles because the only requirement is $\beta_{j,v} < 1.5$ (Although one may obtain the same for larger t by using information on $\lambda_{j,v}$.). Further, the bound on $\beta_{j,v}$ is monotone increasing with t and thus one can easily compute that infact $\beta_{j,v} < 1.5$ at least for $t \in [2, 3.1]$.

Now, we consider the contributions from an edge $e \in \mathcal{E}$. We denote its interior angle by ω_e and introduce cylindrical coordinates (ρ_e, θ_e, z_e) with respect to the edge e. We will no longer be able to assert that $t > 3$ but will be forced to allow values $t > 2$ depending on the edge angle ω_e.

Then we obtain, see [9, Section 17.D] and [17, Theorem 2.5.11] for $t = 2$, or [15, Theorem 4.1] and [18, Section 7] for $t > 2$, that there exist functions $q_{j,e} \in W^{2/t'-j\pi/\omega_e,t}(e)$, such that the singular expansion with respect to the edge e has the form

$$\sum_{\substack{j=1 \\ \frac{j\pi}{\omega_e} \neq 1}}^{j<\frac{2\omega_e}{\pi t'}} C_{e,j}(\rho_e, z_e) s_{e,j}(\rho_e, \theta_e)$$

with the singular function

$$s_{e,j}(\rho_e, \theta_e) = \eta_e(\rho_e) \rho_e^{\frac{j\pi}{\omega_e}} \sin\left(\frac{j\pi}{\omega_e}\theta_e\right)$$

and the coefficient function

$$C_{e,j}(\rho_e, z_e) = (G_j(\rho_e, z_e) * q_{j,e})$$

given by a convolution with a $C^\infty(\Omega)$ function $G_j(\rho_e, z_e, \cdot)$ such that $C_{e,j}(\rho_e, z_e) = C_{e,j}(\rho_e, z_e) = (G_j(\rho_e, z_e) * q_{j,e}) \to q_{j,e}(z_e)$ if $\rho_e \to 0$. The precise form of these functions can, for instance, be found in the references above but is of no importance for the following argument.

For our analysis it is important to note that the singular behavior of $s_{e,j}$ is the same as for the corresponding 2d singular function corresponding to the interior angle $\omega_c = \omega_e$. Thus we only need to show that the coefficient factor $C_{e,j}(\rho_e, z_e)$, which depends on ρ_e, does not interfere.

We proceed exactly as in the 2d case. Note that we still have to let $t \to 2$ to exclude certain singular functions. Let $t > 2$ sufficiently small, and $\frac{2\omega_e}{\pi t'} \notin \mathbb{N}$, then in the above sum only $j = 1$ appears, and the first derivative of $s_{j,e}(\rho_e, \theta_e)$ is unbounded for $\rho_e \to 0$. Further, by transformation one gets that the gradient satisfies

$$\nabla = \begin{pmatrix} \cos\theta_e \\ \sin\theta_e \\ 0 \end{pmatrix} \partial_{\rho_e} + \begin{pmatrix} -\sin\theta_e \\ \cos\theta_e \\ 0 \end{pmatrix} \frac{1}{\rho_e} \partial_{\theta_e} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \partial_{z_e}$$

and thus

$$\|\nabla(C_{e,j} s_{e,j})\|^2 = (\partial_{\rho_e}(C_{e,j} s_{e,j}))^2 + \frac{(C_{e,j} \partial_{\theta_e} s_{e,j})^2}{\rho_e^2} + (\partial_{z_e}(C_{e,j} s_{e,j}))^2 \geq \frac{(C_{e,j} \partial_{\theta_e} s_{e,j})^2}{\rho_e^2}.$$

Now, because

$$\lim_{\rho_e \to 0} C_{e,j} = q_{j,e}$$

and

$$\lim_{\rho_e \to 0} \frac{(\partial_{\theta_e} s_{e,j})^2}{\rho_e^2} = \infty$$

we obtain that boundedness of $\nabla(C_{e,j}s_{e,j})$ implies $q_{j,e} \equiv 0$.

