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Abstract

Recently, a novel approach for the robust discretization of the incompressible Stokes equations was proposed that slightly modifies the nonconforming Crouzeix–Raviart element such that its velocity error becomes *pressure-independent*. The modification results in an $\mathcal{O}(h)$ consistency error that allows straightforward proofs for the optimal convergence of the discrete energy norm of the velocity and of the L^2 norm of the pressure. However, though the optimal convergence of the velocity in the L^2 norm was observed numerically, it appeared to be nontrivial to prove. In this contribution, this gap is closed. Moreover, the dependence of the energy error estimates on the discrete inf-sup constant is traced in detail, which shows that classical error estimates are extremely pessimistic on domains with large aspect ratios. Numerical experiments in 2D and 3D illustrate the theoretical findings.

1 Introduction

For several decades, it was common belief in the numerical analysis community, that in mixed discretizations for the Stokes equations in primal variables velocity $\mathbf{u}$ and pressure p, and with data $f \in L^2(\Omega)$, $g \in L^2_0(\Omega)$ and $\nu > 0$ on a domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$,

$$\begin{aligned} -\nu\Delta\mathbf{u} + \nabla p &= \mathbf{f}, && \mathbf{x} \in \Omega,\\ -\nabla \cdot \mathbf{u} &= g, && \mathbf{x} \in \Omega,\\ \mathbf{u} &= 0, && \mathbf{x} \in \partial\Omega, \end{aligned} \tag{1}$$

a dependence of the discrete velocity on the continuous pressure was more or less practically unavoidable. In other words, it became standard in finite

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element analysis to prove for new mixed discretisations of the incompressible Stokes equations the following kind of finite element error estimate for the discrete velocity

$$\|\mathbf{u}-\mathbf{u}_h\|_{1,h} \leq \frac{C_1}{\beta_h} \inf_{\mathbf{w}_h \in X_h} \|\mathbf{u}-\mathbf{w}_h\|_{1,h} + \frac{1}{\nu} \inf_{q_h \in Q_h} \|p-q_h\|_{L^2}, \tag{2}$$

with some generic constant C_1, since this estimate allows to conclude that the discrete velocity converges with an *asymptotically optimal* convergence order. However, the estimate (2) is not really optimal with respect to two different aspects, i.e., qualitatively, it does not give the best possible theoretical result, which one can hope for: i) The first point concerns the appearance of the inverse of the discrete inf-sup constant β_h in the error estimate. The inf-sup constant is well-known to degenerate for domains with a large aspect ratio [16, 15, 38, 11], e.g., for practically relevant channel-like domains. Therefore, velocity error estimates containing the constant $1/\beta_h$ are extremely pessimistic. Indeed, such estimates can be improved, if an appropriate, *locally defined* Fortin operator for a mixed finite element is known, as demonstrated in this contribution. Further, we will derive several *explicit* a-priori error estimates, where all involved constants (such as $\widetilde{C}_1$ in (3) below) only depend on the angles in the underlying finite element mesh, but not on the inf-sup constant or the value of ν.

ii) The second point concerns the appearance of the pressure-dependent error contribution $\frac{1}{\nu}\inf_{q_h \in Q_h}\|p-q_h\|_{L^2}$. Though mixed finite elements without a pressure-dependent error contribution are rather classical [36, 35, 31, 19], they were not really investigated by numerical analysts for many years. For the pressure-robust Crouzeix–Raviart finite element method, we prove in this contribution the error estimate

$$\|\mathbf{u}-\mathbf{u}_h\|_{1,h} \leq \widetilde{C}_1 \|h_{\mathcal{T}} D^2 \mathbf{u}\|_{L^2}, \tag{3}$$

where $h_{\mathcal{T}}$ is the meshsize function. We remark that the appearance of the pressure-dependent error contribution $\frac{1}{\nu}\inf_{q_h \in Q_h}\|p-q_h\|_{L^2}$ in (2) shows that classical mixed methods do not fulfill a *fundamental invariance property* of the continuous Stokes equations (1) exactly: changing the right hand side by $\mathbf{f} \to \mathbf{f} + \nabla\phi$ changes the Stokes solution by $(\mathbf{u}, p) \to (\mathbf{u}, p+\phi)$, i.e., gradient fields in the momentum balance are absorbed completely by the pressure gradient. A renewed interest [39, 40, 8, 13, 26, 21, 20, 37] in *pressure-robust mixed methods* for the Stokes equations that allow for *pressure-independent* velocity error estimates was incited by the seminal work of S. Zhang [39], who constructed in 2005 the first pressure-robust 3D Stokes element. The lack of robustness of classical mixed methods, whose velocity error is indeed pressure-dependent, was demonstrated in recent years for several flow problems, where the pressure is much more complicated than the velocity [24, 10, 17, 26].

Recently, the observation was made that the appearance of the pressure-dependent error contribution $\frac{1}{\nu}\inf_{q_h\in Q_h}\|p-q_h\|_{L^2}$ is only due to the fact that certain discrete *velocity test functions* in classical mixed methods are not divergence-free in the sense of H(div) [26, 25]. This problem also influences the approximation by adaptive finite element methods for stationary [27] and for nonstationary problems as noted in [4, 5], and special care needs to be taken in the transfer of solutions between different meshes at different points in time to preserve the discrete divergence-free condition.

Employing lowest-order H(div)-conforming Raviart–Thomas elements in certain novel *velocity reconstructions* [26, 25], it was shown that the non-conforming Crouzeix–Raviart element [14] can be slightly modified in its discretisation of the right-hand side such that its velocity error becomes *pressure-independent*. These velocity reconstructions introduce an $\mathcal{O}(h)$ consistency error, which allows for straightforward proofs of the optimal convergence of the discrete velocity in its energy norm and of the L^2-norm of the pressure [26]. However, the optimal convergence of the discrete velocity in the L^2-norm seemed to be difficult to prove, although it was observed in numerical experiments [6]. This gap will be closed in this contribution, using an Aubin–Nitsche type duality argument and a certain higher regularity of the right hand side.

The rest of the paper is outlined as follows. Section 2 intrudces continuous and discrete setting and all necessary notation. Section 3 recalls and refines known a priori error estimates for the energy norm and the L^2 norm of the pressure and eventually presents the proof for the optimal convergence of the L^2 velocity error. Section 4 concludes the paper with three numerical examples.

2 Continuous and Discrete Setting

This section explains the continuous and the discrete setting for the model problem under consideration and employs the standard Sobolev spaces

$$\begin{aligned} V &:= H^1_0(\Omega)^d := \{v\in H^1(\Omega)^d \,:\, v=0 \text{ along } \partial\Omega\},\\ Q &:= L^2_0(\Omega) := \{q\in L^2(\Omega) \,:\, \int_\omega q\,dx = 0\},\\ H(\mathrm{div},\Omega) &:= \{v\in L^2(\Omega)^d \,:\, \nabla\cdot v\in L^2(\Omega)\}. \end{aligned}$$

2.1 Continuous Setting

The weak solution $(\mathbf{u},p)\in V\times Q$ of the continuous steady incompressible Stokes problem with right-hand side $\mathbf{f}\in L^2(\Omega)^d$ and $g\in Q$ satisfies

$$\begin{aligned} a(\mathbf{u},\mathbf{v})+b(\mathbf{v},p) &= l(\mathbf{v}),\\ b(\mathbf{u},q) &= \chi(q) \qquad \text{for all } (\mathbf{v},q)\in V\times Q \end{aligned} \tag{4}$$

with a, b and l defined by

$$\begin{aligned} a : V \times V \to \mathbb{R}, \qquad & a(\mathbf{u}, \mathbf{v}) := \nu \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} \, dx, \\ b : V \times Q \to \mathbb{R}, \qquad & b(\mathbf{u}, q) := -\int_\Omega q \nabla \cdot \mathbf{u} \, dx, \\ l : V \to \mathbb{R}, \qquad & l(\mathbf{v}) := \int_\Omega \mathbf{f} \cdot \mathbf{v} \, dx, \\ \chi : Q \to \mathbb{R}, \qquad & \chi(q) := \int_\Omega g \, q \, dx \end{aligned}$$

With the subset of functions that satisfy the divergence constraint

$$V_g := \{\mathbf{v} \in V : -\nabla \cdot \mathbf{v} = g\}, \tag{5}$$

the saddle point problem (4) transforms into a problem for the velocity alone, i.e., $\mathbf{u} \in V_g$ such that

$$a(\mathbf{u}, \mathbf{v}) = l(\mathbf{v}) \qquad \text{for all } \mathbf{v} \in V_0. \tag{6}$$

