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AN OPTIMAL CONTROL PROBLEM GOVERNED BY A REGULARIZED PHASE-FIELD FRACTURE PROPAGATION MODEL

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Abstract. This paper is concerned with an optimal control problem governed by a fracture model using a phase-field technique. To avoid the non-differentiability due to the irreversibility constraint on the fracture growth, the phase-field fracture model is relaxed using a penalization approach. Existence of a solution to the penalized fracture model is shown and existence of at least one solution for the regularized optimal control problem is established. Moreover, the linearized fracture model is considered and used to establish first order necessary conditions as well as to discuss QP-approximations to the nonlinear optimization problem. A numerical example suggests that these can be used to obtain a fast convergent algorithm.

1. Introduction

This paper presents an optimal control formulation for fracture propagation problems using phase-field methods. Presently, phase-field approaches for fracture propagation are subject of intensive research in both mathematical theory and applications. Based on variational principles, they provide an elegant way to approximate lower-dimensional surfaces and discontinuities. Rewriting Griffith's model [21] for brittle fracture in terms of a variational formulation was first done in [17]. Later, these concepts have been complemented with numerical examples [12] and well-posedness results including fractures with linear [18] and nonlinear elasticity [31]. A summary of the state-of-the-art has been compiled in [13]. In [35, 36], the authors refined modeling and material law assumptions to formulate an incremental thermodynamically consistent *phase-field* model for fracture propagation.

With regard to numerical analysis and computational methods important advances have been made first in [12], which was later supplemented with an analysis of the solution algorithm [11]; for a complete proof of that algorithm, we also refer to [14]. For a general Ambrosio-Tortorelli functional, numerical analysis was done in a second paper by the same authors [15]. Recent results and new features of this solution algorithm have been presented in [32]. Parameter studies and a slight re-interpretation of the original model were performed in [30]. A solution approach using shape optimization has been presented in [1] and phase-field models for structural optimization are discussed in [8]. Sophisticated examples and benchmarks from mechanical engineering, using the refined phase-field modeling, have been studied in [2, 9, 10, 23, 35, 36, 43]. Recent modeling and numerical studies by adding non-homogeneous traction forces acting on the fracture surface were conducted in [40, 41, 45].

Following the model proposed in [35, 36], we consider a time discrete, but spatially continuous phase-field approach to model the growth of the fracture over time. The irreversibility of the fracture growth induces an obstacle like problem in each time-point. The novelty of this paper is the formulation and analysis of an optimal control problem subject to such a fracture model.

Due to the irreversibility constraint on the fracture growth, this optimization problems become mathematical programs with complementarity constraints (MPCC), see, e.g., [7]. Due to the complementarity condition, standard constraint qualifications for nonlinear programs, like [42] or [47] can not be satisfied. Hence a zoo of different stationarity concepts has been introduced. For strong-stationarity, see, e.g., [38].

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Unfortunately, in general, such a system is only necessary if a sufficiently large set of controls is admissible in the optimization problem, see, e.g., [44]. In all other cases, weaker concepts need to be considered to obtain stationarity systems that can sometimes be obtained as limits of relaxed formulations, see for instance [25–27]. For the error due to a finite element discretization of the obstacle problem, we refer to [34]. In contrast to the control of an obstacle problem additional difficulties arise due to the coupling of the phase-field variable with the elasticity problem.

To alleviate the difficulties associated with the complementarity conditions of the lower level fracture-propagation problem, we introduce a penalty term to asymptotically enforce the irreversibility constraint. To avoid any difficulties associated with non-differentiability, we consider a smooth penalty based upon the fourth power of the feasibility violation, compare to [33]. Due to the penalization strategy, however, we can no longer simply assume the phase-field to be in L^∞ and hence the nonlinear coupling between phase-field and displacement needs additional care. Utilizing results from [29] for damage models together with a Stampacchia-type cutoff argument, we show that, indeed, the penalized fracture propagation problem admits a solution.

We continue by analyzing the linearization of the regularized fracture model and show that the linearized differential operator is Fredholm. This is utilized to provide first order necessary optimality conditions for the nonlinear optimization problem and discuss QP-approximations to the former.

Throughout the paper, c denotes a generic constant, which is independent of the relevant quantities, but may take a different value in each appearance, even in the same line. If we would like to emphasize the dependence of such a constant on a particular value, we do so by introducing an appropriate index, i.e., c_ε denotes a constant whose value depends on some parameter ε if the precise dependence is not relevant for the argument.

The outline of this paper is as follows: In Section 2, we formulate the nonlinear optimization problem for fracture propagation utilizing a phase-field ansatz, and introduce the regularization of the irreversibility condition for the growth by a penalty approach with parameter γ. Solvability of both the relaxed fracture propagation problem as well as the optimization problem is discussed in Section 3. In Section 4, we discuss the properties of the linearized relaxed phase-field model, and show that the linearization gives rise to a Fredholm operator. This observation is then used to derive first order necessary conditions for the relaxed nonlinear optimization problem, in Section 5, under a constraint qualification. In addition, in Section 6, we show that quadratic approximations to the nonlinear optimization problem are always well-posed and admit a unique solution that can be characterized by its first order necessary optimality conditions. Then, in Section 7, we present a numerical example indicating that indeed quadratic approximations give rise to a convergent algorithm.

2. The Nonlinear Problem and its Linearization

2.1. **The Phase-Field Model for Fracture Propagation.** We consider a bounded domain $\Omega \subset \mathbb{R}^2$. Its boundary $\partial\Omega$ is decomposed into Γ_D and Γ_N satisfying

$$\mathcal{H}^{d-1}(\Gamma_D) \neq 0 \quad \text{and} \quad \mathcal{H}^{d-1}(\Gamma_N) \neq 0$$

where $\mathcal{H}^{d-1}$ is the $d-1$-dimensional Hausdorff measure. With this we introduce the space of admissible displacements $H^1_D(\Omega;\mathbb{R}^2) := \{v \in H^1(\Omega;\mathbb{R}^2) \,|\, v = 0 \text{ on } \Gamma_D\}$. We assume that $\Omega \cup \Gamma_N$ is regular in the sense of Gröger, cf. [22], compare [24, Remark 1.6] for a characterization in the case $\Omega \subset \mathbb{R}^2$ considered here. By $(\cdot,\cdot)$, we denote the usual L^2 scalar product and by $\|\cdot\|$ the corresponding norms.

Following Griffith's criterion for brittle fracture, we suppose that the fracture propagation occurs when the elastic energy restitution rate reaches its critical value G_c. If q is a force applied on Γ_N, assuming that the fracture $\mathcal{C}$ is not reaching $\partial\Omega$, we define the following total energy

$$E(q;u,\mathcal{C}) = \frac{1}{2}(\mathbb{C}e(u), e(u))_{\Omega\setminus\mathcal{C}} - (q,u)_{\Gamma_N} + G_c\mathcal{H}^{d-1}(\mathcal{C}), \tag{2.1}$$

where u denotes the vector-valued displacement field, and $\mathbb{C}$ the elasticity tensor. For simplicity of the presentation, we assume a linear stress-strain relationship

$$\mathbb{C}e(u) = \sigma(u) = 2\mu_s e(u) + \lambda_s \mathrm{tr} e(u) I,$$

where μ_s and λ_s denote the Lamé coefficients, $e(u) = \frac{1}{2}(\nabla u + \nabla u^T)$, and I the identity in d-dimensions. Furthermore, we restrict ourselves to the consideration of homogeneous Dirichlet data for the displacement u, for simplicity.

In the functional (2.1), the first term describes the bulk energy, the second term traction boundary (Neumann) forces, and the final term the surface fracture energy.

Remark 2.1. *Specific examples of traction forces q acting on Neumann boundary parts, including the fracture, have been discussed in [39,41]. Therein, such integrals have been re-written into domain integrals in order to combine them with other domain terms.*

The energy functional is then minimized with respect to the kinematically admissible displacements u and any fracture set satisfying the fracture growth condition; the latter one being discussed below.

To regularize the Hausdorff-measure, we follow [3, 4] and introduce a time-dependent auxiliary variable (i.e., a phase-field for the fracture) φ, defined on $\Omega \times (0, T)$. Specifically, the fracture region is characterized by $\varphi = 0$ and the non-fractured zone by $\varphi = 1$.