Combining the singularities from edges and vertices, see [18, Section 7.2] or [9, Section 17.D] we obtain the assertion. □

REMARK 2.1. *The analysis of Lemma 2.3 shows that if* Ω *has only vertices (and no edges) then a solution* u *of* (1.2) *that satisfies* $u \in W^{1,\infty}(\Omega)$ *also satisfies* $u \in W^{2,3+\varepsilon}(\Omega) \subset C^1(\overline{\Omega})$. *However, in the presence of edges the situation depends on the angles as the following corollary summarizes.*

COROLLARY 2.4. *In Lemma 2.3 one can obtain* $t > 3$ *provided that all of the interior angles* ω_e *of the edges* e *of* Ω *satisfy*

$$\omega_e \notin \left[\tfrac{4}{3}\pi, \pi\right) \cup \left[\tfrac{3}{2}\pi, 2\pi\right).$$

Proof. In order to obtain $t > 3$ we note that as already mentioned in the proof of Lemma 2.3 the singularities coming from the vertices are not a problem. Hence we have to consider an arbitrary edge e. We denote its interior angle by ω_e and introduce cylindrical coordinates (ρ_e, θ_e, z_e) with respect to the edge e. Then as in the proof of Lemma 2.3 we have that there exist functions $q_{j,e}$ and G_j, such that the singular part of the solution is of the form

$$\sum_{\substack{j=1 \\ \frac{j\pi}{\omega_e} \neq 1}}^{j < \frac{2\omega_e}{\pi t'}} C_{e,j}(\rho_e, z_e) s_{e,j}(\rho_e, \theta_e)$$

with $C_{e,j}$ and $s_{e,j}$ given as in the proof of Lemma 2.3.

Now to proceed, we needed to exclude those singular functions $s_{e,j}$ with exponent $\frac{j\pi}{\omega_e} > 1$.

Again, as in Lemma 2.2 we start by excluding the case $\omega_e = 2\pi$. Then we aimed at removing the indices $j \geq 2$ from the above sum, because for $j \geq 2$ the exponent $\frac{j\pi}{\omega_e} > 1$. To do so we have to choose t small enough to obtain

$$\frac{2\omega_e}{\pi t'} < 2.$$

If on the other hand we need $t > 3$ then immediately $t' < \frac{3}{2}$. Hence we have $\frac{2\omega_e}{\pi t'} \downarrow \frac{4\omega_e}{3\pi}$ if $t \downarrow 3$. This yields that we can achieve $\frac{2\omega_e}{\pi t'} < 2$ if and only if $\frac{4}{3}\frac{\omega_e}{\pi} < 2$. Thus if $\omega_e < \frac{3}{2}\pi$ we can find $t > 3$ such that $j \geq 2$ does not appear in the singular expansion.

Now, it remains to consider those singular functions for which the leading singular function is too regular, i.e., $\frac{\pi}{\omega_e} > 1$. This immediately gives that we have to consider edges with $\omega_e < \pi$ only. Similar to the case before, we need to find $t > 3$ to obtain that

$$\frac{2\omega_e}{\pi t'} < 1.$$

With the considerations above we get that for $\omega_e < \frac{3}{4}\pi$ we can exclude the case $j = 1$ in the singular expansion.

Finally, we consider the case $\omega_e = 2\pi$. Then, the value $j = 2$ does not appear in the singular expansion, and we only have to assert $\frac{2\omega_e}{\pi t'} = \frac{4}{t'} < 3$. Now, as above we have $\frac{1}{t'} \downarrow \frac{2}{3}$ and $4\frac{2}{3} < 3$ hence for some $t > 3$ we have $\frac{2\omega_e}{\pi t'} < 3$. □

REMARK 2.2. *In addition to the results of Lemma 2.2 and Lemma 2.3, we remark that provided certain (countably many) critical values of t are avoided the operator $-\Delta$ is closed from $W^{2,t}(\Omega)\cap H_0^1(\Omega)$ into $L^t(\Omega)$. Moreover, the operator is closed for $t=2$.*

To see this we first consider the case $n=2$, then the result is obtained by [14, Theorem 4.3.2.4] for $t>2$ under the condition $\frac{2\omega_c}{\pi t'}\notin \mathbb{N}$ for all interior angles ω_c. The case $t=2$ is covered by [14, Theorem 4.3.1.4].

The case $n=3$ and $t=2$ is covered by [10, Corollary 3.10], for the case $t>2$ see [13, Theorem 5.8][1].