2.2 Notation

In the following, $\mathcal{T}_\mathrm{h}$ denotes a shape-regular family of triangulations of the domain Ω into triangles for $d = 2$ or tetrahedra for $d = 3$, for simplicity, we assume the domain to be polygonal or polyhedral respectively, so that no special treatment of the boundary is needed. For any element $T \in \mathcal{T}_\mathrm{h}$, $\mathrm{mid}(T)$ denotes the barycenter of T. The set of all simplex faces, i.e., edges of triangles for $d = 2$ and faces of tetrahedra for $d = 3$, is denoted by $\mathcal{F}$. The subset $\mathcal{F}(\Omega)$ denotes the set of interior faces, while $\mathcal{F}(\partial\Omega)$ denotes the set of boundary faces along $\partial\Omega$. For any $F \in \mathcal{F}$, $\mathrm{mid}(F)$ denotes the barycenter of F and $\mathbf{n}_F$ abbreviates a face unit normal vector. The orientation of these normal vectors for the interior faces $F \in \mathcal{F}(\Omega)$ are arbitrary, but fixed. The normal vector $\mathbf{n}_F$ for boundary faces $F \in \mathcal{F}(\partial\Omega)$ points outwards of the domain Ω. For every simplex $T \in \mathcal{T}_\mathrm{h}$, $\mathcal{F}(T)$ denotes the set of faces of this simplex and $\mathbf{n}_T$ denotes the outer unit normal of the simplex $T \in \mathcal{T}_\mathrm{h}$. The piecewise constant function $h_\mathcal{T}$ denotes the local mesh size, i.e., $h_\mathcal{T}|_T := \mathrm{diam}(T)$ for all $T \in \mathcal{T}_\mathrm{h}$. Moreover, we let $h = \|h_\mathcal{T}\|_{L^\infty}$. The function space of $P_k(\mathcal{T}_\mathrm{h})$ contains piecewise polynomials of order k with respect to $\mathcal{T}_\mathrm{h}$. For a piecewise Sobolev function $\mathbf{v} \in H^1(\mathcal{T}_\mathrm{h})^d$ and some face $F \in \mathcal{F}(\Omega)$, the notion $[\mathbf{v} \cdot \mathbf{n}_F]$ denotes the jump of the normal flux over F, while $\{\{\mathbf{v} \cdot \mathbf{n}_F\}\}$ denotes the average value of the normal flux over F. The space of Crouzeix–Raviart velocity trial functions is given by

$$\begin{aligned} \mathrm{CR}(\mathcal{T}_\mathrm{h}) := \big\{ \mathbf{v}_h \in P_1(\mathcal{T}_\mathrm{h})^d : \; & [\mathbf{v}_h](\mathrm{mid}(F)) = \mathbf{0} \text{ for all } F \in \mathcal{F}(\Omega) \\ & \& \; \mathbf{v}_h(\mathrm{mid}(F)) = \mathbf{0} \text{ for all } F \in \mathcal{F}(\partial\Omega) \big\}. \end{aligned}$$

The pressure trial function space reads

$$Q(\mathcal{T}_{\mathrm{h}}) := \left\{ q_h \in P_0(\mathcal{T}_{\mathrm{h}}) : \int_\Omega q_h \, dx = 0 \right\}.$$

The space of lowest order Raviart–Thomas finite element functions reads

$$\begin{aligned} \mathrm{RT}(\mathcal{T}_{\mathrm{h}}) := \Big\{ \mathbf{v}_h \in H(\mathrm{div}, \Omega) : \forall T \in \mathcal{T}_{\mathrm{h}} \, \exists \mathbf{a}_T \in \mathbb{R}^d, b_T \in \mathbb{R}, \\ \mathbf{v}_h|_T(\mathbf{x}) = \mathbf{a}_T + b_T \mathbf{x} \Big\}. \end{aligned}$$

Any Raviart–Thomas function is uniquely defined by its constant face normal fluxes $\mathbf{v} \cdot \mathbf{n}_F \in P_0(F)$ for all $F \in \mathcal{F}$ [7].

The discrete setting employs the *broken gradient*

$$\nabla_h : V \oplus \mathrm{CR}(\mathcal{T}_{\mathrm{h}}) \to L^2(\Omega)^{d \times d}$$

and the *broken divergence*

$$\nabla_h \cdot (\cdot) : V \oplus \mathrm{CR}(\mathcal{T}_{\mathrm{h}}) \to L^2(\Omega)$$

in the sense that

$$(\nabla_h \mathbf{v}_h)|_T := \nabla(\mathbf{v}_h|_T), \quad (\nabla_h \cdot \mathbf{v}_h)|_T := \nabla \cdot (\mathbf{v}_h|_T) \quad \text{for all } T \in \mathcal{T}_{\mathrm{h}}\,.$$

The discrete gradient norm for the space $V \oplus \mathrm{CR}(\mathcal{T}_{\mathrm{h}})$ reads

$$\|\mathbf{v}_h\|_{1,h} := \left(\int_\Omega \nabla_h \mathbf{v}_h : \nabla_h \mathbf{v}_h \, dx \right)^{1/2} = \|\nabla_h \mathbf{v}_h\|_{L^2}. \tag{7}$$

2.3 Interpolation operators

The usual Crouzeix–Raviart interpolation operator $\boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{CR}} : V \to \mathrm{CR}(\mathcal{T}_{\mathrm{h}})$ is defined by

$$(\boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{CR}} \mathbf{v})(\mathrm{mid}(F)) = \frac{1}{|F|} \int_F \mathbf{v} \, ds \quad \text{for all } F \in \mathcal{F}\,.$$

The Raviart–Thomas interpolation operator $\boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}} : V \oplus \mathrm{CR}(\mathcal{T}_{\mathrm{h}}) \to \mathrm{RT}(\mathcal{T}_{\mathrm{h}})$ is defined by

$$\mathbf{n}_F \cdot (\boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}} \mathbf{v})(\mathrm{mid}(F)) = \frac{1}{|F|} \int_F \mathbf{v} \cdot \mathbf{n}_F \, ds \quad \text{for all } F \in \mathcal{F}\,.$$

Note that, due to continuity in the face barycenters, this is well-defined also for $\mathbf{v} \in \mathrm{CR}(\mathcal{T}_{\mathrm{h}})$. Moreover, it holds the identity $\boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}} \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{CR}} \mathbf{v} = \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}} \mathbf{v}$ for any $\mathbf{v} \in V$.

For any $\gamma \in Q$ and $\mathbf{v} \in V_\gamma$, it immediately follows -$\nabla \cdot \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}} \mathbf{v} = \pi_0 \, \gamma$ and -$\nabla_h \cdot \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{CR}} \mathbf{v} = \pi_0 \, \gamma$ by Gauss' theorem. Here, π_0 denotes the L^2 projector onto

$P_0(\mathcal{T}_\mathrm{h})$. Furthermore, there are the well-known stability and approximation properties, elementwise on all $T \in \mathcal{T}_\mathrm{h}$,

$$\|\nabla_h(\boldsymbol{\pi}_\mathrm{h}^{\mathrm{CR}}\,\mathbf{v})\|_{L^2(T)} \leq \|\nabla\mathbf{v}\|_{L^2(T)} \quad \text{for all } v \in H^1(T), \tag{8}$$

$$\|\nabla_h(\mathbf{v} - \boldsymbol{\pi}_\mathrm{h}^{\mathrm{CR}}\,\mathbf{v})\|_{L^2(T)} \leq C_I\|h_T D^2\mathbf{v}\|_{L^2(T)} \quad \text{for all } v \in H^2(T)^d, \tag{9}$$

$$\|\mathbf{v} - \boldsymbol{\pi}_\mathrm{h}^{\mathrm{RT}}\,\mathbf{v}\|_{L^2(T)} \leq C_F\|h_T\nabla_h\mathbf{v}\|_{L^2(T)} \quad \text{for all } v \in H^1(T), \tag{10}$$

where the generic constants C_I and C_F depend only on the shape of the simplices in the triangulation $\mathcal{T}_\mathrm{h}$, but not on their size [7, 1, 9].