The regularized fracture functional reads

$$\Gamma_\varepsilon(\varphi) = \frac{1}{2\varepsilon}\|1 - \varphi\|^2 + \frac{\varepsilon}{2}\|\nabla\varphi\|^2. \tag{2.2}$$

This regularization of $\mathcal{H}^{d-1}(\mathcal{C})$, in the sense of the Γ-limit when $\varepsilon \to 0$, was used in [12, 13].

A key assumption in modeling contains the fact that the fracture can only grow, which is represented by the following irreversibility constraint:

$$\varphi(t_2) \le \varphi(t_1) \quad \forall t_1 \le t_2. \tag{2.3}$$

In the following, we replace the energy functional (2.1) by a global constitutive dissipation functional for a rate independent fracture process. To avoid the degeneracy of the elastic energy inside the fracture ($\{\varphi = 0\}$), we regularize by defining for some value $\kappa \ll \varepsilon < 1$

$$g(\varphi) = g_\kappa(\varphi) := (1 - \kappa)\varphi^2 + \kappa.$$

We then obtain the regularized total energy [12, 13]

$$E_\varepsilon(q; u, \varphi) = \frac{1}{2}\Big(g(\varphi)\mathbb{C}e(u), e(u)\Big) - (q, u)_{\Gamma_N} + G_c\Gamma_\varepsilon(\varphi). \tag{2.4}$$

To discretize in time, we introduce an equidistant partition

$$0 = t_0 < t_1 < \ldots < t_M = T,$$

with corresponding approximations $(u^i, \varphi^i)_{i=0}^M$. Then our irreversibility constraint is given as

$$\varphi^i \le \varphi^{i-1}.$$

To summarize our forward fracture propagation problem, we introduce the spaces

$$V := H^1_D(\Omega; \mathbb{R}^2) \times H^1(\Omega), \qquad Q := L^2(\Gamma_N)$$

for the solution of the fracture problem and for the boundary data, respectively.

Summarizing our time discrete fracture problem for given $q = (q^i)_{i=1}^M \in Q^M$ and given $(u^0, \varphi^0) \in V$ with $0 \le \varphi^0 \le 1$ is to find $\mathbf{u} = (u^i, \varphi^i)_{i=1}^M \in V^M$ solving, for each $i = 1, \ldots, M$,

$$\begin{aligned} &\min_{\mathbf{u}} E_\varepsilon(q^i; u^i, \varphi^i) \\ &\text{s.t. } 0 \le \varphi^i \le \varphi^{i-1} \le 1. \end{aligned} \tag{C}$$

Indeed, the lower bound in C is not relevant, as it is satisfied by the solutions in any case, see, e.g., [3, 4].

2.2. **The Optimization Problem and Further Regularizations of the Fracture.** We would like to consider the following model problem in fracture propagation for given $(u^0, \varphi^0) \in V$ with $0 \leq \varphi^0 \leq 1$, we wish to find $(q, \mathbf{u}) = (q, (u, \varphi)) \in (Q \times V)^M$ solving

$$\text{(NLP)} \qquad \begin{aligned} &\min_{q,\mathbf{u}} J(q, \mathbf{u}) := \frac{1}{2} \sum_{i=1}^{M} \|u^i - u_d^i\|^2 + \frac{\alpha}{2} \sum_{i=1}^{M} \|q^i\|_{\Gamma_N}^2 \\ &\text{s.t. } \mathbf{u}^i \text{ solves (C) given the data } q^i, \text{ for each } i = 1, \ldots, M, \end{aligned}$$

where $u_d \in (L^2(\Omega))^M$ is a given desired displacement. We note that (NLP) does not depend explicitly on the phase-field. Alternatively, the constraint in (NLP) could be relaxed by asking for satisfaction of the first-order necessary conditions to (C), only.

The presence of inequality constraints in the lower-level problem (C) leads to several well known problems, see, e.g., [37,38]. Following a classical approach, see, e.g. [7], we regularize (C) to remove the inequality constraints involved in the fracture-propagation problem. Since only the constraint $\varphi^i \leq \varphi^{i-1}$ is relevant for the problem (C), we only consider this constraint in our regularization. We will see in Section 3.1 that indeed neglecting the lower bound is justified.

To ensure sufficient differentiability of the regularization, we follow [33] obtaining for all time-steps $i = 1, \ldots, M$

$$\text{(C}^\gamma\text{)} \qquad \min_{\mathbf{u}} E_\varepsilon^\gamma(q^i, \varphi^{i-1}; u^i, \varphi^i) := E_\varepsilon(q^i; u^i, \varphi^i) + \gamma R(\varphi^{i-1}; \varphi^i)$$

with $\gamma > 0$ and

$$R(\varphi^{i-1}; \varphi^i) = \frac{1}{4} \|(\varphi^i - \varphi^{i-1})^+\|_{L^4}^4.$$

Formally, any minimizer of (C$^\gamma$) satisfies the Euler-Lagrange equations, that for any $(v, \psi) \in V$ and $i = 1, \ldots, M$,

$$\text{(EL}^\gamma\text{)} \qquad \begin{aligned} &\Big(g(\varphi^i)\mathbb{C}e(u^i), e(v)\Big) - (q^i, v)_{\Gamma_N} = 0, \\ &G_c\varepsilon(\nabla\varphi^i, \nabla\psi) - \frac{G_c}{\varepsilon}(1 - \varphi^i, \psi) \\ &\quad +(1-\kappa)(\varphi^i \mathbb{C}e(u^i) : e(u^i), \psi) \\ &\qquad +\gamma([(\varphi^i - \varphi^{i-1})^+]^3, \psi) = 0. \end{aligned}$$

However, since we relaxed the upper bound $\varphi^i \leq \varphi^{i-1}$ it is no longer clear, if all terms above are well-defined since it is not clear whether $\varphi^i \in L^\infty(\Omega)$. We will, positively, answer this question in the following Section 3.

With this we can further relax our problem, and obtain the regularized nonlinear problem, given $(u^0, \varphi^0) \in V$, $0 \leq \varphi^0 \leq 1$, to find $(q, \mathbf{u}) \in (Q \times V)^M$ solving

$$\text{(NLP}^\gamma\text{)} \qquad \begin{aligned} &\min_{q,\mathbf{u}} J(q, \mathbf{u}) \\ &\text{s.t. } (q^i, \mathbf{u}^i) \text{ satisfy (EL}^\gamma\text{) for each } i = 1, \ldots, M. \end{aligned}$$

3. Existence of Solutions to (NLP$^\gamma$)

We proceed in two steps, starting by analyzing the lower level problem, before discussing the existence of solutions to (NLP$^\gamma$).

3.1. **The Phase-Field Model** (EL$^\gamma$)**.** Due to the fact, that we relaxed the constraint $\varphi^i \leq \varphi^{i-1}$ by a penalty approach, we can no longer assume $\varphi^i \in L^\infty$ as it is usually done in proving existence of solutions to (C). The reason is that naively assuming minimal regularity asserted by the functional in (C$^\gamma$) the

products of the variables, i.e., $\varphi^2\mathbb{C}e(u) : e(u)$, are not in L^1. Hence following ideas of Stampacchia [28], we will, temporarily, relax (C^γ) even further. Let $b > 0$ be an arbitrary given number and define

$$m = m_b \colon \mathbb{R} \to \mathbb{R}; \quad m(x) := \begin{cases} x & -b \le x \le b \\ P_{[-b-2,b+2]}(x) & \text{otherwise} \end{cases}$$

where $P_{[-b-2,b+2]}$ is some smoothed projection onto $[-b-2, b+2]$ of which the precise definition is irrelevant as long as $m \in C^2$ with $0 \le m' \le 1$ and $m(\mathbb{R}) \subset [-b-2, b+2]$. With this, we define the regularized coefficient function

$$g_b(\varphi) = (1-\kappa)m\big((\varphi^i)^2\big) + \kappa \in [\kappa, b+2].$$

We modify the cost functional in (C^γ) to include the cutoff function. Consequently, we consider the following family of problems

$$(\mathrm{C}^{\gamma,b}) \qquad \begin{aligned} \min_{\mathbf{u}^i} \; E_\varepsilon^{\gamma,b}(q^i, \varphi^{i-1}; u^i, \varphi^i) := \frac{1}{2}\Big(g_b(\varphi)\mathbb{C}e(u^i), e(u^i)\Big) \\ - (q^i, u^i)_{\Gamma_N} + G_c\Gamma_\varepsilon(\varphi^i) + \gamma R(\varphi^{i-1}; \varphi^i), \end{aligned}$$

at each time-point $i = 1, \dots, M$ The idea of Stampacchia's method, in essence, is to prove that ($\mathrm{C}^{\gamma,b}$) has all desired properties and, moreover, that for suitable $b \in \mathbb{R}$ the solutions of ($\mathrm{C}^{\gamma,b}$) and (C^γ) coincide and thus our original problem inherits, among other properties, the boundedness of φ^i in L^∞. Let us therefore start by discussing ($\mathrm{C}^{\gamma,b}$), first.