In particular, this implies that for such a t there exists a constant C such that for any $f\in I$ and corresponding solution $u_f\in W^{2,t}(\Omega)\cap H_0^1(\Omega)$ the following holds

$$\|u_f\|_{2,t}\le C\|f\|_t.$$

We are now prepared to show the main result of this section.

Proof. [Proof of Theorem 2.1] The proof is now standard. We define the set

$$Q^{\text{feas}} := \{q\in L^r(\Omega)\,|\, a\le q\le b,\ |\nabla u_q|^2\le\psi \ \text{ a.e. in } \overline{\Omega}\}$$

where u_q denotes the solution to (1.2) corresponding to $f=q$.

By assumption $Q^{\text{feas}}\neq\emptyset$. From Lemma 2.2 and Lemma 2.3 we obtain that for any $q\in Q^{\text{feas}}$ the corresponding state $u_q\in H^2(\Omega)\cap H_0^1(\Omega)$ and the mapping $q\mapsto u_q$ is continuous. With this we can define the reduced cost functional $\hat J(q)=J(q,u_q)$.

Now, because the PDE is linear the set Q^{feas} is closed and convex and is hence closed with respect to weak convergence. Hence taking a minimizing sequence in Q^{feas} and noting that $\hat J$ is weakly lower semicontinuous and coercive yields the desired minimizer. Uniqueness of the minimizer is clear because $\hat J$ is strictly convex.

Again by application of Lemma 2.2 and Lemma 2.3 we get the desired regularity of the optimal state noting that $q\in L^t(\Omega)$ for some $t>2$. □

REMARK 2.3. *Concerning the possible extension to semilinear PDEs we note that the embedding $W^{2,t}(\Omega)\to C^1(\overline{\Omega})$ is compact. Thus Q^{feas} will be weakly sequentially closed even if $q\mapsto u_q$ is not linear.*

3. Optimality Conditions. After having established the existence of a solution we will consider the system of first-order necessary conditions.

LEMMA 3.1. *Let $(\overline{q},\overline{u})$ be the solution of (1.1). Further, assume that there exists a Slater point, i.e., $\hat q\in Q^{ad}$ such that the corresponding state $\hat u=u_{\hat q}$ satisfies $|\nabla\hat u|^2<\psi$ on $\overline{\Omega}$.*

Assume that either $n=2$ or $n=3$ and t obtained in Lemma 2.3 is larger than $n=3$. Further, let t be such that Δ is closed from $W^{2,t}(\Omega)\cap H_0^1(\Omega)\to L^t(\Omega)$. Then there exists a measure $\overline{\mu}\in C(\overline{\Omega})^*$ such that the following holds:

$$\begin{aligned}(u_{\overline{q}}-u^d,u_{\delta q}-u_{\overline{q}})+\alpha(|\overline{q}|^{r-2}\overline{q},\delta q-\overline{q})&\\ +2\langle\overline{\mu},\nabla u_{\overline{q}}\cdot(\nabla u_{\delta q}-\nabla u_{\overline{q}})\rangle_{C^*\times C}&\ge 0 \quad \forall\,\delta q\in Q^{ad}\cap I,\\ \langle\overline{\mu},\varphi\rangle_{C^*\times C}&\ge 0\quad\forall\,\varphi\in C(\overline{\Omega}),\varphi\ge 0,\\ \langle\overline{\mu},|\nabla\overline{u}|^2-\psi\rangle_{C^*\times C}&=0,\end{aligned}\tag{3.1}$$

where I denotes the image of $W^{2,t}(\Omega)\cap H_0^1(\Omega)$ under Δ.

[1]The author would like to acknowledge the support of M. Dauge for an e-mail giving the same result for $t>2$, before the author was able to find a citable source.

Proof. We note that the image I of $W = W^{2,t}(\Omega) \cap H_0^1(\Omega)$ under Δ is closed in $L^t(\Omega)$ by assumption. Hence, because Ω is bounded, $I \cap L^r(\Omega)$ is closed in $L^r(\Omega)$ for any $r \geq t$, too. This means, it is sufficient to consider the optimization problem on the smaller (Banach) space $Q = I \cap L^r(\Omega)$.