2.4 The finite element scheme with divergence-conforming reconstruction

The discrete weak formulation of the model problem employs

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}) &:= \nu \int_\Omega \nabla_h\mathbf{u}_h : \nabla_h\mathbf{v}_h\,dx, \\ b_h(\mathbf{u}_h, q_h) &:= -\int_\Omega q_h\nabla_h \cdot \mathbf{u}_h\,dx, \\ l_h(\mathbf{v}_h) &:= \int_\Omega f \cdot \mathbf{v}_h\,dx. \end{aligned}$$

With this, the discrete Stokes problem seeks $(\mathbf{u}_h, p_h) \in \mathrm{CR}(\mathcal{T}_\mathrm{h}) \times Q(\mathcal{T}_\mathrm{h})$ such that

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) &= l_h(\boldsymbol{\pi}_\mathrm{h}^{\mathrm{RT}}\,\mathbf{v}_h), \\ b_h(\mathbf{u}_h, q_h) &= \chi(q_h) \qquad \text{for all } (\mathbf{v}_h, q_h) \in \mathrm{CR}(\mathcal{T}_\mathrm{h}) \times Q(\mathcal{T}_\mathrm{h}). \end{aligned} \tag{11}$$

In comparison to the classical Crouzeix–Raviart nonconforming finite element method [14], the introduction of $\boldsymbol{\pi}_\mathrm{h}^{\mathrm{RT}}$ in the right-hand side constitutes a variational crime that maps discretely divergence-free test functions to divergence-free functions in $H(\mathrm{div}, \Omega)$ with certain benefits as discussed below.

Like the continuous incompressible Stokes and Navier-Stokes equations, also the discretization (11) can be formulated as an problem [34, 18] within the space of discretely constrained functions

$$V_{g,h} := \{\mathbf{v}_h \in \mathrm{CR}(\mathcal{T}_\mathrm{h}) : -\nabla_h \cdot \mathbf{v}_h = \pi_0\, g\}. \tag{12}$$

Then, $\mathbf{u}_h \in V_{g,h}$ is uniquely defined by

$$a_h(\mathbf{u}_h, \mathbf{v}_h) = l_h(\boldsymbol{\pi}_\mathrm{h}^{\mathrm{RT}}\,\mathbf{v}_h) \quad \text{for all } \mathbf{v}_h \in V_{0,h}. \tag{13}$$

Remark 1. *The pair* $\mathrm{CR}(\mathcal{T}_\mathrm{h}) \times Q(\mathcal{T}_\mathrm{h})$ *satisfies the discrete inf-sup condition*

$$0 < \beta_h := \inf_{q_h \in Q(\mathcal{T}_\mathrm{h})\setminus\{0\}} \sup_{\mathbf{v}_h \in \mathrm{CR}(\mathcal{T}_\mathrm{h})\setminus\{0\}} \frac{\int_\Omega q_h\nabla_h \cdot \mathbf{v}_h\,dx}{\|\mathbf{v}_h\|_{1,h}\|q_h\|_{L^2}}. \tag{14}$$

The inf-sup constant β_h *for the Crouzeix–Raviart element is independent of the mesh [14].*

3 A Priori Error Estimates

This section presents a priori finite element error estimates for the modified Crouzeix–Raviart discretization of the incompressible Stokes equations (11). The analysis is based on the estimates of the consistency error in [1], which apply the Raviart–Thomas interpolation to the best advantage and avoid the use of a trace inequality. However, some slight changes due to the divergence-conforming reconstruction deliver fundamentally improved results, since the scheme (11) allows for an error estimate of the discrete velocity that is independent of the pressure. The proof involves the interpolation error estimates from above and the elementwise Poincaré constant

$$\begin{aligned} C_P :&= \sup\left\{\|\mathbf{v}-\pi_0\,\mathbf{v}\|_{L^2}/\|h_{\mathcal{T}}\nabla_h\mathbf{v}\|_{L^2}\ :\ \mathbf{v}\in V\oplus \mathrm{CR}(\mathcal{T}_\mathrm{h})\right\}\\ &\leq \sup_{h>0}\max_{T\in\mathcal{T}_\mathrm{h}}\sup\left\{\|\mathbf{v}-\pi_0\,\mathbf{v}\|_{L^2(T)}/\|h_T\nabla_h\mathbf{v}\|_{L^2(T)}\ :\ \mathbf{v}\in H^1(T)\right\}. \end{aligned}$$

Lemma 1. *For all* $\mathbf{v}_h\in\mathrm{CR}(\mathcal{T}_\mathrm{h})$*, it holds*

$$\begin{aligned} &i) \qquad b_h(\mathbf{v}_h,q)=b(\boldsymbol{\pi}_\mathrm{h}^{\mathrm{RT}}\,\mathbf{v}_h,q)\quad \textit{for all } q\in L^2(\Omega),\\ &ii) \qquad b(\boldsymbol{\pi}_\mathrm{h}^{\mathrm{RT}}\,\mathbf{v}_h,q)=\int_\Omega \nabla q\cdot(\boldsymbol{\pi}_\mathrm{h}^{\mathrm{RT}}\,\mathbf{v}_h)\,dx\quad \textit{for all } q\in H^1(\Omega). \end{aligned}$$

Proof. The divergence theorem and the definition of $\boldsymbol{\pi}_\mathrm{h}^{\mathrm{RT}}$ shows

$$\int_T \nabla_h\cdot(\mathbf{v}_h-\boldsymbol{\pi}_\mathrm{h}^{\mathrm{RT}}\,\mathbf{v}_h)\,dx=\sum_{F\in\mathcal{F}(T)}\int_F(\mathbf{v}_h-\boldsymbol{\pi}_\mathrm{h}^{\mathrm{RT}}\,\mathbf{v}_h)\cdot\mathbf{n}_T\,ds=0.$$

Since $\nabla_h\cdot(\mathbf{v}_h-\boldsymbol{\pi}_\mathrm{h}^{\mathrm{RT}}\,\mathbf{v}_h)$ is elementwise constant, the above implies the relation $\nabla_h\cdot\mathbf{v}_h=\nabla\cdot(\boldsymbol{\pi}_\mathrm{h}^{\mathrm{RT}}\,\mathbf{v}_h)$ which proves the first identity. An integration by parts, which is allowed since $\boldsymbol{\pi}_\mathrm{h}^{\mathrm{RT}}\,\mathbf{v}_h\in H(\mathrm{div},\Omega)$, shows the second identity and concludes the proof. □

Lemma 2. *For all* $\mathbf{v}\in V\cap H^2(\Omega)^d,\mathbf{w}\in V\oplus\mathrm{CR}(\mathcal{T}_\mathrm{h})$*, it holds*

$$\left|\int_\Omega \nabla_h\mathbf{v}:\nabla_h\mathbf{w}+\Delta\mathbf{v}\cdot\boldsymbol{\pi}_\mathrm{h}^{\mathrm{RT}}\,\mathbf{w}\,dx\right|\leq(2C_F+C_P)\|h_{\mathcal{T}}D^2\mathbf{v}\|_{L^2}\|\mathbf{w}\|_{1,h}.$$

Proof. Let $\mathbf{\Pi}_\mathrm{h}^{\mathrm{RT}}$ denote the rowwise Raviart–Thomas interpolator and Π_0 the L^2 projection onto $P_0(\mathcal{T}_\mathrm{h})^d$. Since the normal fluxes $(\mathbf{\Pi}_\mathrm{h}^{\mathrm{RT}}\,\nabla\mathbf{v})\mathbf{n}_F$ are continuous for all $F\in\mathcal{F}$ and constant on the boundary faces $F\in\mathcal{F}(\partial\Omega)$ and $\mathbf{w}$ is zero at least at the centers of any $F\in\mathcal{F}(\partial\Omega)$, it holds

$$\sum_{T\in\mathcal{T}_\mathrm{h}}\int_{\partial T}\left(\mathbf{\Pi}_\mathrm{h}^{\mathrm{RT}}\,\nabla\mathbf{v}\,\mathbf{n}\right)\cdot\mathbf{w}\,ds=0.$$

An elementwise integration by parts and the commutation property of the divergence with the Raviart–Thomas interpolation $\nabla \cdot (\boldsymbol{\Pi}_{\mathrm{h}}^{\mathrm{RT}} \nabla \mathbf{v}) = \Pi_0(\Delta \mathbf{v})$ show

$$\int_\Omega \boldsymbol{\Pi}_{\mathrm{h}}^{\mathrm{RT}} \nabla \mathbf{v} \cdot \nabla_h \mathbf{w} + \Pi_0(\Delta \mathbf{v}) \cdot \mathbf{w} \, dx = 0.$$

This and elementary calculations reveal

$$\begin{aligned}\int_\Omega & \nabla \mathbf{v} : \nabla_h \mathbf{w} + \Delta \mathbf{v} \cdot \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}} \mathbf{w} \, dx \\ &= \int_\Omega \left(\nabla \mathbf{v} - \boldsymbol{\Pi}_{\mathrm{h}}^{\mathrm{RT}} \nabla \mathbf{v} \right) : \nabla_h \mathbf{w} \, dx + \int_\Omega (\Delta \mathbf{v} - \Pi_0(\Delta \mathbf{v})) \cdot \mathbf{w} \, dx \\ &\quad + \int_\Omega \Delta \mathbf{v} \cdot \left(\boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}} \mathbf{w} - \mathbf{w} \right) dx.\end{aligned}$$