Lemma 3.1. *For any $i = 1, \dots, M$ it holds.*

(1) Given $q^i \in L^2(\Gamma_N)$ and $\varphi^{i-1} \in L^2(\Omega)$, ($\mathrm{C}^{\gamma,b}$) has at least one solution $\bar{\mathbf{u}}^i$.

(2) Further, any (local) minimizer $\bar{\mathbf{u}}^i$ of ($\mathrm{C}^{\gamma,b}$) solves for all $(v, \psi) \in V$

$$(\mathrm{EL}^{\gamma,b}) \qquad \begin{aligned} \Big(g_b(\varphi^i)\mathbb{C}e(u^i), e(v)\Big) - (q^i, v)_{\Gamma_N} = 0 \\ G_c\varepsilon(\nabla\varphi^i, \nabla\psi) + (1-\kappa)(m'((\varphi^i)^2)\varphi^i\mathbb{C}e(u^i) : e(u^i), \psi) \\ -\frac{G_c}{\varepsilon}(1-\varphi^i, \psi) + \gamma([(\varphi^i - \varphi^{i-1})^+]^3, \psi) = 0. \end{aligned}$$

(3) Finally, any solution $\mathbf{u}^i = (u^i, \varphi^i) \in V$ to ($\mathrm{EL}^{\gamma,b}$) satisfies

(a) Assuming $\varphi^{i-1} \ge 0$ a.e. it follows $\varphi^i \ge 0$ a.e..

(b) There exists a constant $c_{b,\kappa}$ depending on b and κ (but not on $\mathbf{u}^i$ and $p > 2$), such that

$$\|u^i\|_{1,p} \le c_{b,\kappa}\|q^i\|.$$

(c) Assuming $\varphi^{i-1} \ge 0$ a.e., then

$$\|\nabla\varphi^i\|^2 + \frac{\|\varphi^i\|^2}{2\varepsilon^2} \le \frac{|\Omega|^2}{2\varepsilon^2}.$$

(d) Under the conditions above, it holds

$$0 \le \varphi^i \le 1.$$

Proof. (1) For any given $\varphi \in H^1(\Omega)$ it is $\kappa \le g_b(\varphi) \le b+2$ and hence, by uniform convexity, there exists a unique minimizer $u = u(\varphi)$ of the elastic energy

$$u \mapsto \frac{1}{2}(g_b(\varphi)\mathbb{C}e(u), e(u)) - (q^i, u)_{\Gamma_N}.$$

It is thus sufficient to consider the reduced energy, compare, e.g., [29]

$$\min_\varphi \mathcal{E}_\varepsilon^\gamma(\varphi) := E_\varepsilon^\gamma(q^i, \varphi^{i-1}; u(\varphi), \varphi).$$

Utilizing the results of [24, Theorem 1.1], we obtain, for any $\varphi \in H^1(\Omega)$, the existence of $p > 2$ such that $u(\varphi) \in W^{1,p}(\Omega; R^2) \cap H^1_D(\Omega; \mathbb{R}^2)$ and it holds

$$\|u(\varphi)\|_{1,p} \leq c_{b,\kappa} \|q^i\|.$$

Noticing that g_b satisfies the assumption [29, (2.10)] and the nonnegative penalty term $R(\varphi^{i-1}, \varphi)$ does not influence the statement, we can apply [29, Lemma 2.1] to see that the reduced energy satisfies

$$-\infty < c \leq \mathcal{E}^\gamma_\varepsilon(\varphi) \to \infty \quad (\|\varphi\|_{1,2} \to \infty).$$

Hence there exists $\varphi^i \in H^1(\Omega)$ and an H^1-weakly convergent sequence $\varphi_k \rightharpoonup \varphi^i$ with

$$\mathcal{E}^\gamma_\varepsilon(\varphi_k) \to \inf_\varphi \mathcal{E}^\gamma_\varepsilon(\varphi) > -\infty.$$

By the compact embedding $H^1(\Omega) \subset L^4(\Omega)$, we can w.l.o.g. assume that $\varphi_k \to \varphi^i$ strongly in $L^4(\Omega)$, and hence convergence of $\gamma R(\varphi^{i-1}; \varphi_k) \to \gamma R(\varphi^{i-1}; \varphi^i)$ follows. By [29, Corollary 2.1] it follows that

$$\varphi \mapsto \mathcal{E}^\gamma_\varepsilon(\varphi) - \gamma R(\varphi^{i-1}; \varphi_k)$$

is weakly lower semi-continuous and hence

$$\inf_\varphi \mathcal{E}^\gamma_\varepsilon(\varphi) \leq \mathcal{E}^\gamma_\varepsilon(\varphi^i) \leq \lim_{k\to\infty} \mathcal{E}^\gamma_\varepsilon(\varphi_k) = \inf_\varphi \mathcal{E}^\gamma_\varepsilon(\varphi).$$

This shows the assertion setting $\mathbf{u}^i = (u(\varphi^i), \varphi^i)$.

(2) We notice that for any $(v, \psi) \in V$ the mapping

$$S\colon \mathbb{R} \to \mathbb{R}; \quad s \mapsto E^\gamma_\varepsilon(q^i, \varphi^{i-1}; \mathbf{u}^i + s(v, \psi))$$

is well defined, differentiable and has a local minimizer at $s = 0$. This shows the assertion by consideration of the necessary optimality condition for a minimizer of S, i.e., $S'(0) = 0$.

(3) (a) To show non-negativity of φ^i for the solutions of $(\mathrm{EL}^{\gamma,b})$, we need to test the second equation in $(\mathrm{EL}^{\gamma,b})$ with $\psi = \min(0, \varphi^i)$. We define the set

$$\Omega^- := \{x \in \Omega \,|\, \varphi^i(x) < 0\}$$

and obtain from $(\mathrm{EL}^{\gamma,b})$

$$\begin{aligned} 0 = G_c\varepsilon\|\nabla\varphi^i\|^2_{\Omega^-} + \frac{G_c}{\varepsilon}\|\varphi^i\|^2_{\Omega^-} - \frac{G_c}{\varepsilon}(1, \varphi^i)_{\Omega^-} \\ + (1-\kappa)(m'((\varphi^i)^2)(\varphi^i)^2\mathbb{C}e(u^i), e(u^i))_{\Omega^-} \\ + \gamma([(\varphi^i - \varphi^{i-1})^+]^3, \varphi^i)_{\Omega^-}. \end{aligned}$$

The first two terms are obviously non negative, and positive, if $|\Omega^-| > 0$. The third term satisfies $-(1, \varphi^i)_{\Omega^-} \geq 0$ by definition of Ω^-. The fourth term is nonnegative by our assumption on m' and $\mathbb{C}$. For the fifth (i.e., the final term), we notice, that by assumption on φ^{i-1}

$$\varphi^i \leq 0 \leq \varphi^{i-1} \quad \text{on} \quad \Omega^-$$

and hence

$$([(\varphi^i - \varphi^{i-1})^+]^3, \varphi^i)_{\Omega^-} = 0.$$

This shows $|\Omega^-| = 0$ and hence the assertion $\varphi^i \geq 0$ a.e.

(b) As in the proof of 1. of this Lemma, the equation

$$(g_b(\varphi)\mathbb{C}e(u^i), e(v)) = (q^i, v)_{\Gamma_N}$$

implies the assertion utilizing [24, Proposition 1.2] noting that the estimates only depend on the lower and upper bounds on g_b and not the distribution of the intermediate values.