Then the mapping $q \mapsto |\nabla u_q|^2$ is differentiable as a mapping $Q \to C(\overline{\Omega})$ by construction, because $W \subset C^1(\overline{\Omega})$. In particular we can equivalently restate problem (1.1) as

$$\begin{aligned} &\min_{q\in Q} \hat{J}(q) := \min_{q\in Q} J(q,u_q) \\ &\text{subject to } \begin{cases} |\nabla u_q|^2 - \psi \leq 0 & \text{in } \overline{\Omega}, \\ a \leq q \leq b & \text{a.e. in } \Omega. \end{cases} \end{aligned} \tag{3.2}$$

Next we note that by the assumptions on Ω and t we can apply Lemma 2.2 or Lemma 2.3 to see that $\hat{u} \in C^1(\overline{\Omega})$ and thus it is strictly feasible, i.e., there is some $\delta > 0$ such that $|\nabla \hat{u}| \leq \psi - \delta < \psi$. Thus the solution is regular (in the sense of Lagrange calculus), and by standard theorems, see, e.g., [22, Theorem 1.6], there exists a Lagrange multiplier $\mu \in C(\overline{\Omega})^*$ such that (3.1) holds. □

Remark 3.1. *The result is almost identical to the smooth case, however, we had to consider $Q = I \cap L^r(\Omega)$ and hence if $I \neq L^r(\Omega)$ we get a non local constraint into the admissible set $Q^{ad} \cap I$.*

We are however not yet done with our calculations. In the next step we will separate the influence of the equality and inequality constraints. To this end we have to consider the adjoint equation.

Lemma 3.2. *Let $\overline{u} \in W = W^{2,t}(\Omega) \cap H_0^1(\Omega)$ with $t > n$ be given. Then for any $\overline{\mu} \in C(\overline{\Omega})$ the problem*

$$(-\Delta\varphi, \overline{z}) = (\overline{u} - u^d, \varphi) + \langle \overline{\mu}, 2\nabla\overline{u}\nabla\varphi\rangle_{C^*\times C} \quad \forall \varphi \in W \tag{3.3}$$

has a (non unique) solution $z \in L^{t'}(\Omega)$.

Proof. To see the solvability of (3.3), we first note that the equation

$$(\nabla\varphi, \nabla z_0) = (\overline{u} - u^d, \varphi) \quad \forall \varphi \in H_0^1(\Omega)$$

possesses a solution $z_0 \in H_0^1(\Omega)$ which then automatically satisfies

$$(-\Delta\varphi, z_0) = (\overline{u} - u^d, \varphi) \quad \forall\, \varphi \in W.$$

Hence it is sufficient to consider solvability of the equation

$$(-\Delta\varphi, z_1) = \langle \overline{\mu}, 2\nabla\overline{u}\nabla\varphi\rangle_{C^*\times C} \quad \forall \varphi \in W. \tag{3.4}$$

It is clear that the right-hand side $\langle \overline{\mu}, 2\nabla\overline{u}\nabla\cdot\rangle_{C^*\times C}$ is an element of W^* for any $t > n$. Now $-\Delta\colon W \to I \subset L^t(\Omega)$ is an isomorphism and hence the same holds true for $-\Delta^*\colon I^* \to W^*$. Setting $I^\perp = \{v \in L^{t'}(\Omega) \,|\, (v,q) = 0\; \forall\, q \in I\}$ where $t' = \frac{t}{t-1}$ we have $I^* \cong L^{t'}(\Omega)/I^\perp$ because I is closed in $L^t(\Omega)$, see, e.g., [27, Theorem III.1.10].

This means there exists a solution $z_1 \in L^{t'}(\Omega)$ to (3.4) which is uniquely determined modulo $I^\perp$. Hence $\overline{z} = z_0 + z_1 \in L^{t'}(\Omega)$ is a solution to (3.3). □

We remark that the non uniqueness of the solution is not entirely unexpected, see [16].