For the first integral, a Cauchy-Schwarz inequality and the rowwise version of (10) yield

$$\int_\Omega \left(\nabla \mathbf{v} - \boldsymbol{\Pi}_{\mathrm{h}}^{\mathrm{RT}} \nabla \mathbf{v} \right) : \nabla_h \mathbf{w} \, dx \le C_F \| h_{\mathcal{T}} D^2 \mathbf{v} \|_{L^2} \| \mathbf{w} \|_{1,h}.$$

For the second integral, the L^2 orthogonality of $\Delta \mathbf{v} - \Pi_0(\Delta \mathbf{v})$ and $\mathbf{w} - \Pi_0 \mathbf{w}$ w.r.t. $P_0(\mathcal{T}_{\mathrm{h}})^d$ and elementwise Poincaré inequalities show

$$\begin{aligned}\int_\Omega (\Delta \mathbf{v} - \Pi_0(\Delta \mathbf{v})) \cdot \mathbf{w} \, dx &= \int_\Omega (\Delta \mathbf{v} - \Pi_0(\Delta \mathbf{v})) \cdot (\mathbf{w} - \Pi_0 \mathbf{w}) \, dx \\ &\le \| \Delta \mathbf{v} \|_{L^2} \| \mathbf{w} - \Pi_0 \mathbf{w} \|_{L^2} \le C_P \| h_{\mathcal{T}} \Delta \mathbf{v} \|_{L^2} \| \mathbf{w} \|_{1,h}.\end{aligned}$$

Another Cauchy-Schwarz inequality and (10) bound the third integral by

$$\int_\Omega \Delta \mathbf{v} \cdot \left(\boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}} \mathbf{w} - \mathbf{w} \right) dx \le C_F \| h_{\mathcal{T}} \Delta \mathbf{v} \|_{L^2} \| \mathbf{w} \|_{1,h}. \tag{15}$$

The combination of the last three estimates concludes the proof. □

The estimate of the consistency error is a corollary to Lemma 2.

Lemma 3 (Pressure-independent consistency error estimate)**.** *Given the solution* $(\mathbf{u}, p) \in H^2(\Omega)^d \times H^1(\Omega)$ *of the continuous Stokes equations* (4)*, it holds*

$$\sup_{\substack{\mathbf{0} \neq \mathbf{w}_h \in V \oplus \mathrm{CR}(\mathcal{T}_{\mathrm{h}}), \\ \pi_0(\nabla_h \cdot \mathbf{w}_h) = 0}} \frac{\left| a_h(\mathbf{u}, \mathbf{w}_h) - l_h(\boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}} \mathbf{w}_h) \right|}{\| \mathbf{w}_h \|_{1,h}} \le \nu (2 C_F + C_P) \| h_{\mathcal{T}} D^2 \mathbf{u} \|_{L^2}.$$

Proof. For all $\mathbf{0} \neq \mathbf{w}_h \in V \oplus \mathrm{CR}(\mathcal{T}_\mathrm{h})$ with $\pi_0(\nabla_h \cdot \mathbf{w}_h) = 0$ it holds $\int_\Omega \nabla p \cdot \boldsymbol{\pi}_\mathrm{h}^{\mathrm{RT}} \mathbf{w}_h \, dx = 0$. This and (4) and show

$$\begin{aligned}
\frac{1}{\nu}\left| a_h(\mathbf{u}, \mathbf{w}_h) - l_h(\boldsymbol{\pi}_\mathrm{h}^{\mathrm{RT}} \mathbf{w}_h) \right| &= \frac{1}{\nu} \left| \int_\Omega \nu \nabla_h \mathbf{u} : \nabla_h \mathbf{w}_h - \mathbf{f} \cdot \boldsymbol{\pi}_\mathrm{h}^{\mathrm{RT}} \mathbf{w}_h \, dx \right| \\
&= \frac{1}{\nu} \left| \int_\Omega \nu \nabla_h \mathbf{u} : \nabla_h \mathbf{w}_h + (\nu \Delta \mathbf{u} - \nabla p) \cdot \boldsymbol{\pi}_\mathrm{h}^{\mathrm{RT}} \mathbf{w}_h \, dx \right| \qquad (16) \\
&= \left| \int_\Omega \nabla_h \mathbf{u} : \nabla_h \mathbf{w}_h + \Delta \mathbf{u} \cdot \boldsymbol{\pi}_\mathrm{h}^{\mathrm{RT}} \mathbf{w}_h \, dx \right| .
\end{aligned}$$

Lemma 2 concludes the proof. □

Remark 2. *Note that Lemma 3 does not hold in a pressure-independent way for the standard Crouzeix–Raviart finite element method, since in (16)* ∇p *and* $\mathbf{w}_h$ *for* $w_h \in V_0 + V_{0,h}$ *are not orthogonal in the* L^2 *scalar product.*

The estimate of the pressure-independent consistency error leads to the following optimal a priori estimates.

Theorem 1. *For the solution* $(\mathbf{u}, p) \in H^2(\Omega)^d \times H^1(\Omega)$ *of the continuous Stokes equations (4) and the discrete solution* $(\mathbf{u}_h, p_h)$ *of (11), it holds*

i) $\|\mathbf{u} - \mathbf{u}_h\|_{1,h} \leq (2C_I + 2C_F + C_P) \|h_\mathcal{T} D^2 \mathbf{u}\|_{L^2}$,

ii) $\|\pi_0 \, p - p_h\|_{L^2} \leq (2C_I + 4C_F + 2C_P) \beta_h^{-1} \nu \|h_\mathcal{T} D^2 \mathbf{u}\|_{L^2}$,

iii) $\|p - p_h\|_{L^2}^2 \leq C_P^2 \|h_\mathcal{T} \nabla p\|_{L^2}^2 + (2C_I + 4C_F + 2C_P)^2 \beta_h^{-2} \nu^2 \|h_\mathcal{T} D^2 \mathbf{u}\|_{L^2}^2$.

Proof of i). Formulation (13) and $\mathbf{w}_h := \mathbf{u}_h - \mathbf{v}_h \in V_{0,h}$ for an arbitrary $\mathbf{v}_h \in V_{g,h}$ yield

$$\begin{aligned}
\nu \|\mathbf{w}_h\|_{1,h}^2 &= a_h(\mathbf{w}_h, \mathbf{w}_h) \\
&= a_h(\mathbf{u}_h - \mathbf{v}_h, \mathbf{w}_h) \\
&= a_h(\mathbf{u} - \mathbf{v}_h, \mathbf{w}_h) + a_h(\mathbf{u}_h, \mathbf{w}_h) - a_h(\mathbf{u}, \mathbf{w}_h) \\
&= a_h(\mathbf{u} - \mathbf{v}_h, \mathbf{w}_h) + l_h(\boldsymbol{\pi}_\mathrm{h}^{\mathrm{RT}} \mathbf{w}_h) - a_h(\mathbf{u}, \mathbf{w}_h) \\
&\leq \nu \|\mathbf{u} - \mathbf{v}_h\|_{1,h} \|\mathbf{w}_h\|_{1,h} + \left| a_h(\mathbf{u}, \mathbf{w}_h) - l_h(\boldsymbol{\pi}_\mathrm{h}^{\mathrm{RT}} \mathbf{w}_h) \right| .
\end{aligned}$$

The triangle inequality for $\|\mathbf{u} - \mathbf{u}_h\|_{1,h} = \|(\mathbf{u} - \mathbf{v}_h) - \mathbf{w}_h\|_{1,h}$ produces Strang's second lemma in the form

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,h} \leq 2 \inf_{\mathbf{v}_h \in V_{g,h}} \|\mathbf{u} - \mathbf{v}_h\|_{1,h} + \frac{1}{\nu} \sup_{\mathbf{w}_h \in V_{0,h}} \frac{\left| a_h(\mathbf{u}, \mathbf{w}_h) - l_h(\boldsymbol{\pi}_\mathrm{h}^{\mathrm{RT}} \mathbf{w}_h) \right|}{\|\mathbf{w}_h\|_{1,h}} .$$

Since $\boldsymbol{\pi}_\mathrm{h}^{\mathrm{CR}} \mathbf{u} \in V_{g,h}$, the first error term can be bounded with (9) by

$$\inf_{\mathbf{v}_h \in V_{g,h}} \|\mathbf{u} - \mathbf{v}_h\|_{1,h} \leq \|\mathbf{u} - \boldsymbol{\pi}_\mathrm{h}^{\mathrm{CR}} \mathbf{u}\|_{1,h} \leq C_I \|h_\mathcal{T} D^2 \mathbf{u}\|_{L^2} .$$