(c) We start by bounding the H^1 norm of φ^i. To this end, we test (EL$^{\gamma,b}$) with $\psi = \varphi^i$ and obtain

$$\begin{aligned} G_c\varepsilon\|\nabla\varphi^i\|^2 + \frac{G_c}{\varepsilon}\|\varphi^i\|^2 + \gamma([(\varphi^i - \varphi^{i-1})^+]^3, \varphi^i) \\ + (1-\kappa)(m'((\varphi^i)^2)(\varphi^i)^2\mathbb{C}e(u^i), e(u^i)) \\ = \frac{G_c}{\varepsilon}(1, \varphi^i) \\ \leq \frac{G_c}{2\varepsilon}|\Omega|^2 + \frac{G_c}{2\varepsilon}\|\varphi^i\|^2. \end{aligned}$$

Since all terms on the left are non negative, note that $\varphi^i \geq 0$, we deduce

$$\|\nabla\varphi^i\|^2 + \frac{\|\varphi^i\|^2}{2\varepsilon^2} \leq \frac{|\Omega|^2}{2\varepsilon^2}.$$

(d) To see the assertion, we test (EL$^{\gamma,b}$) with $\psi = (\varphi^i - 1)^+ = \max(0, \varphi^i - 1)$ and obtain

$$\begin{aligned} 0 = G_v\varepsilon\|\nabla(\varphi^i - 1)^+\|^2 + \frac{G_c}{\varepsilon}\|(\varphi^i - 1)^+\|^2 + \gamma([(\varphi^i - \varphi^{i-1})^+]^3, (\varphi^i - 1)^+) \\ + (1-\kappa)(m'((\varphi^i)^2)\varphi^i\mathbb{C}e(u^i) : e(u^i), (\varphi^i - 1)^+). \end{aligned}$$

Noticing that all summands are non negative, the assertion follows analogously to part (a). □

Now, choosing $b \geq 1$ in the last Lemma allows to transfer these results to (C$^\gamma$).

Corollary 3.2. *(1) Given $q^i \in L^2(\Gamma_N)$ and $\varphi^{n-1} \in L^2(\Omega)$ with $\varphi^{i-1} \geq 0$ there exists at least one solution $\mathbf{u}^i$ of* (EL$^\gamma$).

(2) Further, any solution $\mathbf{u}^i = (u^i, \varphi^i)$ to (EL$^\gamma$) *satisfies*

(a) $0 \leq \varphi^i \leq 1$ a.e.

(b) There exists a constant c_κ depending on κ and $p > 2$, such that

$$\|u^i\|_{1,p} \leq c_\kappa\|q^i\|.$$

(c) It holds

$$\|\nabla\varphi^i\|^2 + \frac{\|\varphi^i\|^2}{2\varepsilon^2} \leq \frac{|\Omega|^2}{2\varepsilon^2}.$$

Proof. (1) The existence of at least one solution follows by Lemma 3.1 taking $b \geq 1$ since then $g_b(\varphi^i) = g(\varphi^i)$ and $m'((\varphi^i)^2) = 1$ for any solution to (EL$^{\gamma,b}$) and hence any such solution solves (EL$^\gamma$) as well.

(2) (a) The proof of Lemma 3.1 3.(a) and 3.(d) can be repeated to yield the desired bounds $0 \leq \varphi^i \leq 1$.

(b) The proof of Lemma 3.1 3.(b) can be applied, noticing that the constant depends on the upper and lower bound of the coefficient, i.e., $\kappa \leq g(\varphi) \leq 1$, only.

(c) The proof of Lemma 3.1 3.(c) carries over to the present setting as well.

□

3.2. **The Problem** (NLP$^\gamma$)**.** We can now finalize the existence of solutions to (NLP$^\gamma$).

Theorem 3.3. *There exists at least one global minimizer $(q, \mathbf{u}) \in (Q \times V)^M$ to* (NLP$^\gamma$).

Proof. The proof is almost straight forward. Since $J(q, \mathbf{u}) \geq 0$ there exists a minimizing sequence $(q_k, \mathbf{u}_k)$ satisfying (EL$^\gamma$), i.e., $J(q_k, \mathbf{u}_k) \to \inf_{q,\mathbf{u}} J(q, \mathbf{u})$. The corresponding control q_k is bounded in Q^M and hence there exists a weakly convergent subsequence, w.l.o.g denoted by q_k, with limit q_∞. By Corollary 3.2 2.(b) and 2.(c), the sequence (u_k, φ_k) is bounded in $(W^{1,p}(\Omega;\mathbb{R}^2) \times H^1(\Omega))^M$ and consequently w.l.o.g. $u_k \rightharpoonup u_\infty$ in $W^{1,p}(\Omega;\mathbb{R}^2)^M$ and $\varphi_k \rightharpoonup \varphi_\infty$ in $H^1(\Omega)^M$. To see that the limit

satisfies the elasticity equation in (EL$^\gamma$), note that due to the compact embedding $H^1(\Omega) \subset L^p(\Omega)$ for any $p < \infty$

$$g(\varphi_k^i)\mathbb{C}e(u_k^i) \rightharpoonup g(\varphi_\infty^i)\mathbb{C}e(u_\infty^i)$$

in $L^2(\Omega;\mathbb{R}^{2\times 2})$ holds, since $g(\varphi_k^i)$ converges strongly. To see that the limiting phase-field φ_∞ satisfies the equation, we notice, that by Corollary 3.2 the phase-field satisfies $\varphi_k^i \leq 1$ and hence we can also consider (EL$^{\gamma,b}$), because it coincides with (EL$^\gamma$) in the relevant points but satisfies the conditions in [29]. Thus by [29, Corollary 2.1] it is

$$\begin{aligned} G_c\varepsilon(\nabla\varphi_k^i,\nabla\cdot) + (1-\kappa)(\varphi_k^i\mathbb{C}e(u_k^i) : e(u_k^i),\cdot) - \frac{G_c}{\varepsilon}(1-\varphi_k^i,\cdot) \\ \rightharpoonup G_c\varepsilon(\nabla\varphi_\infty^i,\nabla\cdot) + (1-\kappa)(\varphi_\infty^i\mathbb{C}e(u_\infty^i) : e(u_\infty^i),\cdot) - \frac{G_c}{\varepsilon}(1-\varphi_\infty^i,\cdot) \end{aligned}$$

weakly in $H^1(\Omega)^*$. For the remaining term $\gamma[(\varphi_k^i - \varphi_k^{i-1})^+]^3$ the compact embedding $H^1(\Omega) \subset L^6(\Omega)$ gives convergence in L^2 and consequently the pair $\mathbf{u}_\infty = (u_\infty, \varphi_\infty)$ solves (EL$^\gamma$).

Hence $(q_\infty, \mathbf{u}_\infty)$ is feasible for (NLP$^\gamma$). Weak lower semicontinuity of J shows that

$$J(q_\infty, \mathbf{u}_\infty) \leq \inf_{q,\mathbf{u}} J(q,\mathbf{u})$$

and thus the assertion is shown, setting $(q,u) = (q_\infty, u_\infty)$. □

Corollary 3.4. *Any minimizer $(q,\mathbf{u})$ of (NLP$^\gamma$) satisfies the additional regularity $\mathbf{u} \in (W^{1,p}(\Omega;\mathbb{R}^2) \times L^\infty(\Omega))^M$. More precisely for any $i = 1,\dots M$ it holds $0 \leq \varphi^i \leq 1$ and $\|u^i\|_{1,p} \leq c_\kappa \|q^i\|$.*

Proof. This is an immediate consequence of Corollary 3.2. □

4. The Linearized Problem

In order to discuss first order necessary optimality conditions, as well as the potential approximation of (NLP$^\gamma$) by a sequence of linear-quadratic problems, let $(q_k, \mathbf{u}_k) = (q_k, u_k, \varphi_k) \in (Q \times V)^M$ be a given point. Considering the regularity of solutions to (EL$^\gamma$), we assume $q_k \in Q^M$, and $(u_k, \varphi_k) \in (V \cap (W^{1,p}(\Omega;\mathbb{R}^2) \times L^\infty(\Omega)))^M$.