By combining this with Lemma 3.1 we get the following

THEOREM 3.3. *Under the assumptions of Lemma 3.1 for a solution* $(\overline{q}, \overline{u})$ *of* (1.1), *there exists a measure* $\overline{\mu} \in C(\overline{\Omega})^*$ *and a function* $\overline{z} \in L^{t'}(\Omega)$ *such that:*

$$
\begin{aligned}
(\nabla \overline{u}, \nabla \varphi) &= (\overline{q}, \varphi) && \forall\ \varphi \in H_0^1(\Omega),\\
(-\Delta \varphi, \overline{z}) &= (\overline{u} - u^d, \varphi) + \langle \overline{\mu}, \nabla \overline{u} \cdot \nabla \varphi \rangle_{C^* \times C} && \forall\ \varphi \in W,\\
\alpha(|\overline{q}|^{r-2}\overline{q}, \delta q - \overline{q}) &\geq -(\delta q - \overline{q}, \overline{z}) && \forall\, \delta q \in Q^{ad} \cap I,\\
\langle \overline{\mu}, \varphi \rangle_{C^* \times C} &\leq 0 && \forall\ \varphi \in C(\overline{\Omega}), \varphi \leq 0,\\
\langle \overline{\mu}, |\nabla \overline{u}|^2 - \psi \rangle_{C^* \times C} &= 0.
\end{aligned}
\tag{3.5}
$$

Proof. The proof is simple. Take any solution $\overline{z}$ to (3.3). Then the assertion follows from the optimality conditions in Lemma 3.1. ☐

We remark that z being determined only modulo $I^{\perp}$ doesn't affect the variational inequality

$$
\alpha(|\overline{q}|^{r-2}\overline{q}, \delta q - \overline{q}) \geq -\langle \delta q - \overline{q}, \overline{z} \rangle_{Z^* \times Z} \quad \forall\, \delta q \in Q^{\mathrm{ad}} \cap I
$$

because the test functions are chosen from I and thus the value on the right hand side is unaffected by choosing any representation $\overline{z}$ solving (3.3) since $\overline{z}$ is uniquely determined up to an element in $I^{\perp}$.

REMARK 3.2. *We note that, in the case* $n = 3$*, there is a gap between the existence Theorem 2.1 and the necessary conditions Lemma 3.1 and Theorem 3.3 because there are certain angles for which we could not obtain* $W^{2,t}$ *regularity of the solution for* $t > 3$*, compare also Corollary 2.4.*

4. Regularity. We will now discuss another issue of importance, namely that of regularity of the solutions to (1.1) in the context of first-order constraints. In order to set things into perspective we will briefly comment on the smooth case, although it has already been discussed in [24, Appendix A].

The Case of a Smooth Domain. We recall that the necessary optimality condition for problem (1.1) on a domain with $C^{1,1}$ boundary has the form

$$
\begin{aligned}
(\nabla \overline{u}, \nabla \varphi) &= (\overline{q}, \varphi) && \forall\ \varphi \in H_0^1(\Omega),\\
(\overline{z}, -\Delta \varphi) &= (\overline{u} - u^d, \varphi) + \langle \overline{\mu}, \nabla \varphi \nabla \overline{u} \rangle_{C^* \times C} && \forall\ \varphi \in W,\\
\langle \overline{\mu}, \varphi \rangle_{C^* \times C} &\leq 0 && \forall\ \varphi \in C(\overline{\Omega}),\ \varphi \leq 0,\\
\alpha(|\overline{q}|^{r-2}\overline{q}, \delta q - \overline{q}) &\geq (-\overline{z}, \delta q - \overline{q}) && \forall\, \delta q\ \in Q^{\mathrm{ad}},\\
\langle \overline{\mu}, |\nabla \overline{u}|^2 - \psi \rangle_{C^* \times C} &= 0.
\end{aligned}
\tag{4.1}
$$

Here $\overline{q} \in \mathrm{L}^r(\Omega)$, $\overline{u} \in W = W^{2,r}(\Omega) \cup H_0^1(\Omega)$, $\overline{\mu} \in C(\overline{\Omega})^*$, and $\overline{z} \in L^s(\Omega)$ for any $s < \frac{n}{n-1}$, see [6].

If the regularity of $\overline{z}$ would be best possible, then this would automatically limit the possibility to obtain higher regularity for the control variable by bootstrapping arguments, because the control and the adjoint state are linked by an algebraic equation. In particular, if $\overline{z}$ has no derivatives, then in general $\overline{q}$ has none either.