The second error term is estimated with Lemma 3. □

Proof of ii). Due to the discrete inf-sup stability (14), we can estimate the second term by

$$\|\pi_0 p - p_h\|_{L^2} \leq \frac{1}{\beta_h} \sup_{\mathbf{v}_h \in \mathrm{CR}(\mathcal{T}_h)} \frac{b_h(\mathbf{v}_h, \pi_0 p - p_h)}{\|\mathbf{v}_h\|_{1,h}}.$$

Since $\nabla_h \cdot v_h$ is constant and $\pi_0 p - p$ is orthogonal on constants, the term in the numerator of this expression equals

$$b_h(\mathbf{v}_h, \pi_0 p - p_h) = b_h(\mathbf{v}_h, p - p_h).$$

Elementary calculations, the application of Lemma 1 i) and ii), and $\mathbf{f} = -\nu\Delta\mathbf{u} + \nabla p$ show

$$\begin{aligned}
b_h(\mathbf{v}_h, p - p_h) &= b_h(\mathbf{v}_h, p) + a_h(\mathbf{u}_h, \mathbf{v}_h) - l_h(\boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}}\, \mathbf{v}_h)\\
&= b(\boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}}\, \mathbf{v}_h, p) + a_h(\mathbf{u}_h, \mathbf{v}_h) - \int_\Omega \mathbf{f} \cdot \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}}\, \mathbf{v}_h \, dx\\
&= \int_\Omega \nabla p \cdot \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}}\, \mathbf{v}_h \, dx + a_h(\mathbf{u}_h, \mathbf{v}_h)\\
&\quad + \int_\Omega (\nu\Delta\mathbf{u} - \nabla p) \cdot \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}}\, \mathbf{v}_h \, dx\\
&= a_h(\mathbf{u}_h - \mathbf{u}, \mathbf{v}_h) + \int_\Omega \nu \left\{ \nabla_h \mathbf{u} : \nabla_h \mathbf{v}_h + \Delta\mathbf{u} \cdot \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}}\, \mathbf{v}_h \right\} dx.
\end{aligned}$$

The first term is estimated by i) with

$$a_h(\mathbf{u}_h - \mathbf{u}, \mathbf{v}_h) \leq (2C_I + 2C_F + C_P)\|h_\mathcal{T} D^2\mathbf{u}\|_{L^2}\|\mathbf{v}_h\|_{1,h}$$

Lemma 2 yields the concluding argument

$$\int_\Omega \nu \left\{ \nabla_h \mathbf{u} : \nabla_h \mathbf{v}_h + \Delta\mathbf{u} \cdot \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}}\, \mathbf{v}_h \right\} dx \leq (2C_F + C_P)\nu\|h_\mathcal{T} D^2\mathbf{u}\|_{L^2}\|\mathbf{v}_h\|_{1,h}.$$

□

Proof of iii). For the pressure estimate, the Pythagoras theorem shows

$$\|p - p_h\|_{L^2}^2 = \|p - \pi_0 p\|_{L^2}^2 + \|\pi_0 p - p_h\|_{L^2}^2.$$

Elementwise Poincaré inequalities with constant C_P bound the first term by

$$\|p - \pi_0 p\|_{L^2} \leq C_P\|h_\mathcal{T} \nabla p\|_{L^2}.$$

The combination with ii) concludes the proof. □

Remark 3. *The constants C_I, C_F and C_P in Theorem 1 are independent of the inf-sup-constant β_h. Estimates for mixed methods that depend on β_h are dramatically pessimistic for channel domains with large aspect ratio [38, 15, 16, 11].*

Remark 4. *Guaranteed upper bounds for all involved constants C_I, C_F and C_P in Theorem 1 are known. The Fortin interpolation constant is bounded by $C_F \leq 0.6215$ for rectangular triangles, for details see the maximum angle estimate from [9, Theorem 5.1]. In 2D, [23] shows $C_P = 1/j_{1,1}$ where $j_{1,1} = 3.8317\ldots$ is the first positive root of the first Bessel function J_1. In 3D, the constant $C_P = 1/\pi$ is valid for every convex domain [29, 3]. Moreover, the constant C_I is in fact also bounded by C_P, since the Crouzeix-Raviart interpolation operator $\boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{CR}}$ has the property $\int_T \nabla_h(\mathbf{v}_h - \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{CR}}\mathbf{v}_h)\,dx = 0$ for all $T \in \mathcal{T}_{\mathrm{h}}$ and so allows for a Poincaré type inequality in (9).*

Remark 5. *The modified Crouzeix–Raviart method (11) or equivalently (13) is usually much more accurate than the standard Crouzeix–Raviart method, see [26]. However, the standard Crouzeix–Raviart method performs better in those (very special) situations, whenever the continuous pressure p vanishes. In order to get an estimate, how the modified Crouzeix–Raviart method behaves in this worst case, let $\widehat{\mathbf{u}}_h$ denote the solution of the standard Crouzeix–Raviart finite element method and let $\mathbf{u}_h$ denote the solution of the modified Crouzeix–Raviart finite element solution from (11). Then, by (10) it holds*

$$\begin{aligned}\nu\|\widehat{\mathbf{u}}_h - \mathbf{u}_h\|_{1,h}^2 &= a_h(\widehat{\mathbf{u}}_h - \mathbf{u}_h, \widehat{\mathbf{u}}_h - \mathbf{u}_h)\\ &= l_h(\widehat{\mathbf{u}}_h - \mathbf{u}_h) - l_h(\boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}}(\widehat{\mathbf{u}}_h - \mathbf{u}_h))\\ &\leq C_F\|h_{\mathcal{T}}\mathbf{f}\|_{L^2}\|\widehat{\mathbf{u}}_h - \mathbf{u}_h\|_{1,h}.\end{aligned}$$

Hence, the difference between the two solutions is at most

$$\|\widehat{\mathbf{u}}_h - \mathbf{u}_h\|_{1,h} \leq \frac{C_F}{\nu}\|h_{\mathcal{T}}\mathbf{f}\|_{L^2}.$$

This estimate reflects the considerations above in the following way: the worst case for the classical Crouzeix–Raviart finite element method is for $\mathbf{f} = \nabla p$ which means $u = 0$. Here, the modified Crouzeix–Raviart method delivers the exact velocity solution $u_h = 0$, while $\widehat{\mathbf{u}}_h$ deteriorates in general with $O(1/\nu)$ for $\nu \to 0$. On the other hand, the worst case for the modified Crouzeix–Raviart finite element method happens for $\mathbf{f} = -\nu\Delta u$ which means $p = 0$. Then, it holds

$$\|\widehat{\mathbf{u}}_h - \mathbf{u}_h\|_{1,h} \leq C_F\|h_{\mathcal{T}}\Delta u\|_{L^2}$$

which is independent of $1/\nu$.

Corollary 1 (Invariance property)**.** *The modified Crouzeix–Raviart finite element method satisfies a continuous invariance property in the sense that for all $\mathbf{f} \in L^2(\Omega)^d$ and all $\phi \in H^1(\Omega)/\mathbb{R}$ it holds*

$$\begin{aligned}&\mathbf{f} \to \mathbf{f} + \nabla\phi \Longrightarrow (\mathbf{u}, p) \to (\mathbf{u}, p + \phi),\\ &\mathbf{f} \to \mathbf{f} + \nabla\phi \Longrightarrow (\mathbf{u}_h, p_h) \to (\mathbf{u}_h, p_h + \pi_0\,\phi).\end{aligned}$$

Proof. This is a direct consequence of Theorem 1. Theorem 1 i) shows, that the discrete solution $\mathbf{u}_h$ for the right-hand side $\nabla\phi$ and for $g = 0$ is zero (because the exact solution is zero) and Theorem 1 ii) shows $\|\pi_0\,\phi - p_h\|_{L^2} = 0$. □

Lemma 4. *Given a right-hand side $\mathbf{r} \in L^2(\Omega)^d$, let $\mathbf{u}_r \in V_0$ denote the solution of*

$$a(\mathbf{u}_r, \mathbf{v}) = (\mathbf{r}, \mathbf{v}) \quad \textit{for all } \mathbf{v} \in V_0,$$

and let $\mathbf{u}_{r,h} \in V_{0,h}$ denote the solution of

$$a_h(\mathbf{u}_{r,h}, \mathbf{v}_h) = (\mathbf{r}, \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}}\,\mathbf{v}_h) \quad \textit{for all } \mathbf{v}_h \in V_{0,h}$$