The linearized problem to (EL$^\gamma$) consists, for given $q \in Q^M$ and $\varphi^0 := 0$, of finding $\mathbf{u} = (u,\varphi) \in V^M$ such that for any $i = 1,\dots,M$ and $(v,\psi) \in V$

$$\text{(EL}^\gamma_{\text{lin}}\text{)} \qquad \begin{aligned} &\Big(g(\varphi_k^i)\mathbb{C}e(u^i), e(v)\Big) \\ &+2(1-\kappa)(\varphi_k^i\mathbb{C}e(u_k^i)\varphi^i, e(v)) = (q^i, v)_{\Gamma_N} \\ &G_c\varepsilon(\nabla\varphi^i,\nabla\psi) + \frac{G_c}{\varepsilon}(\varphi^i,\psi) \\ &+(1-\kappa)(\varphi^i\mathbb{C}e(u_k^i) : e(u_k^i),\psi) \\ &+3\gamma([(\varphi_k^i - \varphi_k^{i-1})^+]^2\varphi^i,\psi) \\ &+2(1-\kappa)(\varphi_k^i\mathbb{C}e(u_k^i) : e(u^i),\psi) = 3\gamma([(\varphi_k^i - \varphi_k^{i-1})^+]^2\varphi^{i-1},\psi). \end{aligned}$$

Existence of Solutions to (EL$^\gamma_{\text{lin}}$). We now discuss the properties of the linearized equation (EL$^\gamma_{\text{lin}}$)

Lemma 4.1. *For any given $(u_k,\varphi_k) \in (V \cap (W^{1,p}(\Omega;\mathbb{R}^2) \times L^\infty(\Omega)))^M$ with $p > 2$ and $q_k \in Q^M$ the linear operators $A_i \colon V \to V^*$ corresponding to (EL$^\gamma_{\text{lin}}$) defined by*

$$\begin{aligned} \langle A_i(u^i,\varphi^i),(v,\psi)\rangle_{V,V^*} &:= a_i(u^i,\varphi^i;v,\psi) \\ &:= \Big(g(\varphi_k^i)\mathbb{C}e(u^i), e(v)\Big) + 2(1-\kappa)(\varphi_k^i\mathbb{C}e(u_k^i)\varphi^i, e(v)) \\ &\quad + G_c\varepsilon(\nabla\varphi^i,\nabla\psi) + \frac{G_c}{\varepsilon}(\varphi^i,\psi) + (1-\kappa)(\varphi^i\mathbb{C}e(u_k^i) : e(u_k^i),\psi) \\ &\quad + 3\gamma([(\varphi_k^i - \varphi_k^{i-1})^+]^2\varphi^i,\psi) + 2(1-\kappa)(\varphi_k^i\mathbb{C}e(u_k^i) : e(u^i),\psi) \end{aligned}$$

are Fredholm of index zero.

Proof. Since $p > 2$, we can find $r \in (2, \infty)$ such that $\frac{1}{p} + \frac{1}{2} + \frac{1}{r} = 1$. By embedding theorems, there exists $0 < s < 1$, such that $H^s(\Omega) \subset L^r(\Omega)$ compactly. Then continuity of a_i on $V \times V$ follows

$$\begin{aligned} a_i(u^i, \varphi^i; v, \psi) &\leq c\|u^i\|_{1,2}\|v\|_{1,2} + c\|\varphi^i\|_{0,r}\|v\|_{1,2} \\ &\quad + c\|\varphi^i\|_{1,2}\|\psi\|_{1,2} + c\|\varphi^i\|\|\psi\| + c\|\varphi^i\|_{0,r}\|\psi\|_{0,r} \\ &\quad + c_\gamma\|\varphi^i\|\|\psi\| + c\|u^i\|_{1,2}\|\psi\|_{0,r} \\ &\leq c\Big(\|u^i\|_{1,2}\|v\|_{1,2} + \|\varphi^i\|_{1,2}\|v\|_{1,2} + \|\varphi^i\|_{1,2}\|\psi\|_{1,2} + \|u^i\|_{1,2}\|\psi\|_{1,2}\Big) \\ &\leq c\|(u^i, \varphi^i)\|_V\|(v, \psi)\|_V, \end{aligned}$$

with generic constants c depending on $(u_k^i, \varphi_k^i) \in W^{1,p}(\Omega; \mathbb{R}^2) \times L^\infty(\Omega)$. To derive a lower bound, we notice that the only possibly non-positive terms are the two starting with $2(1-\kappa)$ and we deduce, using Korn's inequality

$$\begin{aligned} a(u^i, \varphi^i; u^i, \varphi^i) &\geq c\|u^i\|_{1,2}^2 + \|\varphi^i\|_{1,2}^2 - c\|\varphi^i\|_{0,r}\|u^i\|_{1,2} \\ &\geq c\|u^i\|_{1,2}^2 + \|\varphi^i\|_{1,2}^2 - c\|\varphi^i\|_{s,2}^2. \end{aligned}$$

Consequently $a_i(\cdot,\cdot) + c(\cdot,\cdot)_{s,2}$ is coercive on $V \times V$ and thus invertible, and in particular Fredholm of index zero, by the Lax-Milgram lemma. Since $H^1(\Omega) \subset H^s(\Omega)$ is compact, we deduce that the mapping $A_i \colon V \to V^*$ given by $(u^i, \varphi^i) \mapsto A_i(u^i, \varphi^i) = a_i(u^i, \varphi^i; \cdot)$ is Fredholm of index zero as well, see, e.g., [46, Theorem 12.8]. □

Lemma 4.2. *Under the assumptions of Lemma 4.1, any element $(u^i, \varphi^i) \in \ker(A_i) \subset V$ satisfies the additional regularity $(u^i, \varphi^i) \in V \cap (W^{1,p}(\Omega; \mathbb{R}^2) \times L^\infty(\Omega))$.*

Proof. Consider $(u^i, \varphi^i) \in \ker(A_i)$, i.e.,

$$a_i(u^i, \varphi^i; v, \psi) = 0 \quad \forall (v, \psi) \in V.$$

First of all, we notice, that the linearized phase-field φ^i satisfies

$$\begin{aligned} G_c\varepsilon(\nabla\varphi^i, \nabla\psi) + \frac{G_c}{\varepsilon}(\varphi^i, \psi) &\\ = -(1-\kappa)(\varphi^i \mathbb{C}e(u_k^i) : e(u_k^i), \psi) - 3\gamma([(\varphi_k^i - \varphi_k^{i-1})^+]^2\varphi^i, \psi) &\\ - 2(1-\kappa)(\varphi_k^i \mathbb{C}e(u_k^i) : e(u^i), \psi). & \end{aligned}$$

With the definition of r as in the proof of Lemma 4.1, it is $\varphi^i \in L^r(\Omega)$ and $\frac{1}{p} + \frac{1}{r} = \frac{1}{2}$. Let r' be given such that $1 = \frac{1}{r} + \frac{1}{r'} = \frac{1}{r} + (\frac{1}{r} + \frac{2}{p})$, then $1 < r' < 2$. As a consequence, the right hand side of the equation above is an element in $L^{r'}(\Omega)$. To see this, we calculate

$$\begin{aligned} \|\varphi^i \mathbb{C}e(u_k^i) : e(u_k^i)\|_{0,r'} &\leq c\|\varphi^i\|_{0,r}\|e(u_k^i)\|_{0,p}^2, \\ \|[(\varphi_k^i - \varphi_k^{i-1})^+]^2\varphi^i\|_{0,r'} &\leq c\|[(\varphi_k^i - \varphi_k^{i-1})^+]^2\varphi^i\|_{0,r} \leq c\|(\varphi_k^i - \varphi_k^{i-1})^+\|_{0,\infty}^2\|\varphi^i\|_{0,r}, \\ \|\varphi_k^i \mathbb{C}e(u_k^i) : e(u^i)\|_{0,r'} &\leq c\|\varphi_k^i\|_{0,\infty}\|e(u_k^i)\|_{0,p}\|e(u^i)\|. \end{aligned}$$

Utilizing elliptic regularity it follows that $\varphi^i \in H^1(\Omega) \cap L^\infty(\Omega)$.