It will be crucial for our analysis that there exists $t > n$ such that $-\Delta$ is $W^{2,t}$-regular, i.e., for $q \in L^t(\Omega)$ the weak solution $u = u_q \in H_0^1(\Omega)$ to (1.2) belongs in fact to $W^{2,t}(\Omega)$, and $\|u\|_{W^{2,t}} \leq c\|q\|_{L^t}$. If the boundary is of class $C^{1,1}$ this is obtained by classical regularity theory for any t, this dates back to [2], for a more recent exposition see [12].

If the domain is polygonal or polyhedral the existence of such a t requires additional conditions on the domain. If $n = 2$ there is such t provided the domain is convex see [17, Theorem. 4.4.3.7]. If $n = 3$ then one needs to assume in addition, that the angle between any two faces of Ω is bounded strictly above by $\frac{3}{4}\pi$ [10, Corollary 3.7]. If this is the case the optimality conditions (4.1) remain true with some sufficiently small $t > n$ in $W = W^{2,t}(\Omega) \cap H_0^1(\Omega)$. Let $t_{\max} > n$ be defined such that $-\Delta$ is $W^{2,t}$-regular for any $t \in (n, t_{\max})$.

From the necessary optimality conditions (4.1) one derives additional regularity for the adjoint state $\overline{z}$. We simply recall the following from [24, Lemma 1]:

Lemma 4.1. *The solution $\overline{z}$ of* (4.1) *belongs to $W^{1-n/t-\varepsilon,t'}(\Omega)$ for every $\varepsilon > 0$ and $t \in (n, t_{max})$, where we define t' as usual by $\frac{1}{t} + \frac{1}{t'} = 1$.*

Further we require the following result for the algebraic relation between $\overline{q}$ and $\overline{z}$, which is based upon Hölder-continuity of $x \mapsto |x|^{\frac{1}{r-1}}$. For details we refer to [24, Lemma 2].

Lemma 4.2. *Let $f \in W^{s,t'}(\Omega)$ with $s < 1$ and let $r \geq 2$, then*

$$\operatorname{sign}(f)|f|^{\frac{1}{r-1}} \in W^{\frac{s}{r-1},t'(r-1)}(\Omega).$$

Then one can derive the following regularity result for the optimal control from the algebraic relation between $\overline{q}$ and $\overline{z}$.

Corollary 4.3. *For any $\varepsilon > 0$ the optimal control $\overline{q}$ given by* (4.1) *belongs to the space $W^{\gamma,p}$, where $\gamma = (1-n/t-\varepsilon)/(r-1)$ and $p = t'(r-1)$ for any $t \in (n, t_{max})$.*

Proof. The proof is based upon the relation

$$\overline{q} = \max\Big(a, \min\Big(b, \operatorname{sign}\Big(\frac{-1}{\alpha}\overline{z}\Big)\Big|\frac{-1}{\alpha}\overline{z}\Big|^{\frac{1}{r-1}}\Big)\Big).$$

Using this Lemma 4.2 together with Lipschitz continuity of the max and min function in $W^{\frac{s}{r-1},t'(r-1)}(\Omega)$, see [23, Theorem II.A.1], yield the assertion. □

The Case of a Non-Smooth Domain. Let Ω be such that there is some $t > n$ such that the solution $\overline{u}$ to (1.1) is in $W = W^{2,t}(\Omega) \cap H_0^1(\Omega)$. With respect to Lemma 2.2 this assumption is met for any polygonal domain for $n = 2$. In regard of Lemma 2.3 and the subsequent Corollary 2.4 it gives a condition on the interior angles at edges of the domain.

For such a domain, we define $t_{\max} > n$ such that the solution $\overline{u} \in W^{2,t}(\Omega) \cap H_0^1(\Omega)$ for any $t \in (n, t_{\max})$. We note here, that the assumptions of Lemma 3.1 are then satisfied for almost every $t \in (n, t_{\max})$. As we will later try to use some interpolation results, we further require $t_{\max}$ sufficiently small, such that

$$-\Delta\colon W_0^{1,t}(\Omega) \to W^{-1,t}(\Omega)$$

is an isomorphism. By [10, Remark 3.11] it is clear that then $t_{\max} > n$ is still possible.

We will now derive a regularity result for the adjoint variable similar to Lemma 4.1. To do so we will employ the K-Method of interpolation (although any other method would do fine). Hence we define fractional-order Sobolev spaces $W^{s,p}$ by Besov spaces $B^s_{p,p}$. For details on this see, e.g., [26, Definition 4.2.1] or [1, Chapter 7].