Then, for the solutions $\mathbf{u}$ from (4) and $\mathbf{u}_h$ from (11), it holds

$$\begin{aligned}\|\mathbf{u} - \mathbf{u}_h\|_{L^2} \leq \sup_{\mathbf{r}\in L^2(\Omega)^d, \|\mathbf{r}\|_{L^2}=1} \Big\{ &\nu\|\mathbf{u} - \mathbf{u}_h\|_{1,h}\|\mathbf{u}_r - \mathbf{u}_{r,h}\|_{1,h} \\ &+ \left|a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{u}_r) - \left(\mathbf{r}, \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}}\,(\mathbf{u} - \mathbf{u}_h)\right)\right| \\ &+ \left|a_h(\mathbf{u}, \mathbf{u}_r - \mathbf{u}_{r,h}) - \left(\mathbf{f}, \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}}\,(\mathbf{u}_r - \mathbf{u}_{r,h})\right)\right| \\ &+ \left|\left(\mathbf{r}, (\mathbf{u} - \mathbf{u}_h) - \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}}\,(\mathbf{u} - \mathbf{u}_h)\right)\right| \\ &+ \left|\left(\mathbf{f}, \mathbf{u}_r - \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}}\,\mathbf{u}_r\right)\right| \Big\}.\end{aligned}$$

Proof. The proof is based on the duality argument

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^2} = \sup_{\mathbf{r}\in L^2(\Omega)^d\setminus\{0\}} (\mathbf{r}, \mathbf{u} - \mathbf{u}_h)\,/\|\mathbf{r}\|_{L^2}.$$

Elementary algebra yields

$$\begin{aligned}(\mathbf{r}, \mathbf{u} - \mathbf{u}_h) &= a_h(\mathbf{u}_h, \mathbf{u}_{r,h}) - a_h(\mathbf{u}, \mathbf{u}_r) + (\mathbf{r}, \mathbf{u} - \mathbf{u}_h) + \left(\mathbf{f}, \mathbf{u}_r - \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}}\,\mathbf{u}_{r,h}\right) \\ &= -a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{u}_{r,h}) - a_h(\mathbf{u}, \mathbf{u}_r - \mathbf{u}_{r,h}) \\ &\qquad + (\mathbf{r}, \mathbf{u} - \mathbf{u}_h) + \left(\mathbf{f}, \mathbf{u}_r - \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}}\,\mathbf{u}_{r,h}\right) \\ &= a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{u}_r - \mathbf{u}_{r,h}) \\ &\qquad - a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{u}_r) + \left(\mathbf{r}, \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}}\,(\mathbf{u} - \mathbf{u}_h)\right) \\ &\qquad - a_h(\mathbf{u}, \mathbf{u}_r - \mathbf{u}_{r,h}) + \left(\mathbf{f}, \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}}\,(\mathbf{u}_r - \mathbf{u}_{r,h})\right) \\ &\qquad + \left(\mathbf{r}, (\mathbf{u} - \mathbf{u}_h) - \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}}\,(\mathbf{u} - \mathbf{u}_h)\right) \\ &\qquad + \left(\mathbf{f}, \mathbf{u}_r - \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}}\,\mathbf{u}_r\right).\end{aligned}$$

Triangle and Cauchy-Schwarz inequalities conclude the proof. □

Theorem 2. *Assuming that Ω is convex, simply connected, and that for the solution of the continuous Stokes equations (4) holds $(\mathbf{u}, p) \in H^2(\Omega)^d \times H^1(\Omega)$, $\Delta u \in H^2(\Omega)$, we obtain for the discrete solution $(\mathbf{u}_h, p_h)$ of the* *scheme* (11) *the following* L^2 *error estimate of optimal order for the discrete velocity*

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^2} \leq C\, h^2 \left(|\mathbf{u}|_{H^2} + \|\Delta \mathbf{u}\|_{H^2}\right), \tag{17}$$

with a constant C *depending on the shape regularity of the triangulation.*

Proof. Since we assume that the domain Ω is convex. we obtain by classical regularity results for the incompressible Stokes equations that $\mathbf{u}_r \in H^2(\Omega)^d$ for all $\mathbf{r} \in L^2(\Omega)^d$ and that the following a-priori estimates

$$\begin{aligned} \nu\, |\mathbf{u}_r|_{H^2} &\leq C\|\mathbf{r}\|_{L^2}, \\ \nu\|\nabla \mathbf{u}_r\|_{L^2} &\leq C\|\mathbf{r}\|_{L^2} \end{aligned} \tag{18}$$

hold. Then, we apply the abstract error estimate from Lemma 4 , and have to estimate the corresponding five different terms. First, we obtain

$$\begin{aligned} \nu\|\mathbf{u} - \mathbf{u}_h\|_{1,h}\|\mathbf{u}_r - \mathbf{u}_{r,h}\|_{1,h} &\leq \nu\,(Ch|\mathbf{u}|_{H^2}) \cdot (Ch|\mathbf{u}_r|_{H^2}) \\ &\leq Ch^2|\mathbf{u}|_{H^2}\|\mathbf{r}\|_{L^2} \end{aligned}$$

using Theorem 1 and (18). The second term

$$\begin{aligned} \left|a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{u}_r) - \left(\mathbf{r}, \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}}\,(\mathbf{u} - \mathbf{u}_h)\right)\right| &\leq \nu Ch|\mathbf{u}_r|_{H^2}\|\mathbf{u} - \mathbf{u}_h\|_{1,h} \\ &\leq Ch^2|\mathbf{u}|_{H^2}\|\mathbf{r}\|_{L^2} \end{aligned}$$

can be estimated by the consistency error for the adjoint problem from Lemma 3, Theorem 1 and (18). By Lemma 3, we obtain analogously

$$\begin{aligned} \left|a_h(\mathbf{u}, \mathbf{u}_r - \mathbf{u}_{r,h}) - \left(\mathbf{f}, \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}}\,(\mathbf{u}_r - \mathbf{u}_{r,h})\right)\right| &\leq \nu Ch|\mathbf{u}|_{H^2}\|\mathbf{u}_r - \mathbf{u}_{r,h}\|_{1,h} \\ &\leq \nu Ch^2|\mathbf{u}|_{H^2}|\mathbf{u}_r|_{H^2} \\ &\leq Ch^2|\mathbf{u}|_{H^2}\|\mathbf{r}\|_{L^2}, \end{aligned}$$

using the estimate of the consistency error for the original problem. For the fourth term, we obtain by Theorem 1 and (10)

$$\left|\left(\mathbf{r}, (\mathbf{u} - \mathbf{u}_h) - \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}}\,(\mathbf{u} - \mathbf{u}_h)\right)\right| \leq Ch\|\mathbf{u} - \mathbf{u}_h\|_{1,h}\|\mathbf{r}\|_{L^2} \leq Ch^2|\mathbf{u}|_{H^2}\|\mathbf{r}\|_{L^2}.$$

Bounding the fifth term goes beyond standard arguments. We introduce the L^2-interpolation Π_0 into elementwise constants and obtain

$$\begin{aligned} \left|\left(\mathbf{f}, \mathbf{u}_r - \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}}\,\mathbf{u}_r\right)\right| &\leq \left|\nu\left(\Delta \mathbf{u} - \Pi_0\, \Delta \mathbf{u}, \mathbf{u}_r - \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}}\,\mathbf{u}_r\right)\right| \\ &\quad + \left|\nu\left(\Pi_0\, \Delta \mathbf{u}, \mathbf{u}_r - \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}}\,\mathbf{u}_r\right)\right| \end{aligned} \tag{19}$$

since $(\nabla p, \mathbf{u}_r - \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}}\,\mathbf{u}_r) = 0$. For the first summand, standard error estimates for the L^2-projection and $\boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}}$ give the desired bound

$$\left|\nu\left(\Delta \mathbf{u} - \Pi_0\, \Delta \mathbf{u}, \mathbf{u}_r - \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}}\,\mathbf{u}_r\right)\right| \leq ch^2\|\nabla \Delta \mathbf{u}\|_{L^2}\|\mathbf{r}\|_{L^2}$$

To estimate the second term on the right of (19), we notice that $\nabla \cdot \mathbf{u}_r = 0$ and hence, utilizing the exactness of the de Rahm complex on a simply connected domain, there is a function σ_r such that $\nabla \times \sigma_r = \mathbf{u}_r$. Further, since $\mathbf{u}_r \in H^1(\Omega)^d$ it holds $\sigma_r \in H^2(\Omega)$ if $d = 2$ and $\sigma_r \in H^2(\Omega)^3$ if $d = 3$. In both cases, it holds $\|\sigma_r\|_{H^2} \leq c\|\mathbf{r}\|_{L^2}$, see, e.g., [22, Lemma 2.6]. Further, there is a finite element space $\widetilde{V_h}$ and a corresponding interpolation operator $I_h \colon H^2 \to \widetilde{V_h}$ such that $\mathbf{u}_r - \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}}\,\mathbf{u}_r = \nabla \times (\sigma_r - I_h\sigma_r)$. Since the space $\widetilde{V_h}$ takes a different form for different dimensions $d = 2, 3$, we proceed by cases:

2d In this case $\widetilde{V_h}$ consists of piecewise linear polynomials and I_h is the standard nodal interpolation, see, e.g., [2, Table 5.1] or [30] for the original definition of the element.