Now, we can continue to derive the improved regularity of u^i. To this end, we notice, that u^i solves

$$\Big(g(\varphi_k^i)\mathbb{C}e(u^i), e(v)\Big) = -2(1-\kappa)(\varphi_k^i \mathbb{C}e(u_k^i)\varphi^i, e(v)).$$

The right hand side satisfies

$$\|\varphi_k^i \mathbb{C}e(u_k^i)\varphi^i\|_{0,p} \leq c\|\varphi_k^i\|_{0,\infty}\|u_k^i\|_{1,p}\|\varphi^i\|_{0,\infty}$$

and thus $(\varphi_k^i \mathbb{C}e(u_k^i)\varphi, e(\cdot))$ defines an element in $W^{-1,p}(\Omega;\mathbb{R}^2) = \left(W^{1,p'}(\Omega;\mathbb{R}^2)\right)^*$, utilizing again [24, Theorem 1.1], we conclude that $u \in W^{1,p}(\Omega;\mathbb{R}^2)$. □

Remark 4.1. *Utilizing the above regularity provided by Lemma 4.2, we can now define the scalar product* $(\cdot,\cdot)_{\mathbb{C}} = (\mathbb{C}\cdot,\cdot)$ *and corresponding norm* $\|\cdot\|_{\mathbb{C}}$. *The above regularity shows, that the norms* $\|\varphi^i e(u_k^i)\|_{\mathbb{C}}$ *and* $\|\varphi_k^i e(u^i)\|_{\mathbb{C}}$ *are finite for all* $(u^i,\varphi^i) \in \ker A_i$. *Consequently, we can now provide an improved lower bound utilizing the parallelogram identity for the above scalar product*

$$
\begin{aligned}
0 &= a(u^i,\varphi^i;u^i,\varphi^i)\\
&= (1-\kappa)((\varphi_k^i)^2\mathbb{C}e(u^i),e(u^i)) + \kappa(\mathbb{C}e(u^i),e(u^i)) + 2(1-\kappa)(\varphi_k^i\mathbb{C}e(u_k^i)\varphi^i,e(u^i))\\
&\quad + G_c\varepsilon(\nabla\varphi^i,\nabla\varphi^i) + \frac{G_c}{\varepsilon}(\varphi^i,\varphi^i) + (1-\kappa)(\varphi^i\mathbb{C}e(u_k^i):e(u_k^i),\varphi^i)\\
&\quad + 3\gamma([(\varphi_k^i-\varphi_k^{i-1})^+]^2\varphi^i,\varphi^i) + 2(1-\kappa)(\varphi_k^i\mathbb{C}e(u_k^i):e(u^i),\varphi^i)\\
&\geq \kappa\|e(u^i)\|_{\mathbb{C}}^2 + (1-\kappa)\|\varphi_k^i e(u^i)\|_{\mathbb{C}}^2 + 2(1-\kappa)(\varphi_k^i e(u^i),\varphi^i e(u_k^i))_{\mathbb{C}}\\
&\quad + c\|\varphi^i\|_{1,2}^2 + (1-\kappa)\|\varphi^i e(u_k^i)\|_{\mathbb{C}}^2 + 2(1-\kappa)(\varphi_k^i e(u^i),\varphi^i e(u_k^i))_{\mathbb{C}}\\
&\geq \kappa\|e(u^i)\|_{\mathbb{C}}^2 + c\|\varphi^i\|_{1,2}^2 + (1-\kappa)\|\varphi_k^i e(u^i)+\varphi^i e(u_k^i)\|_{\mathbb{C}}^2 + 2(1-\kappa)(\varphi_k^i e(u^i),\varphi^i e(u_k^i))_{\mathbb{C}}.
\end{aligned}
$$

Remark 4.2. *From the previous Remark 4.1, we immediately assert, that for sufficiently small* $\|u_k^i\|_{1,p}$, $\|\varphi_k^i\|_{0,\infty}$, *the mixed term can be absorbed into the squared norms and, consequently, for such* (u_k,φ_k), *we have* $\ker A_i = \{0\}$. *Indeed, this would already be clear from the proof of Lemma 4.1, but the condition provided by Remark 4.1 is tighter.*

Corollary 4.3. *For any given* $(u_k,\varphi_k) \in (V \cap (W^{1,p}(\Omega;\mathbb{R}^2)\times L^\infty(\Omega)))^M$ *and* $q_k \in Q^M$ *the linear operators* $\mathcal{A}\colon V^M \to (V^*)^M$ *defined by*

$$
\begin{pmatrix}
A_1 & & & \\
B_2 & A_2 & & \\
 & \ddots & \ddots & \\
 & & B_M & A_M
\end{pmatrix}
$$

with $A_i\colon V \to V^*$ *as in Lemma 4.1 and* $B_i = 3\gamma[(\varphi_k^i-\varphi_k^{i-1})^+]^2$, *are Fredholm of index zero.*

Proof. By Lemma 4.1 the diagonal is Fredholm, and the off-diagonal B_i are compact as a mapping $V \to V^*$. Thus the assertion follows by [46, Theorem 12.8]. □

5. First Order Necessary Conditions for (NLP$^\gamma$)

We can now state the necessary optimality conditions for (NLP$^\gamma$).

Theorem 5.1. *Let* $(\bar q,\bar{\mathbf{u}}) \in (Q\times V)^M$ *be a minimizer of* (NLP$^\gamma$), *such that* $\ker\mathcal{A} = \{0\}$, *with* $\mathcal{A}$ *as defined in Corollary 4.3 in the point* $(q_k,\mathbf{u}_k) = (\bar q,\bar{\mathbf{u}})$. *Then there exists* $\bar{\mathbf{z}} = (\bar z,\bar\zeta) \in V^M$ *such that*

$$
\begin{aligned}
&(\bar q,\bar{\mathbf{u}}) \text{ satisfy } (\mathrm{EL}^\gamma)\\
\langle\mathcal{A}^*\bar{\mathbf{z}},\varphi\rangle &= \sum_{i=1}^M(\bar u^i - u_d^i,\varphi^i) && \forall\varphi\in V^M\\
\alpha\sum_{i=1}^M(\bar q^i,\delta q^i)_{\Gamma_N} &= -\sum_{i=1}^M(\bar z^i,\delta q^i)_{\Gamma_N} && \forall\delta q\in Q^M.
\end{aligned}
$$

Proof. By Corollary 4.3 $\mathcal{A}$ is Fredholm, since $\ker\mathcal{A} = \{0\}$ $\mathcal{A}$ is an isomorphism, and so is its dual $\mathcal{A}^*$. Consequently, the linearized constraint ($\mathrm{EL}^\gamma_{\mathrm{lin}}$) is surjective as a mapping $(Q\times V)^M \to (V^*)^M$ and the existence of $\bar{\mathbf{z}}$ follows by standard results on the existence of Lagrange multipliers, see, e.g., [47, Theorem 4.1.(a)]. □

6. Quadratic Approximations to (NLP$^\gamma$)

We aim to approximate (NLP$^\gamma$) by a linear-quadratic problem in a given point $(q_k, \mathbf{u}_k) = (q_k, u_k, \varphi_k)$. Considering the regularity of solutions to (EL$^\gamma$), we assume $q_k \in Q^M$, and $(u_k, \varphi_k) \in (V \cap (W^{1,p}(\Omega;\mathbb{R}^2) \times L^\infty(\Omega)))^M$.

In order to keep the notation short, we introduce the (compact) operator $\mathcal{B}\colon Q^M \to (V^*)^M$ for the control action as follows

$$\langle \mathcal{B}q, (v,\psi)\rangle_{(V^*)^M, V^M} := \sum_{i=1}^M (q^i, v^i)_{\Gamma_N}. \tag{6.1}$$

By standard reformulations, we obtain the quadratic problem, up to a fixed additive constant in the cost functional,

$$\begin{aligned} &\min_{(q,\mathbf{u})} J_{\mathrm{lin}}(q,\mathbf{u}) := \frac{1}{2}\sum_{i=1}^M \|u^i - (u_d^i - u_k^i)\|^2 + \frac{\alpha}{2}\sum_{i=1}^M \|q^i + q_k^i\|_{\Gamma_N}^2 \\ &\text{s.t. } (q,\mathbf{u}) \text{ satisfy } (\mathrm{EL}^\gamma_{\mathrm{lin}}), \text{ i.e., } \mathcal{A}\mathbf{u} = \mathcal{B}q, \end{aligned} \tag{QP$^\gamma$}$$

where $\mathcal{A}$ is given in Corollary 4.3 and $\mathcal{B}$ in (6.1).