Lemma 4.4. *Let $t \in (n, t_{max})$ satisfy the assumptions of Lemma 3.1. Then one can select a solution $\overline{z}$ of* (3.3) *which belongs to $W^{1-n/t-\varepsilon,t'}(\Omega)$ for every $\varepsilon > 0$ where we define t' as usual by $\frac{1}{t} + \frac{1}{t'} = 1$.*

Proof. The proof is almost identical to Lemma 4.1, however we have to employ some additional effort due to the non uniqueness of the solution $\overline{z}$ to (3.3).

Let $\widetilde{W}^{2,t} = W^{2,t}(\Omega) \cap H_0^1(\Omega)$ and $\varepsilon > 0$ be given, then

$$\langle \nabla\varphi\nabla\overline{u}, \mu\rangle_{C,C^*} \leq \|\overline{u}\|_{C^1}\|\overline{\mu}\|_{C^*}\|\varphi\|_{C^1} \leq C\|\varphi\|_{W^{1+n/t+\varepsilon,t}}$$

by standard embedding theorems [26, Theorem 4.6.1]. By definition, of the interpolation tuple, it holds

$$(W^{1,t}(\Omega), W^{2,t}(\Omega))_{n/t+\varepsilon,t} \supset (W_0^{1,t}(\Omega), \widetilde{W}^{2,t})_{n/t+\varepsilon,t} =: \widetilde{W}^{1+n/t+\varepsilon,t}$$

and we obtain that the right-hand side of the adjoint equation is an element of $(\widetilde{W}^{1+n/t+\varepsilon,t})^*$.

By assumption on t and $t_{\max}$ we have that both

$$\begin{aligned} A_1 &:= -\Delta\colon W_0^{1,t}(\Omega) \to W^{-1,t}(\Omega),\\ A_2 &:= -\Delta\colon \widetilde{W}^{2,t}(\Omega) \to I \subset L^t(\Omega), \end{aligned}$$

are isomorphisms. Hence the adjoint operators

$$\begin{aligned} A_1^*&\colon\ W_0^{1,t'}(\Omega) \to W^{-1,t'}(\Omega),\\ A_2^*&\colon\ I^* \to (\widetilde{W}^{2,t})^*, \end{aligned}$$

are isomorphisms, too. We note that, as in the proof of Lemma 3.2, $I^* \cong L^{t'}(\Omega)/I^{\perp}$. Especially by selecting an arbitrary element $s \in I^{\perp}$ and using $L^{t'}(\Omega) \cong I^* \oplus I^{\perp}$ we can lift an element $z \in I^*$ to $L^{t'}(\Omega)$ by setting $l_s(z) = z + s$. To ensure that the continuous 'inverse' mapping $l_s \circ (A_2^*)^{-1}\colon (W^{2,t}(\Omega) \cap H_0^1(\Omega))^* \to L^{t'}(\Omega)$ is linear we need to choose $s = 0$.

By interpolation [26, Theorem 4.8.2], we have

$$(\widetilde{W}^{1+n/t+\varepsilon,t})^* = ((W^{1,t}(\Omega), \widetilde{W}^{2,t})_{n/t+\varepsilon,t})^* = (W^{-1,t'}(\Omega), (\widetilde{W}^{2,t})^*)_{n/t+\varepsilon,t'}.$$

We apply interpolation for the continuous operators $(A_1^*)^{-1}$ and $l_s \circ (A_2^*)^{-1}$ to obtain [26, Theorem 1.3.3]

$$\overline{z} \in (W_0^{1,t'}(\Omega), L^{t'}(\Omega))_{n/t+\varepsilon,t'} = W^{1-n/t-\varepsilon,t'}(\Omega).$$

This proofs the assertion. □

Finally we can state the desired regularity result

COROLLARY 4.5. *Let $t \in (n, t_{max})$ satisfy the assumptions of Lemma 3.1. Further, assume that $Q^{ad} = L^r(\Omega)$ and if $n = 3$ that the domain contains only vertices but no (nonsmooth) edges. Then for any $\varepsilon > 0$ the optimal control $\overline{q}$ given by* (1.1) *belongs to the space $W^{\gamma,p}$, where $\gamma = (1 - n/t - \varepsilon)/(r-1)$ and $p = t'(r-1)$.*