Then, we can estimate the remaining term as follows utilizing Green's formula

$$
\begin{aligned}
\left|\left(\Pi_0\,\Delta\mathbf{u}, \mathbf{u}_r - \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}}\,\mathbf{u}_r\right)\right| &= \Big|\sum_{T\in\mathcal{T}_{\mathrm{h}}} \left(\Pi_0\,\Delta\mathbf{u}, \mathbf{u}_r - \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}}\,\mathbf{u}_r\right)_T\Big| \\
&= \Big|\sum_{T\in\mathcal{T}_{\mathrm{h}}} (\Pi_0\,\Delta\mathbf{u}, \nabla \times (\sigma_r - I_h\sigma_r)_T\Big| \\
&\leq \Big|\sum_{T\in\mathcal{T}_{\mathrm{h}}} (\nabla \times \Pi_0\,\Delta\mathbf{u}, \sigma_r - I_h\sigma_r)_T\Big| \\
&\quad + \Big|\sum_{T\in\mathcal{T}_{\mathrm{h}}} (\Pi_0\,\Delta\mathbf{u}, \mathbf{n} \times (\sigma_r - I_h\sigma_r))_{\partial T}\Big|
\end{aligned}
\tag{20}
$$

Given that $\Pi_0\,\Delta\mathbf{u}$ is elementwise constant, the volume term vanishes. For the boundary term, we calculate

$$
|(\Pi_0\,\Delta\mathbf{u}, \mathbf{n} \times (\sigma_r - I_h\sigma_r)_{\partial T}| \leq \|\Pi_0\,\Delta\mathbf{u}\|_{L^\infty(\partial T)}\|\sigma_r - I_h\sigma_r\|_{L^1(\partial T)}.
$$

Standard interpolation estimates for linear polynomials imply

$$
\|\sigma_r - I_h\sigma_r\|_{L^1(\partial T)} \leq ch^2|\sigma_r|_{H^2(T)}.
$$

For the sake of completeness, we derive this estimate step by step. The trace identity for any function $v \in H^1(T)$ and triangle $T = \operatorname{conv}\{E, p\}$ with edge E and opposite node P reads

$$
\int_E v ds = \frac{|E|}{|T|}\int_T v dx + \frac{|E|}{2|T|}\int_T \nabla v \cdot (x - P) dx.
$$

Setting $v := |\sigma_r - I_h\sigma_r|$ in this identity yields

$$
\begin{aligned}
&\int_E |\sigma_r - I_h\sigma_r| ds \\
&\leq |E||T|^{-1}\left(\|\sigma_r - I_h\sigma_r\|_{L^1(T)} + 1/2\,\|x - P\|_{L^2(T)}\|\nabla(\sigma_r - I_h\sigma_r)\|_{L^2(T)}\right) \\
&\leq |E||T|^{-1/2}\left(\|\sigma_r - I_h\sigma_r\|_{L^2(T)} + h_T/2\,\|\nabla(\sigma_r - I_h\sigma_r)\|_{L^2(T)}\right) \\
&\leq ch_T^2|\sigma_r|_{H^2(T)}
\end{aligned}
$$

for some constant c that depends only on the shape of T.

For the term $\|\Pi_0\,\Delta\mathbf{u}\|_{L^\infty(\partial T)}$, we observe, that

$$\|\Pi_0\,\Delta\mathbf{u}\|_{L^\infty(\partial T)} = \frac{1}{|T|}\left|\int_T \Delta\mathbf{u}\;dx\right| \le \|\Delta\mathbf{u}\|_{L^\infty(T)} \le c\|\Delta\mathbf{u}\|_{H^2(T)}.$$

Combining this, we can bound (20) as follows

$$\begin{aligned}\left|\left(\Pi_0\,\Delta\mathbf{u}, \mathbf{u}_r - \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}}\,\mathbf{u}_r\right)\right| &\le \left|\sum_{T\in\mathcal{T}_{\mathrm{h}}} (\Pi_0\,\Delta\mathbf{u}, \mathbf{n}\times(\sigma_r - I_h\sigma_r)_{\partial T}\right| \\ &\le ch^2 \sum_{T\in\mathcal{T}_{\mathrm{h}}} \|\Delta\mathbf{u}\|_{H^2(T)}|\sigma_r|_{H^2(T)} \\ &\le ch^2\|\Delta\mathbf{u}\|_{H^2}|\sigma_r|_{H^2} \\ &\le ch^2\|\Delta\mathbf{u}\|_{H^2}\|\mathbf{r}\|_{L^2}.\end{aligned}$$

Here, the elementwise L^∞ norm of $\Delta\mathbf{u}$ was estimated by $\|\Delta\mathbf{u}\|_{H^2(T)}$ to avoid a dependence on the number of elements.

3d In 3d, $\widetilde{V_h}$ is the space of Nedelec elements with the corresponding interpolation, see, e.g., [2, Table 5.2] or [28] for the original definition of the elements. Unfortunately, $\widetilde{V_h}$ does contain all constants, but not all linear polynomials. Hence, the argument from the 2d case needs to be modified, since the interpolation of σ_r can at most provide one power of h. To this end, we utilize the representation (20)

$$\left|\left(\Pi_0\,\Delta\mathbf{u}, \mathbf{u}_r - \boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{RT}}\,\mathbf{u}_r\right)\right| = \left|\sum_{T\in\mathcal{T}_{\mathrm{h}}} (\Pi_0\,\Delta\mathbf{u}, \mathbf{n}\times(\sigma_r - I_h\sigma_r)_{\partial T}\right|$$

and proceed with a different splitting

$$|(\Pi_0\,\Delta\mathbf{u}, \mathbf{n}\times(\sigma_r - I_h\sigma_r))_{\partial T}| \le \|\Pi_0\,\Delta\mathbf{u}\|_{L^1(\partial T)}\|\sigma_r - I_h\sigma_r\|_{L^\infty(\partial T)}.$$

A straight forward calculation utilizing the shape regularity, i.e., $|\partial T| \le ch^2$, gives

$$\begin{aligned}\|\Pi_0\,\Delta\mathbf{u}\|_{L^1(\partial T)} &= \int_{\partial T} |\,\Pi_0\,\Delta\mathbf{u}|\;ds \\ &\le \|\Pi_0\,\Delta\mathbf{u}\|_{L^\infty(\partial T)}\int_{\partial T}\;ds \\ &\le ch^2\|\Delta\mathbf{u}\|_{L^\infty(T)} \\ &\le ch^2\|\Delta\mathbf{u}\|_{H^2(T)}.\end{aligned}$$

For the interpolation error estimate, we employ a Bramble-Hilbert type argument. For this, we note that by assumption any elements T can be obtained by an affine linear transformation $A_T\colon \widehat{T} \to T$ from some reference element $\widehat{T}$. For any function f on T, we denote by $\widehat{f}$ its pullback onto $\widehat{T}$, i.e., $\widehat{f}(\widehat{x}) = f(A_T\widehat{x})$ for any $\widehat{x} \in \widehat{T}$. By definition of the nodal-variables of the

Nedelec element it is $\widehat{I_h f} = \widehat{I}_h \widehat{f}$ where $\widehat{I}_h$ is the local interpolation operator on the reference element. Utilizing L^∞ stability of $\widehat{I}_h$, we conclude using that σ_r is continuous

$$\begin{aligned}\|\sigma_r - I_h\sigma_r\|_{L^\infty(\partial T)} &\leq \|\widehat{\sigma_r} - \widehat{I}_h\widehat{\sigma_r}\|_{L^\infty(\partial\widehat{T})}\\ &\leq c\|\widehat{\sigma_r}\|_{L^\infty(\widehat{T})}\\ &\leq c\|\sigma_r\|_{L^\infty(T)}\\ &\leq c\|\sigma_r\|_{H^2(T)}\end{aligned}$$

with a constant c depending on the shape regularity of the element only. Analogous to the 2d case, the assertion follows. □

Remark 6. *Again, the significance of Theorem 2 lies in the fact that the velocity error $\|\mathbf{u}-\mathbf{u}_h\|_{L^2}$ is independent of the pressure. The additional regularity assumption, needed for the proof of the optimal $\mathcal{O}(h^2)$ error estimate is a consequence of the variational crime commited in the definition of (11), where piecewise linear discretely divergence-free functions are mapped onto divergence-free piecewise constant Raviart–Thomas functions. It is well-known from the classical theory of variational crimes that higher regularity assumptions than usual are necessary for proving optimal error estimates, when the right-hand side is projected onto a polynomial space of less than optimal order as it happens e.g. with quadrature rules [12].*

4 Numerical Experiments

This section deals with three numerical experiments to validate and confirm the theory. The first example demonstrates the benefits of the modified method. The next two examples focus on the convergence rate of the L^2 velocity error under low regularity. Here, we set in each case $p = 0$, since in this case the classical Crouzeix–Raviart element performs best and we want to compare with those results without being distracted by pressure effects in the non-modified standard method. Please note, that $p = 0$ is the (quite unrealistic) worst-case for the modified Crouzeix–Raviart element.