6.1. Existence of Solutions to (QP$^\gamma$).

Theorem 6.1. *For any given $(u_k, \varphi_k) \in (V \cap (W^{1,p}(\Omega;\mathbb{R}^2) \times L^\infty(\Omega)))^M$ and $q_k \in Q^M$ the problem (QP$^\gamma$) has a unique solution $(\overline{q}, \overline{\mathbf{u}}) \in Q^M \times V^M$.*

Proof. It is immediate that, a pair $(q, \mathbf{u})$ satisfies $(\mathrm{EL}^\gamma_{\mathrm{lin}})$ if and only if

$$\mathcal{A}\mathbf{u} = \mathcal{B}q \quad \text{in} \quad (V^*)^M$$

with $\mathcal{A}$ as defined in Corollary 4.3 and $\mathcal{B}$ as in (6.1). Now, by Corollary 4.3, $\mathcal{A}$ is Fredholm and consequently, see, e.g., [46, Theorem 12.2], has closed range. Moreover, since the codimension of the image of $\mathcal{A}$ is finite the intersection $\mathcal{A}(V^M) \cap \mathcal{B}(Q^M)$ is non empty. Clearly J_{lin} is bounded below, and we can pick a minimizing sequence $(q_{(k)}, \mathbf{u}_{(k)})$ satisfying $\mathcal{A}\mathbf{u} = \mathcal{B}q$. Due to the coercivity of J_{lin} the sequence is bounded and, possibly selecting a subsequence, there is a weak limit $q_{(k)} \rightharpoonup q_{(\infty)}$ in Q^M. By compactness, $\mathcal{B}q_{(k)} \to \mathcal{B}q_{(\infty)}$ in $(V^*)^M$.

Since $\mathcal{A}$ is Fredholm, $\dim\ker\mathcal{A} < \infty$ and consequently, we can decompose $V^M = \ker\mathcal{A} \oplus V^M/\ker\mathcal{A}$. Correspondingly, we split the sequence $\mathbf{u}_{(k)} = \mathbf{u}^{\ker}_{(k)} + \mathbf{u}^0_{(k)}$. Then $\mathcal{A}$ induces an isomorphism as a mapping $\mathcal{A}\colon V^M/\ker\mathcal{A} \to \mathcal{A}(V^M)$ and consequently

$$\mathbf{u}^0_{(k)} = \mathcal{A}^{-1}\mathcal{B}q_{(k)} \to \mathcal{A}^{-1}\mathcal{B}q_{(\infty)} = \mathbf{u}^0_{(\infty)}.$$

Moreover, since J_{lin} is bounded along its minimizing sequence, $\|\mathbf{u}^{\ker}_{(k)}\|$ is bounded, and since $\ker\mathcal{A}$ is finite dimensional, possibly selecting a subsequence, there exists a limit $\mathbf{u}^{\ker}_{(k)} \to \mathbf{u}^{\ker}_{(\infty)} \in \ker\mathcal{A}$. By continuity of $\mathcal{A}$ it is

$$\mathcal{A}\mathbf{u}_{(\infty)} = \mathcal{B}q_{(\infty)},$$

and by weak lower semicontinuity

$$J_{\mathrm{lin}}(q_{(\infty)}, \mathbf{u}_{(\infty)}) \le \inf_{(q,\mathbf{u})} J_{\mathrm{lin}}(q,\mathbf{u}).$$

Uniqueness follows, since J_{lin} is strictly convex on $Q^M \times V^M$. □

6.2. **Necessary (& Sufficient) Optimality Conditions.** To conclude the discussion of the quadratic approximations, we note that we can give necessary, and due to convexity also sufficient, first order optimality conditions.

Theorem 6.2. *For any given* $(u_k, \varphi_k) \in (V \cap (W^{1,p}(\Omega;\mathbb{R}^2) \times L^\infty(\Omega)))^M$ *and* $q_k \in Q^M$ *let* $(\overline{q}, \overline{\mathbf{u}}) \in Q^M \times V^M$ *be a solution to* (QP^γ)*. Then there exists a Lagrange multiplier,* $\overline{\mathbf{z}} = (\overline{z}, \overline{\zeta}) \in V^M$*, such that the system*

$$
(\mathrm{KKT}^\gamma) \qquad \begin{aligned} \mathcal{A}\overline{u} &= \mathcal{B}\overline{q} && \text{in } (V^*)^M, \\ \mathcal{A}^*\overline{\mathbf{z}} &= \overline{u} - (u_d - u_k) && \text{in } (V^*)^M, \\ \alpha(\overline{q} - q_k) + \overline{z} &= 0 && \text{on } \Gamma_N \end{aligned}
$$

is satisfied where $\mathcal{A}$ *is given in Corollary 4.3,* $\mathcal{A}^*$ *denotes its adjoint,* $\mathcal{B}$ *is given by* (6.1)*, and the, compact, embedding* $(L^2)^M \subset (V^*)^M$ *is used without special notation for the right hand side of the adjoint equation. Due to the convexity of* (QP^γ)*, any triplet* $(\overline{q}, \overline{\mathbf{u}}, \overline{\mathbf{z}}) \in Q^M \times V^M \times V^M$ *solving* (KKT^γ) *gives rise to a solution of* (QP^γ)*.*

Proof. We notice that the equality constraint in (QP^γ) is linear, and consequently a constraint qualification is given and the result is a consequence of Farkas'-Lemma, see, e.g., [16, Theorem 10] for its generalization to infinite dimensions. □

7. Numerical Illustration

In this final section, we discuss a prototype test in order to substantiate our theoretical advancements. Moreover, our findings indicate that the QP-approximations discussed above can be used to obtain a (locally) fast convergent Newton (SQP) Algorithm.

The setup is to employ a control q on the top boundary of a two-dimensional square domain, acting in normal direction only, in order to steer the solution towards a manufactured solution u_D defined in the entire domain. The computations are performed with DOpElib [19, 20] utilizing the deal.II finite element library [5, 6].

The domain is given by $\Omega := (-1, 1)^2$ in which a horizontal fracture is prescribed. The initial value for φ^0 is taken such that $\varphi^0 = 0$ on $(-0.1 - h, 0.1 + h) \times (-h, h) \subset \Omega$ (see Figure 1), where h denotes the diameter of the elements. The boundary is divided into three parts $\partial\Omega := \Gamma_N \cup \Gamma_D \cup \Gamma_{\text{free}}$ corresponding to the control boundary Γ_N, the Dirichlet boundary Γ_D, and the rest, where natural boundary conditions for the displacement are attained. These boundary parts are given by

$$
\Gamma_N = \{(x, 1) \,|\, -1 \le x \le 1\} \quad \text{and} \quad \Gamma_{\text{free}} = \{(x, y) \,|\, x \in \{\pm 1\}; -1 \le y \le 1\}.
$$

On Γ_D, we prescribe the Dirichlet values $u = 0$.

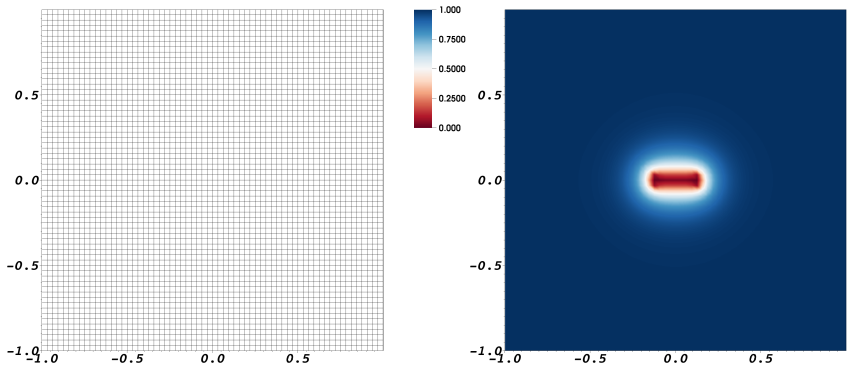

FIGURE 1. Geometry and mesh (left) and initial fracture (in red; right figure).