Proof. Under the assumption $Q^{\mathrm{ad}} = L^r(\Omega)$ the inequality

$$\big(\alpha \operatorname{sign}\overline{q}|\overline{q}|^{r-1}, \delta q - \overline{q}\big) + (\delta q - \overline{q}, \overline{z}) \geq 0 \quad \forall\, \delta q \in Q^{\mathrm{ad}} \cap I$$

becomes an equation

$$\big(\alpha \operatorname{sign}\overline{q}|\overline{q}|^{r-1}, \delta q\big) + (\delta q, \overline{z}) = 0 \quad \forall\, \delta q \in L^r(\Omega) \cap I.$$

Now, as before, we can decompose $L^t(\Omega) = I \oplus I_\perp$ with $I_\perp = \{v \in L^t(\Omega) \,|\, (v,\varphi) = 0 \;\forall \varphi \in I^*\}$ and correspondingly $L^{t'}(\Omega) = I^* \oplus I^\perp$.

We have $\alpha \operatorname{sign} \overline{q} |\overline{q}|^{r-1} + \overline{z} \in L^{t'}(\Omega)$ and thus there exist uniquely determined elements $\overline{r} \in I^*$ and $\overline{s} \in I^\perp$ such that

$$\alpha \operatorname{sign} \overline{q} |\overline{q}|^{r-1} + \overline{z} = \overline{r} + \overline{s}.$$

Further, it is $(\overline{s}, \varphi) = 0$ for all $\varphi \in I$, by definition, and thus

$$\big(\alpha \operatorname{sign} \overline{q} |\overline{q}|^{r-1} + \overline{z} - \overline{s}, \delta q\big) = \big(\operatorname{sign} \overline{q} |\overline{q}|^{r-1} + \overline{z}, \delta q\big) = 0 \quad \forall\, \delta q \in L^r(\Omega) \cap I.$$

On the other hand, it is

$$\big(\alpha \operatorname{sign} \overline{q} |\overline{q}|^{r-1} + \overline{z} - \overline{s}, \delta q\big) = (\overline{r}, \delta q) = 0 \quad \forall \delta q \in I_\perp$$

and thus, because $L^r(\Omega) \cap I \oplus L^r(\Omega) \cap I^\perp = L^r(\Omega)$, it holds

$$\alpha \operatorname{sign} \overline{q} |\overline{q}|^{r-1} + \overline{z} - \overline{s} = 0.$$

From this we deduce that

$$\overline{q} = \operatorname{sign}\Big(\frac{-1}{\alpha}\overline{z} + s\Big) \Big|\frac{-1}{\alpha}\overline{z} + s\Big|^{\frac{1}{r-1}}$$

with some $s \in I^\perp$.

In the following we use the notation of Lemma 2.2 and Lemma 2.3 for the statement of the asymptotic behavior of the singular functions. We obtain a basis for $I^\perp$ using dual singular functions. For $n = 2$ they can be found for instance in [4]. The important condition is that they are the sum of a function in $H^1_0(\Omega)$ and a $C^\infty(\Omega)$ function whose only singularities are at the corners where it behaves like $O(\rho_c^{-\pi/\omega_c})$. Hence we deduce that $s \in W^{1-n/t-\varepsilon,t'}(\Omega)$ for any $\varepsilon > 0$.

In the case $n = 3$ we obtain that for a vertex we obtain the singular behavior $O(\rho_e^{-1-\beta_{j,r}+1/2})$, see, e.g., [3] and once again this yields $s \in W^{1-n/t-\varepsilon,t'}(\Omega)$ for any $\varepsilon > 0$. We note that in both cases it is sufficient to consider the behavior of the dual singular functions alone as there are only finitely many needed to generate $I^\perp$. Hence we can deduce regularity of $\overline{s}$ from the regularity of a basis of $I^\perp$.

From the regularity

$$\overline{z} + s \in W_0^{1-n/t-\varepsilon,t'}(\Omega)$$

we obtain the desired result from Lemma 4.2. □

Remark 4.1. *For edges e it is no longer sufficient to consider the regularity of the dual singular functions as given for instance in [8]. This is due to the fact that the space $I^\perp$ is no longer finite dimensional. Thus a more detailed analysis of the corresponding coefficient function is needed.*

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