4.1 First Example

The first benchmark example studies the Stokes problem with the exact solution $\mathbf{u} = \mathrm{rot}\xi \in P_7(\Omega)^2 \cap V$ for the stream function

$$\xi = x^2(1-x)^2y^2(1-y)^2$$

and the pressure $p = x^3 + y^3 - 1/2$ on the unit square $\Omega = (0,1)^2$. For given viscosity ν, the volume force equals $\mathbf{f} := -\nu\Delta\mathbf{u} + \nabla p$.

ndof	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2}$	order	$\|\mathbf{u}-\mathbf{u}_h\|_{1,h}$	order	$\|p-p_h\|_{L^2}$	order
119	1.9192e-02		2.1347e-01		2.3999e-01	
559	5.1086e-03	2.11	1.1112e-01	1.04	1.1089e-01	1.23
2431	1.1464e-03	2.24	5.2654e-02	1.12	5.1635e-02	1.15
9919	2.9411e-04	2.03	2.6943e-02	1.00	2.5888e-02	1.03
39975	7.4623e-05	2.01	1.3516e-02	1.01	1.2754e-02	1.04
161127	1.8719e-05	2.01	6.7697e-03	1.00	6.3533e-03	1.01

Table 1: Convergence history and convergence order for all error norms for the standard method in the example of Section 4.1 for $\nu = 1$.

ndof	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2}$	order	$\|\mathbf{u}-\mathbf{u}_h\|_{1,h}$	order	$\|p-p_h\|_{L^2}$	order
119	3.6198e-03		4.9825e-02		2.2590e-01	
559	9.1389e-04	2.20	2.4666e-02	1.12	1.0663e-01	1.20
2431	2.2772e-04	2.09	1.2340e-02	1.04	5.0162e-02	1.13
9919	5.7381e-05	2.05	6.1891e-03	1.03	2.5426e-02	1.01
39975	1.4689e-05	2.00	3.1160e-03	1.01	1.2607e-02	1.03
161127	3.6552e-06	2.02	1.5561e-03	1.01	6.3002e-03	1.01

Table 2: Convergence history and convergence order for all error norms for the modified method in the example of Section 4.1 for $\nu = 1$.

Tables 1-2 show the error norms and their convergence orders for $\nu = 1$. The error for the standard method is significantly larger than for the modified method due to the influence of the pressure. Table 3 compares the results on a fixed mesh for different ν towards zero. The velocity errors of the standard method get polluted more and more as indicated by the a priori error estimate, while the modified method is robust and shows no changes in the velocity error.

4.2 Second Example

The second example considers the stream function $w(x,y) = r^2 \log|\log|r||$ with $r(x,y) := \sqrt{x^2+y^2}$ with the exact solution $\mathbf{u} := \operatorname{rot}(w(x,y))$ and right-

	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2}$		$\|\mathbf{u}-\mathbf{u}_h\|_{1,h}$		$\|p-p_h\|_{L^2}$	
ν	(standard)	(modified)	(standard)	(modified)	(standard)	(modified)
1e1	4.0749e-05	8.3527e-05	4.6015e-03	7.4528e-03	1.5977e-02	1.5939e-02
1e0	1.8296e-04	8.3527e-05	1.4125e-02	7.4528e-03	1.2286e-02	1.1921e-02
1e-1	1.7930e-03	8.3527e-05	1.3429e-01	7.4528e-03	1.2244e-02	1.1874e-02
1e-2	1.7927e-02	8.3527e-05	1.3422e+00	7.4528e-03	1.2243e-02	1.1873e-02
1e-3	1.7927e-01	8.3527e-05	1.3422e+01	7.4528e-03	1.2243e-02	1.1873e-02

Table 3: Comparison of error norms for both methods in the example of Section 4.1 for different ν and a fixed mesh with ndof = 8063.

ndof	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2}$	order	$\|\mathbf{u}-\mathbf{u}_h\|_{1,h}$	order	$\|p-p_h\|_{L^2}$	order
1587	1.3437e-02		7.4276e-01		7.5464e-02	
6386	3.3821e-03	2.11	3.7403e-01	1.05	4.0210e-02	0.96
25896	8.4052e-04	2.06	1.8577e-01	1.03	1.9510e-02	1.07
103677	2.1986e-04	1.97	9.4845e-02	0.99	9.7066e-03	1.02

Table 4: Convergence history and convergence order for all error norms for the standard method in the example of Section 4.2.

ndof	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2}$	order	$\|\mathbf{u}-\mathbf{u}_h\|_{1,h}$	order	$\|p-p_h\|_{L^2}$	order
1587	2.8328e-02		1.2626e+00		2.1148e-01	
6386	7.4116e-03	2.05	6.4892e-01	1.02	9.6991e-02	1.19
25896	1.8752e-03	2.03	3.2773e-01	1.01	3.7382e-02	1.41
103677	4.7668e-04	2.01	1.6561e-01	1.00	1.7856e-02	1.08

Table 5: Convergence history and convergence order for all error norms for the modified method in the example of Section 4.2.

hand side $f := -\Delta\mathbf{u}$ on the domain $\Omega := (-3/7, 4/7)^2$. The exact solution satisfies $u \in H^2(\Omega)$ but not $\Delta\mathbf{u} \in L^\infty(\Omega)$. The unstructured meshes for the computations were generated with `Triangle` [32] and the unsymmetric bounds of the domain ensure that the singular point $(0, 0)$ is not a node of the meshes.

Tables 4 and 5 show that all error norms under consideration converge with optimal speed for both methods. The results indicate, that the required regularity $\Delta u \in H^2(\Omega)$ in the statement of Theorem 2 can potentially be relaxed.

4.3 Third Example

The third example concerns the 3d velocity

$$\mathbf{u}(x, y, z) := \frac{13}{5}\begin{pmatrix} 5y - 3z \\ -5x + 2z \\ 3x - 2y \end{pmatrix} r^{-1/2+1/100} \quad \text{with } r^2 := x^2 + y^2 + z^2$$

and the right-hand side $f := -\Delta\mathbf{u}$ on the unit cube $\Omega := (-0.5, 1)^3$. The exact solution satisfies $u \in H^2(\Omega)$ but not $\Delta\mathbf{u} \in L^\infty(\Omega)$. The unstructured meshes for the computations were generated with `TetGen` [33] and the unsymmetric bounds of the domain ensure that the singular point $(0, 0)$ is not a node of the meshes.

Tables 6 and 7 suggest that there is no reduction of the optimal convergence order in case $u \in H^2(\Omega)$ but $\Delta u \notin L^\infty(\Omega)$.

ndof	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2}$	order	$\|\mathbf{u}-\mathbf{u}_h\|_{1,h}$	order	$\|p-p_h\|_{L^2}$	order
113	1.1393e+00	0.00	6.9830e+00	0.00	1.2373e+00	0.00
882	5.1801e-01	1.91	5.0883e+00	0.77	4.8584e-01	2.26
5662	2.1333e-01	2.09	3.5125e+00	0.87	6.5409e-01	-0.70
44752	5.8255e-02	2.23	1.8796e+00	1.07	3.1996e-01	1.23
359529	1.7149e-02	1.92	1.0506e+00	0.91	1.8718e-01	0.84

Table 6: Convergence history and convergence order for all error norms for the standard method in the example of Section 4.3.

ndof	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2}$	order	$\|\mathbf{u}-\mathbf{u}_h\|_{1,h}$	order	$\|p-p_h\|_{L^2}$	order
113	1.8259e+00	0.00	8.9380e+00	0.00	5.0305e+00	0.00
882	8.4509e-01	1.87	6.9863e+00	0.60	1.3127e+00	3.25
5662	3.8754e-01	1.84	5.4600e+00	0.58	1.4177e+00	-0.18
44752	1.0244e-01	2.28	2.9540e+00	1.05	6.5389e-01	1.33
359529	2.6287e-02	2.14	1.5657e+00	1.00	2.4901e-01	1.52

Table 7: Convergence history and convergence order for all error norms for the modified method in the example of Section 4.3.

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