The cost functional is given by

$$J(q,\mathbf{u}) := \frac{1}{2}\sum_{i=1}^{M} \|u^i - u_d^i\|^2 + \frac{\alpha}{2}\|q+q_d\|^2_{\Gamma_N}$$
$$\text{s.t. } (q,\mathbf{u}) \text{ satisfying } (\mathrm{EL}^\gamma),$$

where $u_d^i = 0.001(y+1)$ for all $i = 1,\dots,M$, $\alpha = 10^{-10}$ and a control acting on Γ_N but being the same in all time-steps, i.e, $q^i = q$ for all $i = 1,\dots,M$, and $q_d \equiv 50$. Moreover, $\mathbf{u}^0 = (0;\varphi^0)$ with φ^0 as depicted in Figure 1.

Furthermore, the phase-field regularization parameter is chosen as $\varepsilon = 2h = 0.088$ where $h = 0.0442$ is the element diameter of the mesh for the finite element discretization used for the computations. The bulk regularization parameter is $\kappa = 10^{-10}$, the penalization parameter is $\gamma = 10^8$, the fracture energy release rate is $G_c = 1.0$, Young's modulus is $E = 10^6$ and Poisson's ratio is $\nu = 0.2$. The initial mesh is six times globally refined as shown in Figure 1 and 5 loading steps, i.e., $M = 5$, are performed. The spatial discretization is done using standard Q_1 finite elements for all unknowns.

Our findings are summarized in the following. The initial value of the cost functional is $J_{initial} = 1.247 \times 10^{-5}$ that is obtained by employing the initial control $q \equiv 10$ on Γ_N. In this particular setting, the initial residual of the Newton iteration is small; namely 7.46×10^{-9}. This starting residual is taken as 1 in the relative residual, which is plotted in Table 1. Furthermore, Table 1 shows the iteration history of the Newton steps performed during the solution of the optimization problem. At each step, the Newton residual, the cost functional J and $q_{max} = \max_{\Gamma_N} |q|$ are provided. We observe that the algorithm is convergent, the convergence slows down to a linear rate in the later iterations as it has to be expected since the QP-subproblems are solved only up to an accuracy proportional to the norm of the optimization residual, and consequently only very few, i.e., two, iterations of the linear solver are performed in these Newton steps.

TABLE 1. Results of the nonlinear optimization iterations.

Newton iter.	N-linear iter.	Newton residual (rel.)	$J[\times 10^{-5}]$	q_{max} on Γ_N
0	-	1.00×10^{0}	1.2470	10
1	14	3.57×10^{-2}	1.0487	87
2	11	1.23×10^{-3}	1.0469	84
3	2	3.94×10^{-4}	1.0469	84
4	2	1.27×10^{-4}	1.0469	84

Illustrations of the solutions are provided in the Figures 2–4 displaying the primal and adjoint solutions. Here, expected behavior is shown: the largest y-displacement is on Γ_N. However, the linear growth of this displacement can not be achieved, in contrast to linear elasticity alone, due to the presence of the fracture. It should be noted, that the color-scale in 3 and 4 is adjusted to the size of the displacement in the last Newton step, as it is visible from these pictures the initial displacement is severely smaller and almost invisible in this scale.

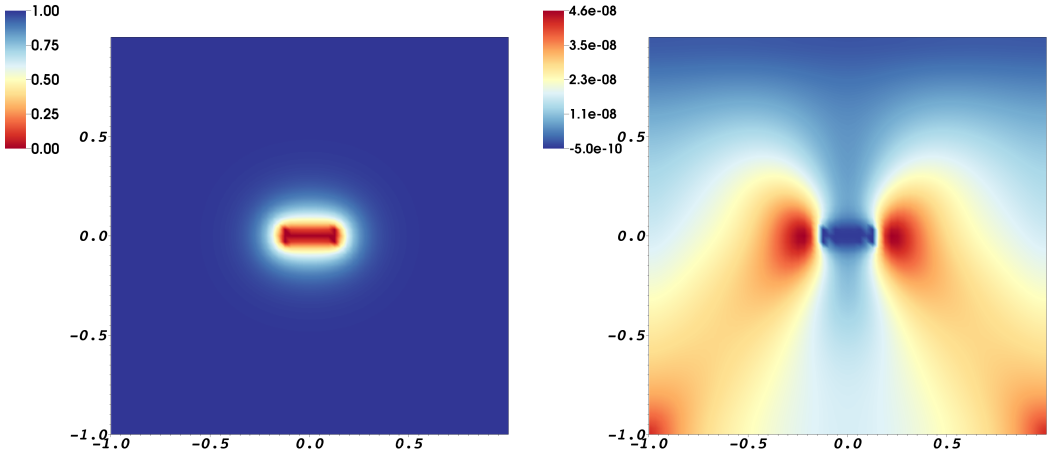

FIGURE 2. Final fracture (in red) and corresponding adjoint phase-field after four Newton iterations at $M = 5$.

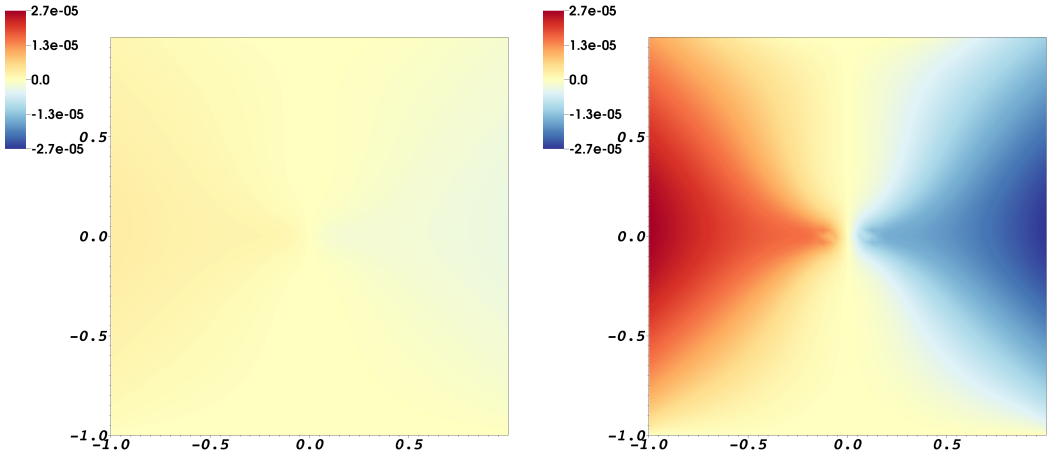

FIGURE 3. Initial x-displacement field and final x-displacement field after four Newton iterations at $M = 5$.

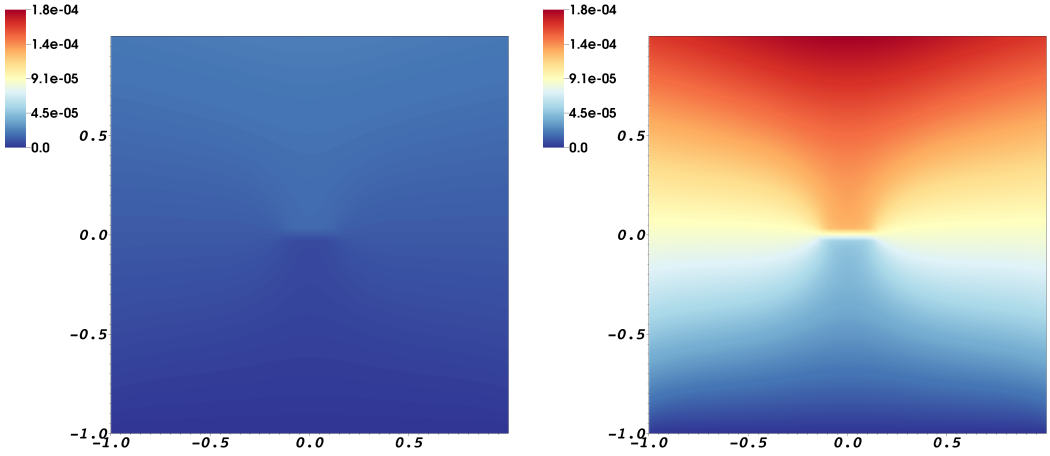

FIGURE 4. Initial y-displacement field and final y-displacement field after four Newton iterations at $M = 5$.

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